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THE ASYMPTOTIC BEHAVIOR OF NORMS OF POWERS OF ABSOLUTELY CONVERGENT FOURIER SERIES

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Let $f(t)$ have an absolutely convergent Fourier series $f(t) = \sum a_k e^{ikt}$ and set $||f|| = \sum |a_k|$. In this paper we will study the asymptotic behavior of $||f^n||$ as $n \to \infty$.

THEOREM. **Let f be absolutely continuous and let f' be of bounded variation on the real line modulo 2π. Let $f(0) = 1$ but $|f(t)| < 1$ for $t \neq 0$ and suppose that for t near 0, $f(t) = \phi(e^{it})$ where ϕ is defined and analytic near $z = 1$. Define the parameters α, p, q, A and β as follows**

$$\alpha = \phi'(1)$$
$$\phi(z) = z^\alpha + Ai^p(z-1)^p + o(1)(z-1)^p,\ z \to 1 (A \neq 0)$$
$$|\phi(e^{it})| = 1 - \beta t^q + o(t^q),\ t \to 0\ (\beta \neq 0)\ .$$

Then, (a) for $p \neq q$

$$||f^n|| \sim (2/\pi)^{1/2+\delta(p)} q^{-1}[p(p-1)]^{1/2}\Gamma(p/2q)\,|A|^{1/2}\beta^{-p/2q} n^{(1-p/q)/2}$$

where $\delta(p) = 0$ if p is even and, $= 1$ if p is odd; (b) for $p = q$

$$\lim_{n\to\infty} ||f^n|| = (1/2\pi)\,||\hat{F}||_1$$

where $\hat{F}$ is the Fourier transform of $F(t) = \exp(At^p)$.

The following results about these parameters are known and easily verifiable: p and q are positive integers, $2 \leqq p \leqq q$, q is even, $\beta > 0$, $ReA \leqq 0$; $p = q$ if and only if $ReA \neq 0$; $p = q$ implies $\beta = -ReA$;

$$(1)\qquad \phi(e^{it}) = \exp[i\alpha t + At^p + \mathcal{O}(t^{p+1})],\ t \to 0 \text{ if } p = q\ ,$$

and

$$(2)\qquad \begin{aligned}\phi(e^{it}) = \exp[i\alpha t + (-1)^p A t^p + i\sum_{r=p+1}^{q} c_r t^r - \beta t^q \\ + \mathcal{O}(t^{q+1})],\ \text{as } t \to 0,\ c_r \text{ real, if } p \neq q\ .\end{aligned}$$

We outline in § 5 how it is possible to relax the condition of analyticity at $t = 0$ and replace it with conditions (1) and (2) where the $\mathcal{O}$ terms satisfy certain smoothness conditions.

G. W. Hedstrom proved in 1966 that under these same hypotheses, there exist two constants c, C such that $c < ||f^n||\, n^{-(1-p/q)/2} < C$

1. Introduction.

The classical result of this type is that $\lim_{n\to\infty} ||f^n||^{1/n} = 1$, but much better estimates, namely $||f^n|| = \mathcal{O}(n)$, $n \to \infty$ and even $||f^n|| = \mathcal{O}(\sqrt{n})$, $n \to \infty$ were relatively easy to obtain [1], [5], [6]. It is also known that this last estimate is the best possible. More precise estimates for $||f^n||$ were separately sought for those functions satisfying respectively

(i) $|\phi(z)|$ attains its maximum on $|z| = 1$ at finitely many points,

(ii) $|\phi(z)| = 1$ for $|z| = 1$.

It is known that if $\phi(z)$ satisfies (ii) and is not a power of z, then $||f^n|| \to \infty$ as $n \to \infty$ [1], and more precisely, there exist positive constants c_1, c_2 such that $c_1\sqrt{n} \leqq ||f^n|| \leqq c_2\sqrt{n}$ [6].

