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REGULAR ELEMENTS IN P.I.-RINGS

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It follows from the proof of Posner's theorem that half-regular elements are regular in prime rings satisfying a polynomial identity (prime P. I.-rings). In this paper we extend these results to semi-prime rings and present counter-examples to several avenues of further generalization.

Throughout this paper all rings will be algebras over a commutative ring. We further assume that the polynomial identities which occur have at least one invertible coefficient. If T is a subset of a ring R then $l(T)$ $(r(T))$ will denote the left (right) annihilator of T. The word "ideal" will mean two-sided ideal. Finally, we recall that if R is semi-prime and if U is an ideal of R then $l(U) = r(U)$. In this case we write $l(U)$, unambiguously, as $\mathrm{Ann}(U)$.

2. We begin with a mild generalization of a result due to Amitsur [1].

LEMMA 1. *Let R be a ring such that Ra satisfies a polynomial identity; then, if $l(a) = 0$, Ra contains a nonzero ideal of R.*

Proof. Among the left ideals Ra^i suppose that Ra^k satisfies an identity of lowest degree. We may assume that this identity is multilinear and has form

$$q(x_1 \cdots, x_n) = q_1(x_1, \cdots, x_{n-1})x_n + q_2(x_1, \cdots\cdot, x_n)$$

where q_1 is of lower degree than q and where x_n does not occur as the last variable of any monomial of q_2. Substitute $r_j a^{2k}$ for x_j for $j=1, \cdots, n-1$ and $r_n a^k$ for x_n, where $r_1, \cdots, r_n$ are arbitrary elements of R, in $q(x_1, \cdots, x_n)$. Since $Ra^{2k} \subset Ra^k$, Ra^{2k} satisfies q and, by our choice of k, no identity of lower degree. Therefore there exist $r_1, \cdots, r_{n-1}$ in R such that $q_1(r_1a^{2k}, \cdots, r_{n-1}a^{2k}) \neq 0$. Freeding this into our identity q we obtain

$$q_1(r_1a^{2k}, \cdots, r_{n-1}a^{2k})r_na^k = -q_2(r_1a^{2k}, \cdots, r_{n-1}a^{2k}, r_na^k)$$

which is contained in Ra^{2k} from the form of q_2. Since $l(a) = 0$ this yields $q_1(r_1a^{2k}, \cdots, r_{n-1}a^{2k})r_n \in Ra^k$. In short, $q_1(r_1a^{2k}, \cdots, r_{n-1}a^{2k})\, R \subset Ra^k$, hence the nonzero ideal $Rq_1(r_1a^{2k}, \cdots, r_{n-1}a^{2k})R$ is contained in Ra^k, and so, in Ra. This proves the result.

The plan now is to study Ra by looking at the ideals of R contained in it. The crucial step is

THEOREM 2. *Suppose that R is a semi-prime ring; if $a \in R$ is such that $l(a) = 0$ and Ra satisfies a polynomial identity then Ra contains an ideal of R whose annihilator is zero.*

Proof. Let U be the sum of the ideals of R which are contained in Ra. We claim that $\mathrm{Ann}(U) = 0$. If not, let $W = \mathrm{Ann}(U) \neq 0$, and $V = \mathrm{Ann}(W)$. Pass to the ring $\bar{R} = R/V$. If $\bar{x}\bar{a} = 0$ in $\bar{R}$ then $xa \in V$ hence $Wxa = 0$; since $l(a) = 0$ this gives $Wx = 0$, and so, $x \in V$, $\bar{x} = 0$. Thus $l(\bar{a}) = 0$. Clearly $\bar{R}\bar{a}$ satisfies a polynomial identity. Therefore $\bar{R}\bar{a}$ contains a nonzero ideal $\bar{T}$ of $\bar{R}$; the inverse image T of $\bar{T}$ thus lies in $Ra + V$. Since $\bar{T} \neq 0$, $T \not\subset V$ therefore $0 \neq WT \subset Ra + WV$. But $WV = 0$. Consequently WT is a nonzero ideal of R lying in Ra. As such, it must be contained in U. But $WU = 0$, so $(WT)^2 \subset W^2T = 0$. Thus semi-primeness of R then forces the contradiction $WT = 0$. With this, the theorem is proved.

