PACIFIC JOURNAL OF MATHEMATICS
Vol. 36, No. 3, 1971

A NON-COMPACT KREIN-MILMAN THEOREM

D. K. OATES

This paper describes a class of closed bounded convex sets which are the closed convex hulls of their extreme points. It includes all compact ones and those with the positive binary intersection property.

Let K be a closed bounded convex subset of a Hausdorff locally convex linear topological space F. Denote by EK the extreme points of K, by $\operatorname{co} EK$ their convex hull and let $\overline{\operatorname{co}}\, EK$ be its closure. We are interested in showing when

$$K = \overline{\operatorname{co}}\, EK .$$

The principal known results are the following:

THEOREM 1.1. *If either*
(a) *K is compact;*
or (b) *K has the positive binary intersection property;*
then

$$K = \overline{\operatorname{co}}\, EK .$$

Case (a) is the Krein-Milman Theorem [3, p. 131]. Case (b) was proved by Nachbin in [6], and he poses in [5, p. 346] the problem of obtaining a theorem of which both (a) and (b) are specializations. This is answered by Theorem 4.2. For the whole of this paper, S is a Stonean (extremally disconnected compact Hausdorff) space.[1]

A simplified version of Theorem 4.2 reads as follows:

THEOREM 1.2. *Let X be a normed linear space. Then any norm-closed ball in the space $\mathfrak{B}(X, C(S))$ of continuous linear operators from X to $C(S)$ is the closure of the convex hull of its extreme points in the strong neighborhood topology.*

The result concerning the unit ball of a dual Banach space in its weak*-topology and that concerning the unit ball in $C(S)$ in its norm topology are special cases of Theorem 1.2.

A *sublinear function* P from a vector space X to a partially ordered space V satisfies

$$P(x + y) \leqq P(x) + P(y)$$

and

[1] Theorem 2.3 and its proof are valid when S is zero-dimensional.

$$P(tx) = tP(x)$$

for all x, y in X and $t \geqq 0$.

A linear operator T from X to V is *dominated by* P if $Tx \leqq Px$ for all x in X. The set of all linear operators from X to V dominated by P will be written $L(P)$.

2. Let P be a sublinear function into $C(S)$, where S is Stonean. We obtain a compact approximation to $L(P)$ by considering a finite partition $\mathscr{U} = \{U_1, \cdots, U_M\}$ of S into disjoint open-and-closed sets. Let $C(S_{\mathscr{U}})$ denote the set of all function in $C(S)$ whose restrictions $f \mid U_k$ are constant. The constant values will be written as $f(U_k)$.

LEMMA 2.1. *Let P be a sublinear function from a vector space X to $C(S_{\mathscr{U}})$ and let $L(P_{\mathscr{U}})$ be the set of all linear operators from X to $C(S_{\mathscr{U}})$ dominated by P. Then*

$$EL(P_{\mathscr{U}}) \subseteqq EL(P) .$$

Proof. Suppose $T \in EL(P_{\mathscr{U}})$. For $k = 1, \cdots, M$ let t_k be chosen arbitrarily in U_k. If $G, H \in L(P)$ and $T = 1/2(G+H)$ define $G', H' \in L(P_{\mathscr{U}})$ by

$$G'x = \sum_{k=1}^{M} (Gx)(t_k)\chi_k \qquad H'x = \sum_{k=1}^{M} Hx(t_k)\chi_k$$

where χ_k is the characteristic function of U_k. Since $1/2(G' + H') = T$ and $T \in EL(P_{\mathscr{U}})$, we have $G' = H' = T$. Hence, for each $x \in X$ and $k = 1, \cdots, M$,

$$G'x(U_k) = H'x(U_k) = Tx(U_k)$$

so that

$$Gx(t_k) = Hx(t_k) = Tx(t_k) .$$

Since t_k was chosen arbitrarily in U_k, $G = H = T$. Hence $T \in EL(P)$.

DEFINITION 2.2. Let X and E be linear topological spaces and let $\mathfrak{B}(X, E)$ be the space of all continuous linear operators from X to E. The *strong neighborhood topology* for $\mathfrak{B}(X, E)$ is the topology with a base given by sets of the form

$$N(T;\ x_1, \cdots, x_n;\ U) = \{S \in \mathfrak{B}(X, E) :\ (T-S)x_i \in U, i = 1, \cdots, n\}$$

where $x_1, \cdots, x_n \in X$ and U is a neighborhood of 0 in E.