It was proved in [1], [3] that if $\phi(z)$ does satisfy (i), then the necessary and sufficient condition for $||f^n|| = \mathscr{O}(1)$ as $n \to \infty$ is that $p = q$ at each point of maximum modulus. In the case where $p = q$ is not satisfied at each point Hedstrom [4] has shown that there exist constants C_1, C_2 such that $C_1n^s \leqq ||\phi^n|| \leqq C_2n^s$ where $s = \max(1 - p/q)/2$, the maximum being taken over all points where $p \neq q$. Further developments and connections with the work of Kahane and Leibenson can be found in Hedstrom's paper where he considers the more general case when $f(t)$ has an absolutely convergent Fourier series. Connections with the conformal invariance of peripheral convergence can be found in a paper by Bajsanski [2] and a recent thesis by Whitford [9]. The main results of Bajsanski, Clunie, and Vermes were rediscovered and appear in a recent paper by Newman [7]. In connection with this paper in § 4 we discuss some partial results concerning the behavior of $||f^n||$ when $|f|$ has several points of maximum on $[-\pi, \pi]$.

2. Preliminary lemmas.

We give in this section some lemmas which will be needed in the proof of the theorem. The proof of the following lemma is explicitly contained in the proof of Lemma 2.1 in [4] and so is not given.

LEMMA 1. *Let $f(t)$ be absolutely continuous and suppose that $f'(t)$ is of bounded variation and $|f(t)| \leqq 1$ for all t. Then, if a_{nk} is the Fourier coefficient of $f^n(t)$, $k = 0, \pm 1, \cdots$, there are constants C_1, C_2 such that $|a_{nk}| \leqq C_1n/k^2$ for $|k| \geqq C_2n$.*

Using (1) and (2) we can write

$$f(t) = \exp(i\alpha t + ip(t) + G_1(t)),\ t \to 0$$

where $p(t)$ is a real polynomial of degree at most q and $G_1(t) = -\beta t^q + G_2(t)$, $G_2(t) = \mathscr{O}(t^{q+1})$, as $t \to 0$. Since $f(t)$ is analytic at $t = 0$, by putting

$$z^{-\alpha}\phi(z) = \exp[ip[(\log z)/i] + G_1[(\log z)/i]],\ z \to 1\ ,$$

we have $f(t) = \phi(e^{it})$ for t in some neighborhood of $t = 0$, where $\phi(z)$ is analytic in a neighborhood of $z = 1$.

The following lemma generalizes Lemma 1 in [1] and plays a fundamental role in the proof of the theorem.

LEMMA 2. *Let* $\Psi(r, t) = Re\,[\log[(re^{it})^{-\alpha}\phi(re^{it})]]$ *and* $r = 1 \pm n^{-c}$, $c = 1 - (p - 1)/q$. *Then, if* ε_0 *is sufficiently small,*

$$\int_{-\varepsilon_0}^{\varepsilon_0} \exp[n\Psi(r, t)]dt = \mathcal{O}(n^{-1/q}),\ n \to \infty\ .$$

Proof. By Taylor's formula

$$\Psi(r, t) = \sum_0^{q-1} C_m(r)t^m + C_q(r; \tau)t^q,\ |t| < \varepsilon_0$$

where

$$C_m(r) = (1/m!)\partial^q\Psi(r, t)/\partial t^m|_{t=0},\ m = 0, 1, \cdots, q - 1$$

and

$$C_q(r, \tau) = (1/q!)\partial^q\Psi(r; t)/\partial t^q|_{t=\tau},\ |\tau| < |t| < \varepsilon_0\ .$$

The derivatives of $C_m(r)$ are given by

$$d^nC_m(r)/dr^n = (1/m!)\partial^{m+n}\Psi(r, t)/\partial r^n\partial t^m|_{t=0}\ ,$$

but $\log z^{-\alpha}\phi(z) = \mathcal{O}(1)(z - 1)^p$, $z \to 1$ and this implies that $\Psi(r, t) = Re[\mathcal{O}(1)(e^{it}(r - 1) + e^{it} - 1)^p]$ so that the partial derivatives of Ψ of order less than p are zero at $r = 1$, $t = 0$. Thus $C_m(r)$ has a zero at the point $r = 1$ of order $\geqq p - m$, for $m = 0, 1, \cdots, p - 1$ and so $|C_m(r)| < C_m|r - 1|^{p-m}$ in a sufficiently small neighborhood of $r = 1$ for constants C_m. In addition $\Psi(1, t) = -\beta t^q + \mathcal{O}(t^{q+1})$ as $t \to 0$, so that $C_m(1) = 0$, $m = p, \cdots, q - 1$ and thus for r sufficiently near 1, $|C_m(r)| \leqq C_m|r-1|$, $m=p, \cdots, q-1$. Furthermore, $d^q\Psi(1, t)/dt^q|_{t=0} = -q!\,\beta < 0$ implies that $\partial^q\Psi(r, t)/\partial t^q < -M$, $M > 0$ for r sufficiently close to 1 and $|t| < \varepsilon_0$, for some $\varepsilon_0 > 0$. Thus, if $r = 1 \pm n^{-c}$ and $|t| \leqq \varepsilon_0$

