PACIFIC JOURNAL OF MATHEMATICS
Vol. 36, No. 2, 1971

REGULAR SEQUENCES AND MINIMAL BASES

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This note records the results of an effort to understand in simple terms a certain theorem of Lichtenbaum and Schlessinger: Let $I \supset J$ be ideals of noetherian local ring. If I and I/J are generated by regular sequences, then so is J. This theorem is closely related to the well known: *If R and R/P are regular local rings, then P is generated by part of a regular system of parameters.* We investigate the implications of " I/J is generated by a regular sequence " and discover an elementary theorem having both of these results as corollaries.

1. Notation and preliminaries.

We consider only commutative rings with $1 \neq 0$ and adopt the convention that local and semilocal rings need not be noetherian. Given a finitely generated ideal (or module) X, we denote by $\nu(X)$ the greatest integer n such that X has no basis of fewer than n elements; a basis of cardinality $\nu(X)$ will be called a *minimal basis*. We denote the following elementary fact as *Lemma A*: *Every basis of a module over a local ring contains a minimal basis.* Recall that a regular sequence in a ring R is a finite sequence $(x_1, \cdots, x_n)$ of elements of R such that $Rx_1 + \cdots + Rx_n \neq R$ and x_i is not a divisor of zero on $R/Rx_1 + \cdots + Rx_{i-1}$ $(1 \leqq i \leqq n)$. If the ideal X is generated by the members of such a sequence, then $\nu(X) = n$, for it is an easy exercise to check that X/X^2 is a free (R/X)-module of rank n. We shall have need of another elementary fact which we shall denote by *Lemma B*: *If an ideal of noetherian local ring is generated by a regular sequence, then any permutation of a minimal basis of the ideal is a regular sequence.* (A good account of all the elementary facts we need can be found, for example, in Kaplansky's book [2]; specifically, Lemma A is contained in Theorem 158, Lemma B in Theorem 129, and § 3-1 deals with regular sequences.) We shall be concerned with results of the type $\nu(I) = \nu(I/J) + \nu(J)$. This equation is clearly equivalent to: *The union of a minimal basis of J with a set of representatives of a minimal basis of I/J is a minimal basis of I.*

2. Noetherian local rings.

In this section $I \supset J$ are ideals of ideals of a noetherian local ring R. The relaxation of the conditions "local" and "noetherian" will be discussed in § 3.

THEOREM. *If I/J is generated by a regular sequence, then* $\nu(I) = \nu(I/J) + \nu(J)$.

COROLLARY 1. (Lichtenbaum and Schlessinger [3, 3.3.4]). *If both I and I/J are generated by regular sequences, then so is J.*

COROLLARY 2. *If R/J is regular, then J is generated by part of a minimal basis of the maximal ideal.* (Hence, if R and R/J are regular, then J is generated by part of a regular system of parameters.)

Corollary 2 is well known; its geometric content is: if p is a simple point of a subvariety W of a variety V, then "locally at p" W "looks" like the intersection of V with a linear space.

Proofs. The corollaries follow immediately from the theorem; the first by Lemma B, and the second by the well known fact that the maximal ideal of a regular local ring is generated by a regular sequence. To prove the theorem notice that by the hypothesis and Lemma A, I has a minimal basis $\{x_i, \cdots, x_n, y_1, \cdots, y_m\}$, where the y's lie in J and x's form a regular sequence mod J. By passing to $R/Ry_1 + \cdots + Ry_m$, we may assume that $\nu(I) = \nu(I/J)$ and must prove that $J = 0$. We proceed by induction on n. For $n = 1$, since x_1 is not a divisor of zero mod J, $J = Jx_1$, whence $J = 0$ by Nakayama's Lemma. For $n > 1$, an application of the case $n = 1$ to the ring $R/Rx_1 + \cdots + Rx_{n-1}$ shows that the canonical image of J is 0; that is, $Rx_1 + \cdots + Rx_{n-1} \supset J$, whence $J = 0$ by the induction hypothesis.

REMARK 1. The theorem also admits a proof by the methods of Lichtenbaum and Schlessinger. The exact sequence of the triple $(R, R/I, R/J)$ applied to the (R/J)-module R/J together with the fact that I/J is generated by a regular sequence gives the exactness of $0 \to J/IJ \to I/I^2 \to I/J + I^2 \to 0$ [3, 3.2.1 and proof of 3.3.4]. Since $I/J + I^2$ is (R/I)-free, the sequence splits, whence by "local", $\nu(I/I^2) = \nu(J/IJ) + \nu(I/J + I^2)$. The theorem follows since $\nu(I/I^2) = \nu(I)$, $\nu(J/IJ) = \nu(J)$, and $\nu(I/J + I^2) = \nu(I/J)$ by Nakayama's Lemma.

