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A CONSTRUCTIVE PROOF OF SARD'S THEOREM

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The theorem of Sard asserts that if a mapping F from a region in R^m to R^p is smooth enough, then the set of critical values of F has measure zero in R^p. The purpose of this paper is to give a constructive proof of this theorem. By a constructive proof is meant one which has numerical content, as explained in E. Bishop's Foundations of Constructive Analysis. In particular, it is shown that in every open ball in R^p one can compute a point which is not a critical value of F.

The proof is based on one given by Milnor, which is a modification of a proof of Pontryagin. These proofs, as well as all other known proofs, are nonconstructive, and it is not obvious that they can be constructivized. One difficulty lies in the fact that, given two real numbers a and b, one cannot, in general, prove constructively that either $a \geqq b$ or $a < b$; one can only prove, for arbitrary $\varepsilon > 0$, that either $a > b - \varepsilon$ or $a < b$. This fact forces, among other things, the consideration of 'nearly critical values' instead of critical values, and the derivation of a slightly more general result. Once a proper interpretation for "nearly critical values" has been found, Milnor's proof can be followed, replacing various nonconstructive arguments by constructive ones.

1. Preliminaries.

1.1. Suppose f is a C^k function (in other words, that f has continuous partial derivatives of all orders not exceeding k) on a compact subset U of R^m. If the positive constant M is a bound for the absolute values of the partial derivatives of f (of order at most k and including f itself) on U, then we say M is a modulus of k-smoothness of f on U. Given natural numbers m and k, and a positive real number β, there are functions $\Phi_{k,+}$, $\Phi_{k,\times}$, $\Phi_{k,/}$, $\Phi_{k,s}$ from $(0, \infty) \times (0, \infty)$ to $(0, \infty)$ and a function Ψ_k from $(0, \infty)$ to $(0, \infty)$ with the following properties:

(i) Suppose f, g are C^k functions on $U = [a_1, b_1] \times \cdots \times [a_m, b_m] \subset R^m$, and suppose M_f and M_g are respectively moduli of k-smoothness of f and g on U. Then the numbers $\Phi_{k,+}(M_f, M_g)$ and $\Phi_{k,\times}(M_f, M_g)$ are moduli of k-smoothness for $f + g$ and fg respectively on U. If further $|g| \geqq \beta$, then $\Phi_{k,/}(M_f, M_g)$ is a modulus of k-smoothness for f/g on U.

(ii) Suppose f and g are respectively C^k functions on $U = [a_1, b_1] \times \cdots \times [a_m, b_m]$ and on

$$U' = [a_1, b_1] \times \cdots \times \widehat{[a_i, b_i]} \times \cdots \times [a_m, b_m]$$

(a hat signifies omission) with $g(y) \in [a_i, b_i]$ for each $y \in U'$. If M_f and M_g are moduli of k-smoothness for f and g respectively, then the function h defined on U' by $h(y) = f(x^1, \cdots, x^{i-1}, g(y), x^{i+1}, \cdots, x^m)$ (for each $y = (x^1, \cdots, \widehat{x^i}, \cdots, x^m)$ in U') will have $\Phi_{k,s}(M_f, M_g)$ as a modulus of k-smoothness.

(iii) Suppose f is a C^k function on $U = [a_1, b_1] \times \cdots \times [a_m, b_m]$ with M_f as a modulus of k-smoothness and $\partial f/\partial x_i \geqq \beta$. If g is a function from $U' = [a_1, b_1] \times \cdots \times \widehat{[a_i, b_i]} \times \cdots \times [a_m, b_m]$ to $[a_i, b_i]$ such that $f(x^1, \cdots, x^{i-1}, g(y), x^{i+1}, \cdots, x^m) = 0$ for every $y = (x^1, \cdots, \hat{x}^i, \cdots, x^m)$ in U', then $\Psi_k(M_f)$ is a modulus of k-smoothness for g on U'.

Existence of $\Phi_{k,+}$, $\Phi_{k,\times}$, $\Phi_{k,/}$, and $\Phi_{k,s}$ are obvious. By taking their maximum we can assume that they all equal the same function Φ_k. With notation as in (iii), the Implicit Function Theorem gives $\partial g/\partial x_j = -(\partial f/\partial x_j)/(\partial f/\partial x_i)$, and similar formulas give higher order partial derivatives of g. From these the existence of Ψ_k follows. Explicit forms of Φ_k and Ψ_k can be found (e.g., $\Phi_{k,+}(M, M') = M + M'$) but we shall not need them.