If E is normed, then we write

$N(T;\ x_1, \cdots, x_n;\ \varepsilon)$ for $N(T;\ x_1, \cdots, x_n;\ U)$ when U is the open ε-ball about 0.

THEOREM 2.3. *Let $\mathscr{W}$ be a finite partition of S into nonempty open-and-closed subsets. Let P be a sublinear function from a linear space X into $C(S_{\mathscr{W}})$. Then $L(P) = \overline{\text{co}}\ EL(P)$, with the closure in the strong neighborhood topology of $\mathfrak{B}(X, C(S))$.*

Proof. Let $\mathscr{U}$ be any finite partition of S into nonempty open-and-closed sets. From Lemma 2.1, $\overline{\text{co}}\ EL(P) \supseteqq \overline{\text{co}}\ EL(P_{\mathscr{U}})$. Now $L(P_{\mathscr{U}})$ can be linearly identified with a certain compact convex subset of a finite product $X^* \times \cdots \times X^*$, where X^* is the algebraic dual of X with the topology $w(X^*, X)$. Hence, from the Krein-Milman Theorem, $\overline{\text{co}}\ EL(P_{\mathscr{U}}) = L(P_{\mathscr{U}})$.

Let $T \in L(P)$ and let $N(T;\ x_1, \cdots, x_n;\ \varepsilon)$ be a strong neighborhood of T. The functions $\{Tx_i\colon\ i = 1, \cdots, n\}$ are continuous so for each fixed i there is a finite covering

$$\mathscr{V}^{(i)} = \{V_1^i \cdots,\ V_{N_i}^i\}$$

of S by open sets such that

$$\text{Var}\,(Tx_i,\ V_k^i) < \varepsilon$$

for all k.

Since S is zero-dimensional, there is a finite partition

$$\mathscr{U} = \{U_1, \cdots, U_M\}$$

of S into nonempty open-and-closed sets that simultaneously refines $\mathscr{V}^{(1)}, \cdots, \mathscr{V}^{(n)}$. By taking a further refinement if necessary, $\mathscr{U}$ may also be assumed to be a refinement of $\mathscr{W}$ and then the functions $P(x)$ are constant on each of the sets U_k.

For each $k = 1, \cdots, M$ define a sublinear functional q_k on X by $q_k(x) = \sup\{Tx(t)\colon\ t \in U_k\}$. From the Hahn-Banach Theorem, there exists a linear functional ϕ_k on X dominated by q_k. Define $T_1\colon X \to C(S_{\mathscr{U}})$ by

$$T_1 x = \sum_{k=1}^{M} \phi_k(x)\, \chi_{U_k}\ .$$

Then $T_1 \in L(P_{\mathscr{U}})$ and, for $i = 1, \cdots, n$,

$$\|(T_1 - T)x_i\| \leqq \sup_{R} \text{Var}\,(Tx_i,\ U_k) < \varepsilon\ .$$

DEDUCTION of THEOREM 1.2. With X and S as in the statement of the theorem, let $\mathfrak{B}_1$ be the closed unit ball in $\mathfrak{B}(X, C(S))$.

The set $\mathfrak{B}_1$ is $L(P)$, where P is the sublinear function $P(x) = \|x\| e$, e being the unit function in $C(S)$. By Theorem 2.3 $\mathfrak{B}_1 = \text{co } E\mathfrak{B}_1$ and the result for any closed ball then follows by a scalar multiplication and translation.

3. Nachbin's problem. Let K be a closed bounded convex subset of a linear topological space E. Recall that K has the *positive binary intersection property* if every pairwise-intersecting subfamily of

$$\{x + \lambda K \colon x \in E,\ \lambda \geqq 0\}$$

has nonempty intersection.

If K is bounded and has the above property, it may be shown to be centrally symmetric with a unique centre c, and to have the *binary intersection property* where the restriction $\lambda \geqq 0$ is removed. This is proved in [6].

