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MULTIPLIERS AND UNCONDITIONAL CONVERGENCE OF BIORTHOGONAL EXPANSIONS

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We solve in the affirmative a problem raised by B. S. Mityagin in 1961, namely, we prove that if (x_n, f_n) is a biorthogonal system for a Banach space E with (f_n) total over E, such that the set of multipliers $M(E, (x_n, f_n))$ contains all sequences (ε_i) with $\varepsilon_i = \pm 1$ for each i, then (x_n) is an unconditional basis for E.

Let E be a Banach space, and let (x_n, f_n) be a biorthogonal system for E (i.e., $(x_n) \subset E$, $(f_n) \subset E^*$ and $f_n(x_m) = \delta_{nm}$) which has (f_n) total over E (i.e., $f_n(x) = 0$ for all n implies $x = 0$). A scalar sequence (γ_n) is called a *multiplier* of an element x in E with respect to (x_n, f_n) (write $(\gamma_n) \in M(x, (x_n, f_n))$) if there is an element y of E such that $f_n(y) = \gamma_n f_n(x)$ for all n (call this element $x_{(\gamma_n)}$). The set of multipliers for E with respect to (x_n, f_n) is

$$M(E, (x_n, f_n)) = \cap \{M(x, (x_n, f_n)) \mid x \in E\} .$$

Here we consider the following two problems:

P 1: (Mityagin [6], Kadec-Pelczynski [4], Pelczynski [7]). Let E be separable and suppose that $M(E, (x_n, f_n))$ contains all sequences (ε_i) with $\varepsilon_i = \pm 1$ for each i. Is (x_n) an unconditional basis for E?

P 2: (Kadec-Pelczynski [4]). Let E be separable and suppose $M(x, (x_n, f_n))$ contains all sequences (ε_i) with $\varepsilon_i = \pm 1$ for each i. Does the formal expansion $\sum_n f_n(x)x_n$ converge unconditionally to x?

Problem 2 (and hence also problem 1) is known to have an affirmative answer in the following cases [4]:

1°. $M(x, (x_n, f_n)) \supset m$ (the space of bounded sequences).

2°. E contains no subspace isomorphic to c_0 (the space of sequences converging to 0) and $M(x, (x_n, f_n)) \supset c_0$.

3°. $sp(f_n)$ (= linear span of (f_n)) is norming (i.e.,

$$|x| = \sup \{|f(x)| \mid f \in sp(f_n), \|f\| \leqq 1\}$$

defines a norm on E equivalent to the original norm on E).

Problem 1 is known to have an affirmative answer in the case when $[x_n] = E$, where $[x_n]$ denotes the closed linear span of $\{x_n\}$ ([5]; see also [1], Theorem 3.4, implication (4) $\Rightarrow$ (3)).

In the present paper we give an affirmative solution for problem

1. Our method also provides a more elementary proof of 3° than that given in [4].

THEOREM 1. *Let E be a separable Banach space and let (x_n, f_n) be a biorthogonal system for E with (f_n) total over E. If $M(E, (x_n, f_n))$ contains all sequences (ε_i) with $\varepsilon_i = \pm 1$ for each i, then (x_n) is an unconditional basis for E.*

If the hypothesis $[x_n] = E$ is added then a much simpler proof of the theorem is obtained (see the Remark following Lemma 3 below).

LEMMA 1. *$M(E, (x_n, f_n)) \supset \{(\varepsilon_i) \mid \varepsilon_i = \pm 1$ for all $i\}$ if and only if $M(E, (x_n, f_n)) \supset \{(\varepsilon_i) \mid \varepsilon_i = 0$ or 1 for all $i\}$.*

Proof. Obvious.

LEMMA 2. *Suppose $(\varepsilon_i) \in M(E, (x_n, f_n))$, where $\varepsilon_i = 0$ or 1 for all i and define $S_{(\varepsilon_i)} = E \to E$ by*

$$S_{(\varepsilon_i)}(x) = x_{(\varepsilon_i)} \qquad (x \in E)\,. \tag{1}$$

Then $S_{(\varepsilon_i)}$ is a continuous linear mapping.

Lemma 2 is well known (see e.g. [8]).

In the particular case when $\varepsilon_i = 1$ for $i = 1, \cdots, n$ and $\varepsilon_i = 0$ $i = n + 1, n + 2, \cdots$ we shall use for $S_{(\varepsilon_i)}$ the notation S_n. Obviously,

$$S_n(x) = \sum_{i=1}^{n} f_i(x) x_i \qquad (x \in E,\ n = 1, 2, \cdots)\,. \tag{2}$$

If σ is a subset of the positive integers, we define the mapping S_σ: $E \to E$ by

$$S_\sigma = S_{(\varepsilon_i)}, \tag{3}$$

where $\varepsilon_i = 1$ for $i \in \sigma$ and $\varepsilon_i = 0$ for $i \notin \sigma$.