If $F = (F^1, \cdots, F^p)$ is a C^k function from a compact set U in R^m to R^p, then a positive real number M is said to be a modulus of k-smoothness of F if it is at the same time moduli of k-smoothness of each of $F^1, \cdots, F^p$. The partial derivative $\partial F^i/\partial x_j$ will be written F^i_j.

1.2. If m, p are natural numbers, $t = \min(m, p)$, and if (a^i_j) is a $p \times m$ matrix, then $D((a^i_j))$ will denote

$$\max\{|\det S|: S \text{ is a } t \times t \text{ submatrix of } (a^i_j)\} .$$

Suppose F is a C^1 function from a compact subset U of R^m to R^p. Then J_F will denote the function defined on U by $J_F(x) = D((F^i_j(x)))$. It can easily be shown classically that $J_F(x) = 0$ if and only if $F(x)$ is a critical value (i.e., if and only if the matrix $(F^i_j(x))$ is of less than full rank.) Thus, roughly speaking, $F(x)$ is a nearly critical value if $J_F(x)$ is very small. Clearly there exists a function which assigns to every positive real number M an operation $\omega(M): (0, \infty) \to (0, \infty)$ such that if M is a modulus of 2-smoothness of F then $\omega(M)$ is a modulus of continuity of J_F.

1.3. To simplify the notation we introduce the symbol $\langle m, p, K, \Delta, F, M\rangle$ to mean the statement:

m, p are natural numbers with $p \geqq 1$; $K = [a_1, b_1] \times \cdots \times [a_m, b_m]$ where $0 \leqq a_i \leqq b_i \leqq 1$ for each $i \in \{1, \cdots, m\}$, Δ is a positive real number; F is a $C^{k(m,p)}$ function from the closed neighborhood $U = [a_1 - \Delta, b_1 + \Delta] \times \cdots \times [a_m - \Delta, b_m + \Delta]$ to R^p with M as a modulus of $k(m, p)$-smoothness on U, where $k(m, p) = 2 + 2^{-1}(m - p)(m - p + 1)$.

2. Sard's Theorem.

THEOREM 2.1. *Given natural numbers m, p and positive real numbers Δ, M, and ε, there exists a natural number $n = n(m, p, \Delta, M, \varepsilon)$ and a positive real number $\nu = \nu(m, p, \Delta, M, \varepsilon)$ such that if $\langle m, p, K, \Delta, F, M\rangle$ then the set $\{F(x): x \in K, J_F(x) \leqq \nu\}$ is contained in n cubes in R^p whose edges are equal and whose total volume is less than ε.*

Proof of the theorem will be by induction, which is broken down into several lemmas.

LEMMA 2.2. *Assume 2.1 is proved for a particular natural number $m - 1$ ($m \geqq 1$). Then for every natural number $p \geqq 1$, and positive real numbers Δ, M, ε, and β, there exist a natural number $n' = n'(m, p, \Delta, M, \varepsilon, \beta)$ and a positive real number $\nu' = \nu'(m, p, \Delta, M, \varepsilon, \beta)$ with the following properties: if $\langle m, p, K, \Delta, F, M\rangle$, and if for some $i \in \{1, \cdots, p\}$ and $j \in \{1, \cdots, m\}$ the partial derivative F^i_j has absolute value always greater than β on U, then $\{F(x): x \in K, J_F(x) \leqq \nu'\}$ is contained in n' cubes in R^p whose edges are equal and whose total volume is less than ε.*

Proof. In case $p = 1$ we can take $\nu' = \beta/2$ and $n' = 0$. For by assumption $J_F \geqq |F^1_j| \geqq \beta > \nu'$ on U and so our set $\{F(x): x \in K, J_F(x) \leqq \nu'\}$ is void. Thus we can assume $p \geqq 2$. Without loss of generality we can also assume $M > 1$ and $\beta < 1/2$.