Results in [4] and [2] then show that the set $K_0 = K - c$ generates a subspace of E which is a hyperconvex normed space and isomorphic to $C(S)$, with S Stonean.

THEOREM 3.1. *Let E be a locally convex Hausdorff linear topological space containing a closed bounded convex subset K with the positive binary intersection property. Let p be a continuous sublinear functional on a locally convex Hausdorff linear topological space X.*

If L is the set of linear maps $T \colon X \to E$ such that for all x in X

$$Tx \in \frac{1}{2}[p(x) - p(-x)]\, e + \frac{1}{2}[p(x) + p(-x)]\, K_0$$

where e is any extreme point of K_0, then $L = \overline{\text{co}}\, L$, with the closure taken in $\mathfrak{B}(X, E)$ with the strong neighborhood topology.

Proof. Because p is continuous the set $L(P)$ is closed in the space $\mathfrak{B}(X, E)$ in the strong neighborhood topology. Since K is centrally symmetric, K_0 has the binary intersection property and is linearly isomorphic to the unit ball in a space $C(S)$ with S Stonean. The isomorphism may be chosen as in [4] so that e is mapped onto the unit function of $C(S)$. Using e to denote also this unit function, we may define a sublinear function $P(x) = p(x)\, e$ from X to $C(S)$, which is the situation of Theorem 3.1. with $\mathscr{W} = \{S\}$.

Given $T \in L(P)$, $x_1, \cdots, x_n \in X$ and $\varepsilon > 0$ there exists $A \in \text{co } EL(P)$ with

$$(T - A)\, x_i \in \varepsilon K_0 \quad (i = 1, \cdots, n) .$$

Given a neighborhood U of 0 in E, there exists $r > 0$ with $K_0 \subseteqq rU$, since K is bounded. So choosing $\varepsilon = r^{-1}$ there exists $A \in \mathrm{co}\, EL(P)$ with

$$(T - A)\, x_i \in r^{-1} K_0 \subseteqq U \quad (i = 1, \cdots, n) ,$$

which completes the proof.

DEDUCTION OF THEOREM 1.1. (a) Let p_K be the sublinear functional defined on F^* by

$$p_K(f) = \sup \{f(k) \colon k \in K\} .$$

Then, from the bipolar theorem,

$$L = \{g \in F^{**} \colon g(f) \leqq p_K(f) \quad \text{for all } f \in F^*\}$$

is identical with the canonical image $\hat{K}$ of K under the evaluation map. Now apply Theorem 3.1 with $E = \mathbf{R}$, $K = [-1, 1]$, $e = 1$ and $X = F^*$, taken with the topology of uniform convergence on compact subsets of F. This shows that $\hat{K}$ is the closure of $\mathrm{co}\, E\hat{K}$ in the topology $w(F^{**}, F^*)$, which is equivalent to K being the $w(F, F^*)$ and hence the strong closure of $\mathrm{co}\, EK$ in F.

(b) Apply Theorem 3.1 with $X = \mathbf{R}$ and $E = F$. Then, under the natural isomorphism of $\mathfrak{B}(X, E)$ and E, K_0 corresponds to L, which satisfies $L = \overline{\mathrm{co}}\, EL$. Since E is a linear topological space we have

$$K = \overline{\mathrm{co}}\, EK .$$

REFERENCES

1. N. Aronszajn and P. Panitchpakdi, *Extensions of unifomly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. **6** (1956), 405-439.

2. J. L. Kelley, *Banach spaces with the extension property*, Trans. Math. Soc. **72** (1952), 323-326.

3. J. L. Kelley, I. Namioka, et al., *Linear Topological Spaces*, van Nostrand, 1963.

4. L. Nachbin, *A theorem of the Hahn-Banach type for linear transformations*, Trans. Amer. Math. Soc. **68** (1950), 28-46.

5. ———, *Some problems in extending and lifting continuous linear transformations*, Proc. Int. Symposium on Linear Spaces, Jerusalem 1960, 340-350.

6. ———, *Sur l'abondance des points extrémaux d'un ensemble convexe borné et fermé*, Anais Acad. Brasileira Ciên. **34** (1962), 445-448.

Received August 11, 1969.

UNIVERSITY OF EXETER
U. K.