From Theorem 2 many good things flow.

THEOREM 3. *Suppose that R is a semi-prime P.I.-ring. If $a \in R$ satisfies $l(a) = 0$ then*

1. $r(a) = 0$
2. *Ra is essential.*

Proof. 1. Let U be the ideal in Ra of Theorem 2. If $ax = 0$ then $Ux = 0$, which is not possible unless $x = 0$. Thus $r(a) = 0$.

2. If I is a nonzero left ideal then $0 \neq UI \subset U \cap I \subset Ra \cap I$.

A ring R is said to be *von Neumann finite* if for $x, y \in R$, $xy = 1$ implies $yx = 1$. If R_n is v. N. finite for all n, we call R N-finite.

COROLLARY. *A P. I.-ring is N-finite.*

Proof. The result follows easily from the following two observations:

1. if R is a P. I.-ring then R_n is a P. I.-ring [3].
2. R is v. N. finite if and only if $R/J(R)$ is, where $J(R)$ is the Jacobson radical of R.

Hence we can reduce to the semi-simple (and so, semi-prime) case. If $xy = 1$ then $l(x) = 0$ where, by Theorem 3, $r(x) = 0$. Since $x(1 - yx) = 0$ we get $yx = 1$.

Theorem 2 also tells us something about the nature of the identities satisfied by R and Ra.

THEOREM 4. *If R is a semi-prime ring and if $a \in R$ satisfies $l(a) = 0$ then R satisfies any polynomial identity satisfied by Ra.*

Proof. The argument follows one by Goldie [2]. Since R is semi-prime, $0 = \cap P_\alpha$ where P_α are prime ideals. Let $U \subset Ra$ be an ideal of R such that $\mathrm{Ann}(U) = 0$. Now $U \not\subset P_\beta$ for some prime ideal P_β. Divide the prime ideals of R into two parts: those which contain U and those which do not. The intersection of the primes in the first part contains U and is annihilated by the intersection of the primes in the second part. But $\mathrm{Ann}(U) = 0$, so this latter intersection must be 0. Hence $0 = \cap P_\gamma$ where the P_γ are prime ideals and $U \not\subset P_\gamma$ for each γ. We find, then, that $R_\gamma = R/P_\gamma$ has a nonzero ideal $(U + P_\gamma)/P_\gamma$ which satisfies an identity. Since R_γ is prime, it satisfies the same identity as $(U + P_\gamma)/P_\gamma$ [1]. To finish up, we note that R is a subdirect sum of the R_γ, hence satisfies any identity of U, therefore any identity of Ra.

3. In this section we present several counter-examples to possible generalizations of the results in § 2. We begin with examples to show that "semi-prime" is needed in Theorem 3.

Let F be a field and $F[x]$ the polynomial ring in x over F. Form the ring $S^{(1)} = \begin{pmatrix} F[x] & F \\ 0 & F \end{pmatrix}$, where $F[x]$ acts on F in the usual way (identifying $F = F[x]/(x)$ as an $F[x]$-module). $S^{(1)}$ satisfies the identity $(ab - ba)^2 = 0$. It is easy to see that $l\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = 0$, but $r\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \neq 0$.

Now form the ring $S^{(2)} = \begin{pmatrix} F[x] & F[x] \\ 0 & F \end{pmatrix}$ with the obvious actions on $F[x]$. $S^{(2)}$ satisfies the same identity as $S^{(1)}$. The element $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ is regular in $S^{(2)}$ but $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} S^{(2)} \cap \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} = 0$—that is, the right ideal $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} S^{(2)}$ is not essential. We pause to note that this implies that $S^{(2)}$ does not satisfy the right Ore condition. Yet $S^{(2)}$ possesses a ring of left quotients which even is Artinian.