$$\Psi(r, t) \leqq \sum_{m=0}^{p-1} C_m n^{-(p-m)c}|t|^m + \sum_{m=p}^{q-1} C_m n^{-c}|t|^m - Mt^q$$

and so we have

$$\int_{-\varepsilon_0}^{\varepsilon_0} \exp[n\Psi(r, t)]dt \leqq n^{-1/q}\int_{-\infty}^{\infty} \exp[Q(u)]du\ ,$$

where $Q(u) = \sum_{m=0}^{p-1} C_m n^{1-c(p-m)-m/q}|u|^m + \sum_{m=p}^{q-1} C_m n^{1-c-m/q}|u|^m - Mu^q$. But $Q(u)$ is bounded by

$$\sum_{m=0}^{q-1} C_m \mid u \mid^m - Mu^q$$

since $1 - c - m/q \leqq 0$ for $m = p, \cdots, q-1$ and $1 - c(p-m) - m/q \leqq 0$ for $m = 0, \cdots, p-1$; the result follows.

LEMMA 3. *Let $h(t) = at + P(t)$ be a real polynomial where $a \neq 0$ and $P(t) = c_p t^p + \cdots + c_q t^q$, $q \geqq p \geqq 2$, q even and let $g(t) = P'(t)/h'(t)$. Then, if for $t \in J = [s_1, s_2]$, P'' is of constant sign, $h'(t) \neq 0$ and $G(t)$ is such that $Re\, G(t) \leqq \beta t^q/2$ $(\beta > 0)$ and $\mid G'(t) \mid \leqq C \mid t \mid^{q-1}$, there is a constant M depending only on C, β and q such that*

$$\begin{aligned} I &\equiv \left| \int_J \exp\left[ih(t) + G(t)\right] dt \right| \\ &\leqq M \mid a \mid^{-1} \max\left[\max_J \mid g(t) - 1 \mid, \mid g(s_1) \mid, \mid g(s_2) \mid\right] . \end{aligned}$$

Proof. Integration by parts yields

$$\begin{aligned} I \leqq \left| \exp\left[ih(t) + G(t)\right]/h'(t) \Big|_{s_1}^{s_2} \right| + \left| \int_J \exp\left[ih(t) + G(t)\right](G'(t)/h'(t)) dt \right| \\ + \left| \int_J \exp\left[ih(t) + G(t)\right]\left[h''(t)/(h'(t))^2\right] dt \right| = I_1 + I_2 + I_3 . \end{aligned}$$

We have

$$I_1 \leqq \mid a \mid^{-1} \max_J \mid a/h'(t) \mid = \mid a \mid^{-1} \max_J \mid g(t) - 1 \mid$$

since $\mid \exp G(t) \mid \leqq \exp\left[-\beta t^q/2\right] < 1$.

$$\begin{aligned} I_2 &\leqq \max_J \mid 1/h'(t) \mid \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\beta t^q\right) c \mid t \mid^{q-1} dt \\ &\leqq M_1 \mid a \mid^{-1} \max_J \mid g(t) - 1 \mid \end{aligned}$$

and

$$\begin{aligned} I_3 &\leqq \int_J (\mid h''(t) \mid / \mid h'(t) \mid^2) dt \\ &= \mid a \mid^{-1} \int_J \mid g'(t) \mid dt = \mid a \mid^{-1} \mid g(s_2) - g(s_1) \mid \end{aligned}$$

since $g'(t)$ does not change sign. The result follows.

LEMMA 4. *Let $(g_n(x))$ be a sequence of real valued function on an interval $[a, b]$ such that*

(a) g'_n is continuous, for each n.
(b) $\lim_{n\to\infty} g'_n(x) = g'(x)$ uniformly
(c) $\lim_{n\to\infty} g_n(a) = g(a)$
(d) $g'(x) \geqq \delta > 0$.

Then, if $f(x)$ is continuous on $[a, b]$,

$$\lim_{n\to\infty} \int_a^b f(x) \,|\cos(n^\mu g_n(x) + \theta)|\, dx = \frac{2}{\pi}\int_a^b f(x)dx$$

for any $\mu > 0$ and any real θ.