(i) Choose a natural number q and then a positive real number ε' for which the following inequalities hold:

$$q > 6\beta^{-1}\Delta^{-1}mM; \ \varepsilon'(q + 1)^{m-1}4^pM < \beta\varepsilon .$$

Since 2.1 is proved for $m - 1$, there exist a natural number $n_0 = n(m - 1, p - 1, q^{-1}, \Phi_k(M, \Psi_k(M)), \varepsilon')$ and a positive real number $\nu_0 = \nu(m - 1, p - 1, q^{-1}, \Phi_k(M, \Psi_k(M)), \varepsilon')$ having the properties as described in 2.1. Let ε'' be a positive real number and d a natural number such that

$$\varepsilon'' = (\varepsilon'/n_0)^{1/(p-1)} ,$$

$$d^{-1} < (p - 1)^{-1}M^{-1}\varepsilon'' ,$$

and

$$d^{-1} < \omega(M)(2^{-1}\beta\nu_0)$$

where ω was defined in 1.2. Since $d\varepsilon'' > 1$ and $M\beta^{-1}d > 1 \cdot 2 \cdot 1 > 1$, we can find natural numbers ρ and D such that

$$3d\varepsilon'' < \rho < 4d\varepsilon'' ,$$

and

$$2M\beta^{-1}d < D < 4M\beta^{-1}d .$$

Now let

$$n' = n_0\rho^{p-1}D(q+1)^{m-1}$$

and let

$$\nu' = 2^{-1}\beta\nu_0 .$$

We shall show n' and ν' have the desired properties.

(ii) Thus suppose $\langle m, p, K, \Delta, F, M\rangle$ and (by relabelling) $|F_1^1| \geqq \beta$ on U. Without loss of generality assume $K = [0, 1]^m$ and $F_1^1 \geqq \beta$ on U. Let $\{\theta_i: i = 1, \cdots, (q+1)^{m-1}\}$ be a family of cubes (in $[0, 1]^{m-1}$) of edges q^{-1} which covers $[0, 1]^{m-1}$.

In (iii) and (iv) let i be an arbitrary (but fixed) member of $\{1, \cdots, (q+1)^{m-1}\}$.

(iii) Since $D > 2M\beta^{-1}d$ and since $|F^1| \leqq M$ on $[0, 1] \times \theta_i$ we can find D points $x_h = (x_h^1, \cdots, x_h^m)(h = 1, \cdots, D)$ in $[0, 1] \times \theta_i$ such that $\{F^1(x_h)\}$ form a βd^{-1}-net for $F^1([0, 1] \times \theta_i)$. Let $h \in \{1, \cdots, D\}$ be arbitrary. Let U_i be the cube in R^{m-1} with edge $3q^{-1}$ and same center as θ_i, the edges of U_i being parallel to corresponding ones of θ_i. Then for every $y = (x^2, \cdots, x^m)$ in U_i we have

$$|y^j - x_h^j| \leqq 3q^{-1} \qquad (j = 2, \cdots, m) .$$

If we let $c_h = F^1(x_h)$ it follows that

$$|F^1(x_h^1, y) - c_h| \leqq 3q^{-1}mM < \beta\Delta/2 .$$

Therefore, since $F_1^1 \geqq \beta > 0$, there exists $g^h(y) \in [-\Delta, 1 + \Delta]$ such that

$$F^1(g^h(y), y) - c_h = 0 .$$

By the Implicit Function Theorem and by definition of Ψ_k, the function g^h is a C^k function with $\Psi_k(M)$ as a k-smoothness modulus. ($k = k(m, p) = k(m-1, p-1)$.) Now define a function

$$G^h: U_i \longrightarrow R^{p-1}$$

by

$$G^h(y) = (F^2(g^h(y), y), \cdots, F^p(g^h(y), y)) .$$

Then G^h is a C^k function with $\Phi_k(M, \Psi_k(M))$ as a k-smoothness modulus. Therefore, by the definitions of n_0, ν_0, and ε'', the set

$$B_h = \{G^h(y): y \in \theta_i, J_{G^h}(y) \leqq \nu_0\}$$

is contained in n_0 cubes in R^{p-1}, each having edges ε''. Since $\rho > 3\varepsilon''d$, the ε''-neighborhood of B_h is contained in $n_0\rho^{p-1}$ cubes of edges d^{-1}. Label these cubes $\eta_{h,1}, \cdots, \eta_{h,n_1}$, where $n_1 = n_0\rho^{p-1}$.