We conclude this section with a simple example of a right Noetherian ring which lacks a right ring of quotients. Let R be any commutative Noetherian ring with the following property: there exists an element $a \in R$ which is not regular but its image, $\bar{a}$, is regular in $\bar{R} = R/N$ where N is the nil radical of R. (An example of such is $\dfrac{F[x, y]}{(x^2, xy)}$ where $a = y + (x^2, xy)$.) Our example is $S^{(2)} = \begin{pmatrix} \bar{R} & \bar{R} \\ 0 & R \end{pmatrix}$.

The element $\begin{pmatrix} \bar{a} & 0 \\ 0 & 1 \end{pmatrix}$ is quickly seen to be regular in $S^{(3)}$. If the right Ore condition were valid we would have an equation

$$\begin{pmatrix} \bar{a} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{d} & \bar{c} \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & \bar{1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{r} & \bar{s} \\ 0 & t \end{pmatrix}$$

where $\begin{pmatrix} \bar{r} & \bar{s} \\ 0 & t \end{pmatrix}$ was regular. This forces t to be regular in R. Writing the relations out explicitly, we have $\bar{a}\bar{c} = \bar{t}$, which means that $ac = t + n$ where $n \in N$. But t is regular, hence $t + n$ is and so ac is regular. This contradicts our choice of a.

4. To finish up, we present a result on the rank of free modules over P. I.-rings which, for commutative rings, is a well-known theorem on homogeneous systems of linear equations. The proof we give may be of additional interest in that we cannot, of course, use determinants.

Denote by ${}_RR^{(n)}$ the external direct sum of n copies of ${}_RR$, that is, the free module on n basis elements.

THEOREM 5. *If R is a* P. I.*-ring, then $R^{(n)} \subset R^{(m)}$ implies $n \leqq m$.*

Proof. Suppose that $n > m$. First note that this forces $R^{(t)} \subset R^{(m)}$ for arbitrary t. To see this, write $R^{(n)} = R^{(m)} \oplus R^{(n-m)}$. We can find a copy of $R^{(n)}$ in the first summand, so $R^{(n)} \oplus R^{(n-m)} \subset R^{(m)}$. We now repeat the process on the "new" $R^{(n)}$. In particular, we obtain $R^{(2m)} \subset R^{(m)}$. This means that $R^{(m)}$ contains a set, $\alpha_1, \cdots, \alpha_{2m}$, of $2m$ linearly independent elements. We can consider the α's as $1 \times m$ row vectors and form the $m \times m$ matrices X and Y where the rows of X are $\alpha_1, \cdots, \alpha_m$ and those of Y are $\alpha_{m+1}, \cdots, \alpha_{2m}$. In R_m it is immediate that $l(X) = 0$ and $l(Y) = 0$ since $\alpha_1, \cdots, \alpha_{2m}$ are independent. But R_m is a P. I.-ring, so by Lemma 1 R_mX contains a nonzero ideal U. Now, since $l(Y) = 0$, $UR_mY \neq 0$ and is contained in R_mX. This yields nonzero matrices A and B such that $AX = BY$. Writing this out explicitly gives a dependence relation among the α's, a contradiction. The proof is complete.

REFERENCES

1. S. A. Amitsur, *Prime rings satisfying polynomial identities with arbitrary coefficients*, Proc. London Math. Soc. (3) **17** (1967), 470-486.

2. A. W. Goldie, *A note on prime rings with polynomial identities*, London Math. Soc. (2), **1** (1969), 606-608.

3. C. Procesi and L. Small, *Endomorphism rings of modules over P. I.-algebras*, Math. Zeit. **106** (1968), 178-180.

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