LEMMA 3. *Let (x_n, f_n) be a biorthogonal system for E (not necessarily separable), with (f_n) total over E. If $M(E, (x_n, f_n))$ contains all sequences (ε_i) with $\varepsilon_i = \pm 1$ for all i, then $(\|S_n\|)$ is bounded.*

Consequently, (x_n) is an unconditional basic sequence (i.e., *an unconditional basis of its closed linear span $[x_n]$*) *and hence, if $[x_n] = E$, then (x_n) is an unconditional basis of E.*

Proof. Assume that $(\|S_n\|)$ is unbounded. Let (n_k) be an increasing sequence of integers such that $\|S_{n_k}\| \geqq 2^k + \|S_{n_{k-1}}\|$, whence

$\|S_{n_k} - S_{n_{k-1}}\| \to \infty$. Let $(M_p;\ p = 1, 2, \cdots)$ be a countable collection of pairwise disjoint, infinite subsets of the positive integers, $I_k = \{n_{k-1} + 1, \cdots, n_k\}$, and $\sigma_p = \bigcup_{k \in M_p} I_k$. The projection S_{σ_p} is continuous by Lemma 2. Moreover, if k is in M_p and x is in E, we have

$$\|(S_{n_k} - S_{n_{k-1}})x\| = \left\| \sum_{i=n_{k-1}+1}^{n_k} f_i(x)x_i \right\| = \left\| \sum_{i=n_{k-1}+1}^{n_k} f_i(S_{\sigma_p}x)x_i \right\|$$
$$= \|S_{n_k} - S_{n_{k-1}})S_{\sigma_p}x\| \leqq \|S_{n_k} - S_{n_{k-1}}\|_{X_p} \|S_{\sigma_p}\| \|x\|$$

where

$$X_p = \{x \in E \mid f_j(x) = 0 \quad \text{if} \quad j \notin \sigma_p\}.$$

It follows that $\|S_{n_k} - S_{n_{k-1}}\|_{X_p}$ is unbounded as k runs through M_p. Choose $u_p \in X_p$, $k_p \in M_p$ such that $\|u_p\| \leqq 2^{-p}$ and $\|(S_{n_{k_p}} - S_{n_{k_p-1}})u_p\| \geqq 1$. Let $\sigma = \bigcup_{p=1}^{\infty} I_{k_p}$. Now $\sigma \cap \sigma_p = I_{k_p}$ so that if $y_p \in X_p$ then $f_i(S_\sigma y_p) = f_i[(S_{n_{k_p}} - S_{n_{k_p-1}})y_p]$ for all i, whence $S_\sigma y_p = (S_{n_{k_p}} - S_{n_{k_p-1}})y_p$. Thus $\sum_p u_p$ converges while $S_\sigma(\sum_p u_p) = \sum_p S_\sigma(u_p) = \sum_p (S_{n_{k_p}} - S_{n_{k_p-1}})u_p$ doesn't converge, contradicting Lemma 2, that S_σ is continuous. Thus (x_n) is [2] a basic sequence. Since the same argument remains valid for every permutation $(x_{\rho(n)})$ of (x_n), it follows that (x_n) is an unconditional basic sequence, which completes the proof.

REMARK. One can give a much simpler proof of the fact that under the hypotheses of Lemma 3 we have

$$\sup_n \|S_n|_{[x_j]}\| < \infty, \tag{4}$$

whence (x_n) is an unconditional basic sequence (and, if $[x_n] = E$, then (x_n) is an unconditional basis of E). Indeed, if (4) does not hold, then there exist increasing sequences of positive integers (p_n), (q_n) with $p_{n-1} + 1 \leqq q_{n-1} + 1 \leqq p_n$ $(n = 1, 2, \cdots;\ p_0 = q_0 = 0)$ and a sequence (u_n) with $u_n \in [x_{q_{n-1}+1}, \cdots, x_{q_n}]$ $(n = 1, 2, \cdots)$ such that $\|S_{p_n}u_n\| = 1$, $\|u_n\| \leqq 1/2^n$ $(n = 1, 2, \cdots)$, whence $(\sum_{j=1}^n u_j)$ is convergent, but for $\sigma = \{1, \cdots, p_1, q_1 + 1, \cdots, p_2, \cdots\}$ the sequence $(S_\sigma(\sum_{j=1}^n u_j)) = (\sum_{j=1}^n S_{p_j}u_j)$ is not convergent. Thus, S_σ is not continuous, which contradicts Lemma 2, completing the proof.