(iv) Suppose $x = (x^1, x^2, \cdots, x^m) \in [0, 1] \times \theta_i$ is such that $J_F(x) \leqq \nu'$. By definition of $\{x_h: h = 1, \cdots, D\}$ there exists $h \in \{1, \cdots, D\}$ such that

$$|F^1(x) - c_h| < \beta d^{-1} .$$

This implies, if we write $y = (x^2, \cdots, x^m)$, that

$$|F^1(x^1, y) - F^1(g^h(y), y)| < \beta d^{-1} .$$

Therefore, since $F_1^1 \geqq \beta$, we have

$$|g^h(y) - x^1| < d^{-1} .$$

We shall compute $J_{G^h}(y)$. For each $u \in \{2, \cdots, p\}$ and $v \in \{2, \cdots, m\}$, by definition of G^h we have

$$\begin{aligned}(G^h)_v^u(y) &= F_v^u(g^h(y), y) + F_1^u(g^h(y), y)g_v^h(y)\\ &= \{[F_1^1F_v^u - F_v^1F_1^u]/F_1^1\}(g^h(y), y) .\end{aligned}$$

Let $t = \min(m-1, p-1)$ and let S be any $t \times t$ submatrix of $((G^h)_v^u(y))$, say $S = ((G^h)_{v(s)}^{u(r)}(y))$ (where $r, s = 1, \cdots, t$). Then

$$\begin{aligned}|\det S| &= |\det([F_1^1F_{v(s)}^{u(r)} - F_{v(s)}^1F_1^{u(r)}]/F_1^1)(g^h(y), y)|\\ &= \left|(F_1^1)^{-1}\det\begin{bmatrix} F_1^1, & F_{v(1)}^1 & , \cdots, & F_{v(t)}^1\\ F_1^{u(1)}, & F_{v(1)}^{u(1)}, & \cdots, & F_{v(t)}^{u(1)}\\ \cdot & \cdot & \cdot & \cdot\\ F_1^{u(t)}, & F_{v(1)}^{u(t)}, & \cdots, & F_{v(t)}^{u(t)}\end{bmatrix}(g^h(y), y)\right|\\ &\leqq \beta^{-1}J_F(g^h(y), y) .\end{aligned}$$

The second equality was obtained by an identity in the theory of determinants. Since S was arbitrary we see

$$J_{G^h}(y) \leqq \beta^{-1}J_F(g^h(y), y) .$$

But $|g^h(y) - x^1| < d^{-1} < \omega(M)(2^{-1}\beta\nu_0)$ and $\omega(M)$ is by definition a continuity modulus for J_F. Hence

$$\begin{aligned}J_F(g^h(y), y) &\leqq J_F(x) + 2^{-1}\beta\nu_0\\ &\leqq \nu' + 2^{-1}\beta\nu_0 = \beta\nu_0 .\end{aligned}$$

Combining, we have

$$J_{G^h}(y) \leqq \nu_0$$

and so $G^h(y) \in B_h$. But

$$\begin{aligned}
&\|G^h(y) - (F^2(x), \cdots, F^p(x))\| \\
&= \|(F^2(g^h(y), y), \cdots, F^p(g^h(y), y)) - (F^2(x^1, y), \cdots, F^p(x^1, y))\| \\
&\leqq (p-1)M|g^h(y) - x^1| \\
&< (p-1)Md^{-1} < \varepsilon'' .
\end{aligned}$$

Therefore $(F^2(x), \cdots, F^p(x))$ is in the ε''-neighborhood of B_h and so by (iii) is in one of the cubes $\eta_{h,1}, \cdots, \eta_{h,n_1}$. On the other hand,

$$|F^1(x) - c_h| \leqq \beta d^{-1} \leqq 2^{-1}d^{-1} .$$

Combining, we see

$$\{[c_h - 2^{-1}d^{-1}, c_h + 2^{-1}d^{-1}] \times \eta_{h,j}: j = 1, \cdots, n_1; h = 1, \cdots, D\}$$

are cubes of edges d^{-1}, which together contain the set

$$\{F(x): x \in [0, 1] \times \theta_i, J_F(x) \leqq \nu'\} .$$

(v) Repeating the arguments in (iii) and (iv) for each i in $\{1, \cdots, (q+1)^{m-1}\}$, we can enclose the set $\{F(x): x \in K, J_F(x) \leqq \nu'\}$ in $n'(=(q+1)^{m-1}n_1D)$ cubes of edges d^{-1}. It remains to check that $n'd^{-p} < \varepsilon$. But this is immediate.

