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# INTERSECTIONS OF NILPOTENT HALL SUBGROUPS

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**A family $\mathscr{H}$ of subgroups of a finite group $G$ is said to *satisfy (property)* $B^*$ if whenever $U = H_1 \cap \cdots \cap H_r$ is a representation of $U$ as intersection of elements of $\mathscr{H}$ of minimal length $r$, then $r \leqq 2$. The aim of this paper is to prove**

**THEOREM 1. Let $H$ be a nilpotent Hall $\pi$-subgroup of a group $G$ and assume that if $H_1, H_2 \in S_\pi(G)$ then $H_1 \cap H_2 \lhd H_1$. Then $S_\pi(G)$ satisfies $B^*$.**

All groups in this work are finite. A family $\mathscr{H}$ of subgroups of a group $G$ will be said *to satisfy (property)* $B$ if there exist $H_1$ and $H_2$ in $\mathscr{H}$ such that

$$H_1 \cap H_2 = \bigcap \{H \mid H \in \mathscr{H}\} .$$

We will denote by $S_p(G)$ the family of Sylow $p$-subgroups of $G$ and the (possibly empty) family of Hall $\pi$-subgroups of $G$ will be denoted by $S_\pi(G)$. It was shown by Brodkey [1] that if $G$ possesses an Abelian Sylow $p$-subgroup, then $S_p(G)$ satisfies $B$. Itô has shown in [3] that if $G$ is of odd order, hence solvable by [2], then $S_p(G)$ satisfies $B$ for all primes. He has also shown that if $G$ is solvable, then $S_p(G)$ satisfies $B$ in several other cases.

As indicated above, we will consider here a more restrictive condition $B^*$ on families of subgroups of $G$. It follows from our main result, Theorem 1, that even the property $B^*$ is satisfied by $S_\pi(G)$, whenever $G$ possesses an Abelian or Hamiltonian (i.e., Dedekind) Hall $\pi$-subgroup. Theorem 1 yields the following

COROLLARY 1. *Let $H$ be a nilpotent Hall subgroup of the group $G$ and suppose that the index $[H\colon H \cap H^x]$ is prime for all $x \in G - N_G(H)$. Then either $H \lhd G$ or for all $x$, $y \in G$ such that $xy^{-1} \notin N_G(H)$ we have*

$$H^x \cap H^y = B \lhd G$$

*and $[H\colon B] = p$, a prime. $B$ is independent of $x$ and $y$.*

**2. Generalizations.** As a matter of fact, we will prove a more general result than Theorem 1. We will say that a group $N$ *satisfies* (*property*) $D_\pi$, where $\pi$ is a set of primes, if $N$ contains at least one Hall $\pi$-subgroup, any two Hall $\pi$-subgroups of $N$ are conjugate and each $\pi$-subgroup of $N$ is contained in a Hall $\pi$-subgroup of $N$. The set of all maximal $\pi$-subgroups of a group $G$ will be denoted by $\mathrm{Syl}_\pi(G)$. Theorem 1 follows from

THEOREM 2. *Let the group $G$ satisfy the following conditions.*

( i ) *If $H_1, H_2 \in \mathrm{Syl}_\pi(G)$ then $C = H_1 \cap H_2 \lhd H_1$* and

( ii ) *If $C = H_1 \cap H_2 \ntriangleleft G$, then $N_G(C)$ satisfies $D_\pi$. Then* $\mathrm{Syl}_\pi(G)$ *satisfies $B^*$.*

A subgroup $N$ of the group $G$ will be called *a* $\pi$-local *subgroup of* $G$ if $N = N_G(H)$, where $H$ is a nontrivial $\pi$-subgroup of $G$. An *$N_\pi$-group* is a group all of whose $\pi$-local subgroups are solvable. Theorem 1 yields the following

THEOREM 3. *Let $G$ be an $N_\pi$-group and suppose that all its maximal $\pi$-subgroups are Dedekind groups. Then* $\mathrm{Syl}_\pi(G)$ *satisfies $B^*$.*

As a consequence, we have

COROLLARY 2. *Let $G$ be a nonsolvable $N_\pi$-group and suppose that each $H \in Syl_\pi(G)$ is a Dedekind group. Then there exist $H_1, H_2 \in$* $\mathrm{Syl}_\pi(G)$ *such that*:

$$o(G) \geqq o(H_1)o(H_2) .$$

**3. Proofs.** We begin with a proof of Theorem 2. Let $U = H_1 \cap \cdots \cap H_r$ be a representation of $U$ as intersection of elements of $\mathrm{Syl}_\pi(G)$ of minimal length and suppose that $r > 2$. It follows from assumption ( i ) and the minimality of $r$ that

$$U \subsetneqq H_1 \cap H_2 = C \lhd H_1, H_2 . \tag{1}$$

The minimality of $r$ also implies that $C \not\subset H_3$ and consequently $C \ntriangleleft G$. By assumption (ii) $N = N_G(C)$ has the $D_\pi$-property and by (1) $H_1, H_2 \in S_\pi(N)$. There exists $Q \in S_\pi(N)$ containing $H_3 \cap \cdots \cap H_r \cap N$ and since the elements of $S_\pi(N)$ are conjugate in $N$, there exists $R \in S_\pi(N)$ such that $Q \cap R = C$. Since $Q$ and $R$ are conjugates of $H_1$, they belong to $\mathrm{Syl}_\pi(G)$. However,

$$\begin{aligned} U &= Q \cap R \cap H_3 \cap \cdots \cap H_r \supset R \cap N \cap H_3 \cap \cdots \cap H_r \\ &= R \cap H_3 \cap \cdots \cap H_r \supset U \end{aligned}$$

in contradiction to the minimality of $r$. The proof of Theorem 2 is complete.

Theorem 1 follows immediately from Theorem 2 and Wielandt's Theorem [4] which states that if $G$ contains a nilpotent Hall $\pi$-subgroup then $G$ satisfies $D_\pi$. Theorem 3 also follows immediately from Theorem 2.

Corollary 1 follows from Theorem 1. Suppose that $H \ntriangleleft G$, and let $H_1, H_2 \in S_\pi(G)$, $H_1 \neq H_2$. Since by our assumptions and the above mentioned Theorem of Wielandt $[H_1: H_1 \cap H_2]$ is prime, it follows that $H_1 \cap H_2 \triangleleft H_1$. Let $B = \bigcap \{H^x \mid x \in G\}$; obviously $B \triangleleft G$ and since $H \ntriangleleft G$, $B \subsetneqq H$. By Theorem 1 there exist $u, v \in G$ such that for all $x, y \in G$ we have:

$$H^u \cap H^v = B \subset H^x \cap H^y .$$

Suppose that $xy^{-1} \notin N_G(H)$; then $H^x \neq H^y$ and

$$p = [H^u: B] = [H^x: B] = [H^x: H^x \cap H^y][H^x \cap H^y: B] .$$

Since $[H^x: H^x \cap H^y] > 1$, it follows that $H^x \cap H^y = B$. The proof of Corollary 1 is complete.

Finally, Corollary 2 follows from Theorem 3. Since $G$ is a non-solvable $N_\pi$-group, $\bigcap \{H \mid H \in \mathrm{Syl}_\pi(G)\} = \{1\}$. Consequently, by Theorem 3, there exist $H_1, H_2 \in \mathrm{Syl}_\pi(G)$ such that $H_1 \cap H_2 = \{1\}$. Obviously $o(G) \leqq o(H_1)o(H_2)$ and the proof is complete.

## REFERENCES

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