Proof of Theorem 1. We prove that $S_n x \to x$ for each x in E. This will prove the theorem by noting that the same proof works to show that each permutation of (x_n) is a basis for E, so that (x_n) is an unconditional basis for E. Choose x in E such that $(S_n x)$ does not converge (if it converges, its limit must be x by totality of the sequence (f_n)). Let (n_k), (m_k) be sequences of integers such that $m_k + 1 \leqq n_k \leqq m_{k+1}$ for all k and such that there is $\varepsilon > 0$ with $\varepsilon < \|S_{n_k} - S_{m_k})x\|$ for all k. Let $u_k = (S_{n_k} - S_{m_k})x = \sum_{i=m_k+1}^{n_k} f_i(x)x_i$. For each sequence (η_i) such that $\eta_i = 1$ or 0 for each i there is an element of E, denoted here by $\Sigma\eta_i u_i$, such that $(S_{n_k} - S_{m_k})(\Sigma\eta_i u_i) = \eta_k u_k$ for every $k(\Sigma\eta_i u_i$ is $x_{(\varepsilon_j)}$ where $\varepsilon_j = \eta_k$ for $m_k + 1 \leqq j \leqq n_k$, $k = 1, 2, \cdots$ and 0 for the other j). Since E is separable, and since the set $\{\Sigma\eta_i u_i \mid \eta_i = 1 \text{ or } 0\}$ in E is uncountable, there is a sequence $(y_n)_0^\infty$ with $y_n = \Sigma\eta_i^{(n)} u_i$ such that $y_n \neq y_m$ if $n \neq m$ and $y_n \rightarrow y_0 = \Sigma\eta_i^{(0)} u_i$. Let K be a bound on $\|(S_{n_k} - S_{m_k})\|$ as guaranteed by Lemma 3. Then for p large, and all k, $\|(S_{n_k} - S_{m_k})(y_p - y_0)\| \leqq K\|y_p - y_0\| < \varepsilon$, but

$$(S_{n_k} - S_{m_k})(y_p - y_0) = (\eta_k^{(p)} - \eta_k^{(0)})u_k ,$$

whence

$$\|(S_{n_k} - S_{m_k})(y_p - y_0)\| = \begin{cases} 0 & \text{if } \eta_k^{(p)} = \eta_k^{(0)} \\ \|u_k\|\,|\eta_k^{(p)} - \eta_k^{(0)}| = \|u_k\| & \text{otherwise.} \end{cases}$$

Since $y_p \neq y_0$ for all $p \neq 0$, there is a $k = k(p)$ for which

$$\|(S_{n_k} - S_{m_k})(y_p - y_0)\| = \|u_k\| > \varepsilon ,$$

which is impossible for large p. Therefore $S_n x \rightarrow x$, which completes the proof of Theorem 1.

REMARK. Using the same method, one can also give a more elementary proof of the result 3° mentioned in the Introduction (actually, of a slightly more general result), than that given in [4]. As above, it is sufficient to show that $(S_n x)$ converges. If not, let (n_k), (m_k), $\varepsilon > 0$ and (u_k) be as in the above proof. Since $sp(f_n)$ is norming, by a technique of [3], or, equivalently, by [4], p. 311, lemma and p. 317, Lemma 5, we may assume (dropping to subsequences of (n_k) and (m_k) if necessary) that the natural projection P_k of $[x_1, \cdots, x_{n_k}] \oplus [f_1, \cdots, f_{m_{k+1}}]_\perp$ onto $[x_1, \cdots, x_{n_k}]$ is of norm $\|P_k\| \leqq C$, where $C > 1$ is a constant independent of k (actually, only this projection property is used in the sequel and therefore we obtain a slightly more general result than 3°). As in the above proof of Theorem 1 there is an element of E, denoted by $\Sigma\eta_i u_i$, which is in each of the subspaces $[x_1, \cdots, x_{n_k}] \oplus [f_1, \cdots, f_{m_{k+1}}]_\perp$, such that $(P_k - P_{k-1})(\Sigma\eta_i u_i) = \eta_k u_k$. The proof is completed in precisely the same manner as before, where now $P_k - P_{k-1}$ take the role of $S_{n_k} - S_{m_k}$.

Note. After this work had been completed, we have learned of the recent paper of G. F. Bachelis and H. P. Rosenthal "On unconditionally converging series and biorthogonal systems in a Banach space" (to appear in Pacific J. Math), where Problem 2 (and hence also Problem 1) is solved, even with the hypothesis "Let E be separable" replaced by the weaker hypothesis "Let E contain no subspace isomorphic to m". However, our methods are completely different and use more elementary tools.

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