LEMMA 2.3. *Assume* 2.1 *is proved for a particular natural number* $m-1$ $(m \geqq 1)$. *Given a natural number* $p \geqq 1$, *positive real numbers* Δ, M, ε, *and* β, *there exist a natural number* $n'' = n''(m, p, \Delta, M, \varepsilon, \beta)$ *and a positive real number* $\nu'' = \nu''(m, p, \Delta, M, \varepsilon, \beta)$ *with the following properties:*
if

(i) $[m/p] > 1$,

(ii) $\langle m, p, K, \Delta, F, M\rangle$,

(iii) *for some* $h \in \{1, \cdots, [m/p] - 1\}$, *some* $(h+1)$*st partial derivative* $F^j_{i_1,\cdots,i_{h+1}}$ *has absolute value not less than* β *on* U, *and*

(iv) $B_{\nu''}$ *denotes the set* $\{x \in K$: *for all* $r \in \{1, \cdots, p\}$, $h' \in \{1, \cdots, h\}$, *every* $h'-th$ *partial derivative of* F^r *has absolute value at* x *not exceeding* $\nu''\}$, *then the set* $F(B_{\nu''})$ *is contained in* n'' *cubes in* R^p *with equal edges and whose total volume is less than* ε.

Proof. (i) Choose a natural number q and then a positive real number ε' such that the following inequalities hold:

$$q > 12\beta^{-1}\Delta^{-1}(m-1)M ;$$

$$3^p(q+1)^{m-1}\varepsilon' < \varepsilon .$$

Let $k' = k(m-1, p)$. Since we assume 2.1 is proved for $m-1$, there exist a natural number $n_0 = n(m-1, p, q^{-1}, \Phi_{k'}(M, \Psi_{k'}(M)), \varepsilon')$ and a positive real number $\nu_0 = \nu(m-1, p, q^{-1}, \Phi_{k'}(M, \Psi_{k'}(M)), \varepsilon')$ with properties as described in 2.1. Let Q be a positive real number so small that whenever the entries of a $p \times (m-1)$ matrix A are bounded in absolute value by Q then $D(A) \leqq \nu_0$. Let

$$\varepsilon'' = (\varepsilon'/n_0)^{1/p} ,$$

$$n'' = (q+1)^{m-1} n_0 ,$$

and let ν'' be a positive real number so small that

$$\nu''(1 + M\beta^{-1})^2 < Q ,$$

$$\nu'' M p^{1/2} < \beta\varepsilon'' ,$$

and

$$\nu'' < \beta\Delta/8 .$$

We shall show that n'' and ν'' have the desired properties.

(ii) Therefore suppose $\langle m, p, K, \Delta, F, M \rangle$, suppose $[m/p] > 1$, and suppose (after relabelling) $F^1_{i_1,\cdots i_h,1} \geqq \beta$ on U. Without loss of generality assume $K = [0, 1]^m$. Write $f = F^1_{i_1,\cdots,i_h}$. We can easily verify that

$$\begin{aligned} k(m, p) - h &\geqq k(m, p) - [m/p] + 1 \\ &\geqq k(m-1, p) = k' . \end{aligned}$$

Hence f is a $C^{k'}$ function with M as a k'-smoothness modulus. Let $\{\theta_i : i = 1, \cdots, (q+1)^{m-1}\}$ be a family of cubes (in $[0, 1]^{m-1}$) of edges q^{-1} covering $[0, 1]^{m-1}$. For each i let

$$m_i = \min\{|f(x)| : x \in [0, 1] \times \theta_i\} .$$

Partition $\{1, \cdots, (q+1)^{m-1}\}$ into subsets P and S such that

$$m_i < 2\nu'' \quad \text{if} \quad i \in P ,$$

$$m_i > \nu'' \quad \text{if} \quad i \in S .$$

For each $i \in P$ choose $x_i = (x_i^1, \cdots, x_i^m) \in [0, 1] \times \theta_i$ such that $|f(x_i)| < 2\nu''$. Write $y_i = (x_i^2, \cdots, x_i^m)$.