The lemma is a straightforward generalization of Exercise 118 in [8] and so the proof will not be given.

3. Proof of the theorem. The proof of our result is divided into several parts. We want to determine the asymptotic behavior of

$$\|f^n\| = \sum |a_{nk}| = \sum \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f^n(t)e^{-ikt}dt\right| .$$

The essential ideas in the proof are these: depending upon the parameters α, p, q introduced above and n, there are only small ranges of values for k near $n\alpha$ which are significant in the determination of the asymptotic behavior; and for these values of k there are only small neighborhoods of $t = 0$ in the integrals above which are of significance. Part 1 will provide an initial reduction of the expression above. Part 2 will further this reduction for the case $p \neq q$. In the remaining three parts the asymptotic constants will be obtained separately for the cases $p \neq q$, p even; $p \neq q$, p odd; $p = q$.

We shall omit the phrase "for n sufficiently large" finitely many times in the course of the proof.

The following notation will also be used consistently:

$$c = 1 - (p-1)/q \qquad b_n = [(1 - p/q)/2 + 1]\log n$$

$$\mu = n\alpha - k \qquad \gamma = (n\alpha - k)n^{-1/q}$$

$$S(n) = \{k\colon n^c b_n \leqq |\mu| \leqq n^{1+1/q}\}$$

$$T(n) = \{k\colon b_n < |\mu| < n^c b_n\}$$

$$U(n) = \{\gamma\colon n^{-1/q}b_n < \gamma < n^{c-1/q}b_n\} .$$

We now choose ε_0 sufficiently small so that $|G_2(t)| \leqq \beta t^q/2$, for $|t \leqq \varepsilon_0$, $\phi(re^{it})$ is analytic for $|t| \leqq \varepsilon_0$ and r sufficiently close to 1, and $|\phi(re^{it})| \leqq 1 - \delta < 1$ for $t = \pm\varepsilon_0$ and r sufficiently near 1, for some $\delta > 0$.

PART 1

We first show that

$$(3)\qquad 2\pi\sum_{-\infty}^{\infty}|a_{nk}| = \sum\left|\int_{-\varepsilon_0}^{\varepsilon_0} f^n(t)e^{-ikt}dt\right| + o(n^{(1-p/q)/2}) ,$$

as $n\to\infty$, where the range of summation is for $k\in T(n)\cup S(n)$. Let $|\mu|>n^{1+1/q}$. Then for some $K>0$, $|k|>Kn^{1+1/q}$ and so by Lemma 1

$$\sum_{|\mu|>n^{1+1/q}}|a_{nk}|\leqq nC\sum_{|k|>n^{1+1/q}K}1/k^2 = \mathcal{O}(n^{-1/q}) , \quad n\to\infty .$$

If we write

$$2\pi a_{nk} = \left(\int_{-\varepsilon_0}^{\varepsilon_0} + \int_{[-\pi,\pi]\backslash[-\varepsilon_0,\varepsilon_0]}\right)f^n(t)e^{-ikt}dt = a_{nk}^{(1)} + a_{nk}^{(2)} ,$$

then

$$|a_{nk}^{(1)}|\leqq\int_{-\varepsilon_0}^{\varepsilon_0}\exp(-n\beta t^q/2)dt = \mathcal{O}(n^{-1/q}) , \quad n\to\infty$$

and since for all $t\in[-\pi,\pi]\backslash[-\varepsilon_0,\varepsilon_0]$ there is a $\eta>0$ such that $|f(t)|\leqq 1-\eta<1$, $|a_{nk}^{(2)}|\leqq 2\pi(1-\eta)^n$ and we obtain

$$\sum_{|\mu|\leqq b_n}|a_{nk}^{(1)}| = \mathcal{O}(n^{-1/q}b_n) , \quad n\to\infty$$

and

$$\sum_{|\mu|\leqq n^{1+1/q}}|a_{nk}^{(2)}| = \mathcal{O}(n^{1+1/q}(1-\eta)^n) , \quad n\to\infty .$$

Each of these last sums is $o(n^{(1-p/q)/2})$ as $n\to\infty$ and so (3) follows.