REMARK 2. Let $A \supset B$ be finitely generated R-modules, and let S denote the sequence $0 \to B \to A \to A/B \to 0$. Observe that for any ideal N, $S \otimes R/N$ is exact if, and only if, $B \cap NA = NB$. With this in mind one routinely abstracts the argument of the last two sentences of the preceding paragraph to prove the equivalence of:

(1) $\nu(A) = \nu(A/B) + \nu(B)$.

(2) For M the maximal ideal of R, $S \otimes R/M$ is exact (if, and only if, $B \cap MA = MB$).

(3) There exists an ideal N such that $S \otimes R/N$ is exact (if, and only if, $B \cap NA = NB$) and $A/B \otimes R/N$ is (R/N)-free.

(4) There exists an ideal N such that $S \otimes R/N$ is split exact. Generalization of the theorem beyond the realm of ideals generated by regular sequences is probably best studied through (4); that is, one could look for classes of pairs $A \supset B$ for which reasonable candidates for N would present themselves. In general (3) is probably not likely to be an improvement over (2) since one cannot reasonably expect the freeness of $A/B \otimes R/N$ for many N's.

REMARK 3. In view of the previous remarks we see that the Lichtenbaum-Schlessinger proof of the theorem amounts to showing that $J \cap I^2 = IJ$. Following the referee's suggestion to examine the appropriate Koszul complex we see where this point lies buried in our proof. The choice of the x's guarantees that the complex obtained by tensoring with R/J the Koszul complex they generate has vanishing homology in dimension 1; and the long exact sequence of "Koszul homology" associated to $0 \to J \to R \to R/J \to 0$ then gives the exactness of $0 \to J/KJ \to R/K \to R/I \to 0$, where $K = Rx_1 + \cdots + Rx_n$ (for details see, e.g. [1, 1.2 and 1.5]). Thus $J \cap K = KJ$ from which $J \cap I^2 = IJ$ immediately follows. (Actually our proof reduced to the case $I = K$ and showed $J = IJ$ which this argument also shows.)

3. On extending the results of § 2. *Graded rings.* We consider only graded rings R such that R_0 is a field and $R_i = 0$ for $i < 0$. As a rule theorems about local rings translate into theorems in the graded situation, and that is true of the results of § 2. The translations are accomplished via the graded versions of Lemmas A and B which are valid for homogeneous ideals and bases even without the noetherian hypothesis in Lemma B. The proof of the theorem works in the graded case without any assumption other than that I and J be homogeneous ideals; and it follows that the Lichtenbaum-Schlessinger result holds for homogeneous ideals in graded rings. The translation of Corollary 2 requires that $R = R[R_1]$ and that R_1 be of finite R_0-dimension (i.e., R is the quotient of a polynomial ring by a homogeneous ideal); whence: *If J is a homogeneous ideal such that R/J is isomorphic (as an abstract ring) to a polynomial ring, then J is generated by a subspace of R_1.*

Local rings. Our theorem is valid for arbitrary local rings provided that J is finitely generated, for that is the extent to which the proof requires "noetherian". That some vestige of "noetherian" must remain is clear—consider any valuation ring of Krull dimension greater than 1. Remark 2 is clearly valid for arbitrary local rings. Concerning Remark 3, observe that "$0 \to J/IJ \to I/I^2 \to I/J + I^2 \to 0$ is split exact" is a formal consequence of the vanishing of the appropriate "Koszul homology" in dimension 1; no other hypothesis is needed (not even "local") since that condition implies the (R/I)-freeness of $I/J + I^2$ as well as the exactness of $0 \to J/KJ \to R/I \to R/K \to 0$. (Of course this condition on the homology is equivalent in the noetherian local case to "the x's form a regular sequence mod J" [1, 2.8].) Stated for modules over local rings our theorem becomes: *If* $A \supset B$ *are finitely generated submodules of* C *such that* $A = B + Cx_1 + \cdots + Cx_n$, where *the* x's *form a* (C/B)*-regular sequence, then* $\nu(A) = \nu(A/B) + \nu(B)$. The proof given in § 2 works here—as does the proof of the generalization assuming only the vanishing in dimension 1 of the homology of the complex obtained by tensoring with C/B the Koszul complex generated by the x's. Recalling Remark 2 and letting $N = Rx_1 + \cdots + Rx_n$, one can observe that here we have a case in which $S \otimes R/N$ is split exact with $A/B \otimes R/N$ not necessarily (R/N)-free.

Semilocal rings. An easy application of the Chinese Remainder Theorem shows that for semilocal R, $\nu(X) = \max\{\nu(X \otimes R_M\}$, where M runs over the maximal ideals of R. With this fact and the local version of the theorem one readily proves that our theorem holds for arbitrary semilocal rings provided that I is a radical ideal (i. e., contained in every maximal ideal). Then because Lemma B holds for radical ideals in noetherian rings, it follows that the Lichtenbaum-Schlessinger result is valid for radical ideals in noetherian semilocal rings.

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Received March 16, 1970.

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