(iii) Take any $i \in P$. Let U_i be a cube in R^{m-1} with same center as θ_i and edges $3q^{-1}$, the edges being parallel to corresponding ones of θ_i. Then for each $y = (y^2, \cdots, y^m) \in U_i$ we have

$$|y^j - x_i^j| \leqq 3q^{-1} \qquad (j = 2, \cdots, m) .$$

Hence

$$\begin{aligned} |f(x_i^1, y)| &\leqq |f(x_i^1, y_i)| + 3q^{-1}(m-1)M \\ &< 2\nu'' + \beta\Delta/4 < \beta\Delta/2 . \end{aligned}$$

Therefore, since $f_1 \geqq \beta$ on U, there is a point $g^i(y) \in [-\Delta, 1 + \Delta]$ such that $f(g^i(y), y) = 0$. By the Implicit Function Theorem and by the definition of $\Psi_{k'}$, we see g^i is a $C^{k'}$ function with $\Psi_{k'}(M)$ as a k'-smoothness modulus. Now define a function

$$G^i\colon U_i \longrightarrow R^p$$

by

$$G^i(y) = F(g^i(y), y)$$

for each $y \in U_i$. Then G^i is a $C^{k'}$ function with $\Phi_{k'}(M, \Psi_{k'}(M))$ as a k'-smoothness modulus on U_i. Therefore, by definition of n_0 and ν_0, there are cubes $\xi_{i,1}, \cdots, \xi_{i,n_0}$ in R^p of edges ε'' which cover $\{G^i(y)\colon y \in \theta_i, J_{G^i}(y) \leqq \nu_0\}$.

(iv) Now suppose $x = (x^1, \cdots, x^m) \in B_{\nu''}$. Write $y = (x^2, \cdots, x^m)$. Then $y \in \theta_i$ for some $i \in \{1, \cdots, (q+1)^{m-1}\}$. Since for this i we have $m_i \leqq |f(x)| \leqq \nu''$, it follows that $i \in P$. Now $f_1 \geqq \beta$ on U. Hence it follows from

$$|f(x^1, y)| \leqq \nu''$$

and

$$f(g^i(y), y) = 0$$

that

$$|x^1 - g^i(y)| \leqq \nu''\beta^{-1} .$$

Therefore, for each $r \in \{1, \cdots, p\}$ and $s \in \{2, \cdots, m\}$ we have

$$\begin{aligned} |(G^i)^r_s(y)| &= |F^r_s(g^i(y), y) + F^r_1(g^i(y), y)g^i_s(y)| \\ &\leqq |F^r_s(x)| + \nu''\beta^{-1}M + |[F^r_1 f_s/f_1](g^i(y), y)| \\ &\leqq \nu'' + \nu''\beta^{-1}M + |F^r_1(x)| \cdot |[f_s/f_1](g^i(y), y)| + \nu''\beta^{-2}M^2 \\ &\leqq \nu''(1 + 2\beta^{-1}M + \beta^{-2}M^2) \\ &= \nu''(1 + \beta^{-1}M)^2 \leqq Q . \end{aligned}$$

Consequently, by definition of Q, we have

$$J_{G^i}(y) = D((G^i)^r_s(y)) \leqq \nu_0.$$

Thus, by (iii), there is an α in $\{1, \cdots, n_0\}$ such that $G^i(y) \in \xi_{i,\alpha}$. Let $\xi'_{i,\alpha}$ be the cube with the same center as $\xi_{i,\alpha}$, and with edges $3\varepsilon''$, the edges being parallel to the corresponding ones of $\xi_{i,\alpha}$. Then, since

$$\begin{aligned} ||F(x) - G^i(y)|| &= ||F(x^1, y) - F(g^i(y), y)|| \\ &\leqq \nu''\beta^{-1}Mp^{1/2} < \varepsilon'' , \end{aligned}$$

the point $F(x)$ is in $\xi'_{i,\alpha}$. Summing up, we see that the $n'' = (q+1)^{m-1}n_0$ cubes $\{\xi_{i,\alpha}\colon i = 1, \cdots, (q+1)^{m-1}; \alpha = 1, \cdots, n_0\}$ (where we take $\xi'_{i,\alpha}$ to be some arbitrary cube in R^p with edges $3\varepsilon''$ if $i \notin P$) form a cover for $F(B_{\nu''})$. It remains to show $n''(3\varepsilon'')^p < \varepsilon$. But this is immediate.