We next show that

$$\sum_{k\in S(n)}|a_{nk}^{(1)}| = o(n^{(1-p/q)/2})$$

as $n\to\infty$ by using the analyticity of $f(t)$ at $t=0$ to deform the contour of integration for the function ϕ. Indeed, $a_{nk}^{(1)}$ is the integral of $\phi^n(z)z^{-k-1}$ along the path $z=e^{it}$, $|t|\leqq\varepsilon_0$. We replace this contour by contours Γ_{nk} chosen to be functions of n, α and k in the following way: let

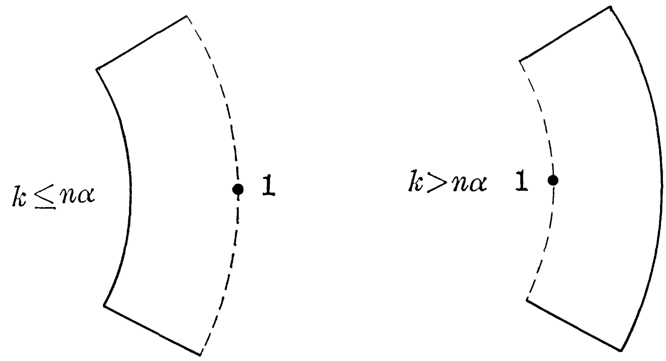
$$\Gamma_{nk} = R_{nk}^{-}\cup Q_{nk}\cup R_{nk}^{+}$$

where

$$Q_{nk}\colon |z| = \begin{cases}1-n^{-c}, & k\leqq n\alpha,\ |\arg z|\leqq\varepsilon_0\\ 1+n^{-c}, & k>n\alpha,\ |\arg z|\leqq\varepsilon_0 ,\end{cases}$$

$$R_{nk}^{\pm}\colon \arg z = \pm\varepsilon_0\begin{cases}1-n^{-c}\leqq|z|\leqq 1, & k\leqq n\alpha\\ 1\leqq|z|\leqq 1+n^{-c}, & k>n\alpha .\end{cases}$$

These paths are illustrated below.



For definiteness we will assume that $\alpha > 0$. The cases for $\alpha < 0$ and $\alpha = 0$ follow in a similar fashion. We write

$$\sum |a_{nk}^{(1)}| < \sum_{m=1}^{3} J_m$$

where

$$J_1 = \sum \left| \int_{R_{nk}^{+}} \right| , \quad J_2 = \sum \left| \int_{R_{nk}^{-}} \right| , \quad J_3 = \sum \left| \int_{Q_{nk}} \right| ;$$

the integrand in each case is $\phi^n(z) z^{-k-1}$ and the range of summation is for $k \in S(n)$.

We estimate each J_m:

$$\begin{aligned} J_1, J_2 &< \sum_{k \in S(n)} \int_{R_{nk}^{\pm}} | \phi^n(z) z^{-k-1} || dz | \\ &\leqq \sum_{k \leqq n\alpha} \int_{1-n^{-c}}^{1} | \phi(re^{\pm i\varepsilon_0}) |^n r^{-k-1} dr \\ &+ \sum_{k > n\alpha} \int_{1}^{1+n^{-c}} | \phi(re^{\pm i\varepsilon_0}) |^n r^{-k-1} dr = \sum\nolimits_1 + \sum\nolimits_2 . \end{aligned}$$

Now

$$\sum\nolimits_1 \leqq 2n^{-c}(1-\delta)^n r_0^{-1} \sum_{k \leqq n\alpha} r_0^{-k} , \quad r_0 = 1 - n^{-c}$$

and

$$\sum_{k \leqq n\alpha} r_0^{-k} = \mathcal{O}[n^c \exp [\alpha n^{(p-1)/q}]] , \quad n \to \infty ,$$

so

$$\sum\nolimits_1 = \mathcal{O}[(1-\delta)^n \exp (\alpha n^{(p-1)/q})] = o(1) , \quad n \to \infty .$$

Further,

$$\sum_2 \leqq 2n^{-c}(1-\delta)^n \sum_{n\alpha}^{n^{1+1/q}} 1 = o(1) , \quad n \to \infty$$

and so J_1 and J_2 are both $o(n^{(1-p/q)/2})$ as $n \to \infty$. To estimate J_3 we treat two cases:

(i) Let $\sum_3$ denote the sum in J_3 restricted to $k \in S(n)$, $k > n\alpha$. Then

$$\begin{aligned}\sum_3 &= \sum r^{n\alpha-k} \left| \int_{-\varepsilon_0}^{\varepsilon_0} [(re^{it})^{-\alpha}\phi(re^{it})]^n e^{i(n\alpha-k)t} dt \right| \\ &\leqq \sum r^{n\alpha-k} \int_{-\varepsilon_0}^{\varepsilon_0} \exp[n\psi(r, t)]dt \\ &= \mathcal{O}(n^{-1/q}) \sum r^{n\alpha-k} , \quad r = 1 + n^{-c}\end{aligned}$$

by applying Lemma 2 and since

$$\sum_{k \in S(n), k > n\alpha} r^{n\alpha-k} \sim n^c \exp(-b_n) ,$$

$$\sum_3 = \mathcal{O}[n^{(1-p/q)/2-1}] = o(n^{(1-p/q)/2}) , \quad n \to \infty .$$

(ii) If $\sum_4$ is the sum in J_3 restricted to $k < n\alpha$,

$$\sum_4 = \mathcal{O}(n^{-1/q}) \sum r^{n\alpha-k} , \quad r = 1 - n^{-c}$$

as $n \to \infty$. This last sum is also of order $n^c \exp(-b_n)$ and so $\sum_4 = o(n^{(1-p/q)/2})$ as $n \to \infty$ completing the estimate for J_3. Thus, we can write

$$2\pi \sum_{k=-\infty}^{\infty} | a_{nk} | = \sum_{k \in T(n)} | a_{nk}^{(1)} | + o(n^{(1-p/q)/2}) , \quad n \to \infty . \tag{4}$$

Part 2

By making a simple change of variable we can write

$$\begin{aligned}&\sum_{k \in T(n)} | a_{nk}^{(1)} | \\ &= \sum_{\gamma \in U(n)} n^{-1/q} \Bigg[\left| \int_{-\varepsilon_0 n^{1/q}}^{\varepsilon_0 n^{1/q}} \exp[i\gamma t + inp(tn^{-1/q}) + nG_1(tn^{-1/q})]dt \right| \\ &\quad + \left| \int_{-\varepsilon_0 n^{1/q}}^{\varepsilon_0 n^{1/q}} \exp[-i\gamma t + inp(tn^{-1/q}) + nG_1(tn^{-1/q})]dt \right| \Bigg]\end{aligned} \tag{5}$$

where

$$p(t) = bt^p + \sum_{k=p+1}^{q} b_k t^k .$$

If $b < 0$, we can write $p(t)$ using $-|b|$ instead of b and then by taking conjugates (5) can be put into the same form as above with

$|b|$ instead of b. Hence, whether b is positive or negative (5) can be written in the form

$$\sum_{\gamma \in U(n)} n^{-1/q}\left[\left|\int_{-\varepsilon_0 n^{1/q}}^{\varepsilon_0 n^{1/q}} \exp\left[ih_1(t) + G(t)\right]dt\right| + \left|\int_{-\varepsilon_0 n^{1/q}}^{\varepsilon_0 n^{1/q}} \exp\left[ih_2(t) + G(t)\right]dt\right|\right] \tag{6}$$

where

$$\begin{aligned} h_1(t) &= \gamma t + nP(tn^{-1/q}) \\ h_2(t) &= -\gamma t + nP(tn^{-1/q}) \\ P(t) &= bt^p + \sum_{k=p+1}^{q} c_k t^k\,, \quad b > 0 \\ G(t) &= -\beta t^q + n^{-1/q} G_3(t) \end{aligned}$$

and $G_3(t) = \mathcal{O}(t^{q+1})$, $tn^{-1/q} \to 0$, $n \to \infty$ and $|G_3(t)| \leqq \beta t^q/2$, for $|tn^{-/q}| \leqq \varepsilon_0$. In what follows we will establish the existence and uniqueness of zeros of $h_1'(t)$ and $h_2'(t)$ in certain intervals which depend upon n, and then show that, after splitting off from (6) the sum of integrals in small neighborhoods of these zeros, the remainder will be $o(n^{(1-p/q)/2})$, as $n \to \infty$. We define

$$\lambda_n = (2/\beta)^{1/q}(\log n^{\omega})^{1/(p-1)}$$

where $\omega > (1 - p/q)/2$ is to be determined as follows: there is an $\omega > (1 - p/q)/2$ such that