2.4. *Proof of* 2.1. The theorem is clearly true for $m = 0$. Assume it is proved for some particular number $m - 1$ ($m \geqq 1$).

(i) We are given a natural number $p \geqq 1$ and positive real numbers Δ, M, and ε. Let

$$\lambda = [m/p] ,$$
$$\varepsilon' = 2^{-p-2}(\lambda + 1)^{-1}\varepsilon .$$

Let q_λ be a natural number so large that

$$q_\lambda^{\lambda+1-(m/p)}(\varepsilon')^{1/p} \geqq 2^{(m/p)+2}M((\lambda + 1)!)^{-1}m^{\lambda+1}$$

and let δ_λ be a positive real number so small that

$$\delta_\lambda \sum_{h=1}^{\lambda} m^h/h! \leqq 2^{-(m/p)-2}q_\lambda^{-m/p}(\varepsilon')^{1/p} .$$

Inductively define q_h, ε_h, δ_h, and $n_h (h = \lambda - 1, \lambda - 2, \cdots, 0)$ in the following way: For each natural number h such that $0 \leqq h < \lambda$, let q_h be a natural number such that

$$m^{-1/2}q_h > \Delta^{-1} \quad \text{and} \quad q_h > 24mM\delta_{h+1}^{-1} ,$$

and let ε_h be a positive real number so small that

$$(q_{h+1})^m\varepsilon_h < \varepsilon' .$$

If $0 < h < \lambda$ let $n_h = n''(m, p, q_h^{-1}, M, \varepsilon_h, 2^{-3}\delta_{h+1})$ and $\delta_h = \nu''(m, p, q_h^{-1}, M, \varepsilon_h, 2^{-3}\delta_{h+1})$ where n'' and ν'' are as given in 2.3. If $0 = h < \lambda$ let $n_h = n'(m, p, q_h^{-1}, M, \varepsilon_h, 2^{-3}\delta_{h+1})$ and $\delta_h = \nu'(m, p, q_h^{-1}, M, \varepsilon_h, 2^{-3}\delta_{h+1})$ where n' and ν' are as given in 2.2. Without loss of generality we may take $\delta_h < \delta_{h+1}$. Finally, it is obvious that we can find a natural number n with the following properties:

If we have $(q_\lambda + 1)^m$ cubes in R^p of equal edges and of total volume not exceeding ε', and if for each natural number h with $0 \leqq h < \lambda$ we are given $(q_h + 1)^m n_h$ cubes in R^p of equal edges and of total volume not exceeding ε', then we can find n cubes in R^p of equal edges and of total volume not exceeding $2^{p+1}(\lambda + 1)\varepsilon'$, which cover all the given cubes.

Take such an n. We shall show n and $\nu = \delta_0$ have the desired properties.

(ii) Thus suppose $\langle m, p, K, \Delta, F, M\rangle$. Without loss of generality we may assume $K = [0, 1]^m$. If $\lambda = 0$ let $C_0 = K$. If $\lambda > 0$ let

$$C_\lambda = \{x \in K\text{: all partial derivatives of } F \text{ of order } h$$
$$(1 \leqq h \leqq \lambda) \text{ have absolute value at } x \text{ bounded by } \delta_\lambda\} ,$$

let for each $h(0 < h < \lambda)$

$$C_h = \{x \in K\colon \text{all partial derivatives of } F \text{ of order } h' \ (1 \leqq h' \leqq h) \text{ have absolute values at } x \text{ bounded by } \delta_h, \text{ but some } (h+1)st \text{ partial derivative has absolute value at } x \text{ greater than } 2^{-1}\delta_{h+1}\} ,$$

and let

$$C_0 = \{x \in K\colon \text{some first partial derivative of } F \text{ has absolute value at } x \text{ greater than } 2^{-1}\delta_1, \text{ but } J_F(x) \leqq \nu\} .$$

Then, since $\delta_h < \delta_{h+1}(h = 0, \cdots, \lambda - 1)$, the set $\{x \in K\colon J_F(x) \leqq \nu\}$ is contained in $C_0 \cup C_1 \cup \cdots \cup C_\lambda$.

(iii) Let $\{\theta_i\colon i = 1, \cdots, (q_\lambda + 1)^m\}$ be a family of cubes in $[0, 1]^m$ of edges q_λ^{-1}, which covers $[0, 1]^m$. For each i choose a point $x_i \in \theta_i$. Let $x \in C_\lambda$. Then x is in θ_i for some i. Taylor's Theorem gives, for each $j \in \{1, \cdots, p\}$,

$$\begin{aligned} &|F^j(x_i) - F^j(x)| \\ &\leqq \sum_{k=1}^{\lambda} (k!)^{-1} \sum_{i_1=1}^{m} \cdots \sum_{i_k=1}^{m} |F^j_{i_1 \cdots i_k}(x)| \cdot |x_i^{i_1} - x^{i_1}| \cdots |x_i^{i_k} - x^{i_k}| \\ &\quad + ((\lambda + 1)!)^{-1} m^{\lambda+1} M q_\lambda^{-\lambda-1} \\ &\leqq \delta_\lambda \sum_{k=1}^{\lambda} (k!)^{-1} m^k q_\lambda^{-k} + ((\lambda + 1)!)^{-1} m^{\lambda+1} M q_\lambda^{-\lambda-1} \\ &\leqq (2q_\lambda)^{-m/p} (\varepsilon')^{1/p} 2^{-2} . \end{aligned}$$

Therefore $F(C_\lambda)$ is contained in $(q_\lambda + 1)^m$ cubes in R^p of equal edges and of total volume not exceeding ε'.

(iv) Let h be any natural number such that $0 \leqq h < \lambda$. Let $\{\theta_i\colon i = 1, \cdots, (q_h + 1)^m\}$ be a family of cubes in $[0, 1]^m$ of edges q_h^{-1}, which covers $[0, 1]^m$. Let P, Q be a partition of $\{1, \cdots, (q_h + 1)^m\}$ with the following properties: if $i \in P$ then some $(h + 1)st$ partial derivative of F has absolute value greater than $2^{-2}\delta_{h+1}$ at some point in θ_i; if $i \in Q$ then every $(h + 1)st$ partial derivative of F is bounded in absolute value by $2^{-1}\delta_{h+1}$ on θ_i. Then clearly $C_h \subset \bigcup_{i \in P}(\theta_i \cap C_h)$. Let U_i be the cube of edge $3q_h^{-1}$ and with the same center as θ_i, the edges of U_i being parallel to corresponding edges of θ_i. Then every $(h + 1)st$ partial derivative of F varies by at most $3q_h^{-1}mM$, or less than $2^{-3}\delta_{h+1}$, on U_i. Thus $i \in P$ implies some $(h + 1)st$ partial derivative has absolute value greater than $2^{-3}\delta_{h+1}$ on U_i. But $\langle m, p, \theta_i, q_h^{-1}, F, M\rangle$. Therefore, by the definitions of n_h and δ_h and by 2.2 and 2.3, the set $F(C_h)$ is contained in $(q_h + 1)^m n_h$ cubes of equal edges and of total volume not exceeding $(q_h + 1)^m \varepsilon_h$ which is less than ε'.

(v) Combining (ii), (iii), and (iv) we see that $\{F(x)\colon x \in K, J_F(x) \leqq \nu\}$

is contained in a union of $(q_\lambda + 1)^m$ cubes of equal edges of total volume not exceeding ε', and of $(q_h + 1)^m n_h$ cubes of equal edges of total volume not exceeding $\varepsilon'(0 \leqq h < \lambda)$. Therefore, by definition of n, the set $\{F(x): x \in K, J_F(x) \leqq \nu\}$ is contained in a union of n cubes in R^p, whose edges are equal and whose total volume does not exceed $2^{p+1}(\lambda + 1)\varepsilon'$ which is less than ε.

In 2.1 the mapping is assumed to be $2 + 2^{-1}(m - p)(m - p + 1)$ times continuously differentiable. The classical theorem ([4], [5]) assumes it is only $m - p + 1$ times continuously differentiable if $m \geqq p$ (while, of course, no differentiability is needed if $m < p$). The author has not been able to obtain a constructive proof without assuming higher differentiability than $m - p + 1$.

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