$$b_n/pb\lambda_n^{p-1} \leqq \delta < \frac{1}{2}\,. \tag{7}$$

Indeed, $b_n/pb\lambda_n^{p-1} = (1/pb)(\beta/2)^{(p-1)/q}[(1 - p/q)/2 + 1]/\omega$. Let $V = (1/pb)(\beta/2)^{(p-1)/q}$. Then, if $V < 1/2$, pick $\omega \geqq (1 - p/q)/2 + 1$ and if $V \geqq 1/2$, pick ω so large that $[(1-p/q)/2+1]/\omega < 1/2V$. We establish the existence of zeros of h_1' and h_2' by use of the following inequalities: for $\varepsilon > 0$ sufficiently small P, P' and P'' are monotone on each of $[-\varepsilon, 0]$ and $[0, \varepsilon]$ and in each of these intervals

$$\frac{1}{2}pb\,|t|^{p-1} \leqq |P'(t)| \leqq 2pb\,|t|^{p-1}\,,$$

$$\frac{1}{2}p(p-1)b\,|t|^{p-2} \leqq |P''(t)| \leqq 2p(p-1)b\,|t|^{p-2}\,.$$

For n sufficiently large, $\lambda_n n^{-1/q} < \varepsilon$, and then in the case of even p, for $t \in [-\lambda_n, 0]$, we have,

$$\gamma + 2pbn^{1-p/q}t^{p-1} \leqq \gamma + n^{1-1/q}P'(tn^{-1/q}) \leqq \gamma + \frac{1}{2}pbn^{1-p/q}t^{p-1}$$

and both of these bounding polynomials have zeros in $[-\lambda_n, 0]$. Hence, $h_1'(t)$ has a zero, say $-t_{nk}'$ in $[-\lambda_n, 0]$. Furthermore, since $P'(tn^{-1/q})$ is monotone there, this zero is unique and for $t \in [0, \lambda_n]$ we have $|h_1'(t)| \geqq \gamma$. In a similar way we can show that $h_2'(t)$ has a unique zero, t_{nk}, in $[0, \lambda_n]$ and $|h_2'(t)| \geqq \gamma$ for $t \in [-\lambda_n, 0]$. If p is odd, the same type of arguments yield $|h_1'(t)| \geqq \gamma$ for $t \in [-\lambda_n, \lambda_n]$ and $h_2'(t)$ has unique zeros in each of $[-\lambda_n, 0]$ and $[0, \lambda_n]$, say, respectively, $-\tau_{nk}$ and t_{nk}.

Let t_0 be a zero of $h_1'(t)$ or $h_2'(t)$. Then $|t_0| < \lambda_n$ and

$$n^{1-1/q}P'(t_0 n^{-1/q}) = n^{1-p/q}pbt_0^{p-1}[1 + \mathcal{O}(n^{-1/q}\log n)]$$

since $\lambda_n = \mathcal{O}(\log n)$ as $n \to \infty$. Thus,

$$|t_0| = (\gamma/n^{1-p/q}pb)^{1/(p-1)}(1 + \mathcal{O}(n^{-1/q}\log n))$$

as $n \to \infty$ from which it follows that

$$\lim_{n\to\infty}[|t_0|/(\gamma/n^{1-p/q}pb)^{1/(p-1)}] = 1$$

uniformly in k.

We now introduce the following intervals:

$$\begin{aligned} I_{nk} &= [t_{nk}(1 - n^{-d}), t_{nk}(1 + n^{-d})] \\ I_{nk}' &= [-t_{nk}'(1 + n^{-d}), -t_{nk}'(1 - n^{-d})] \\ I_{nk}'' &= [-\tau_{nk}(1 + n^{-d}), -\tau_{nk}(1 - n^{-d})] \end{aligned}$$

for $d = 7(1 - p/q)/16$. Also let $D_1 = [\lambda_n, \varepsilon_0 n^{1/q}]$, $D_2 = [-\varepsilon_0 n^{1/q}, -\lambda_n]$, $D_3 = [0, \lambda_n]$, $D_4 = D_5 \backslash I_{nk}'$, $D_5 = [-\lambda_n, 0]$, $D_6 = D_3 \backslash I_{nk}$, $D_7 = D_4 \backslash I_{nk}''$. Our purpose is to show that if p is even

$$\begin{aligned} \sum_{k \in T(n)} |a_{nk}^{(1)}| = \sum_{\gamma \in U(n)} n^{-1/q}\Big[\Big|\int_{I_{nk}} \exp[ih_2(t) + G(t)]dt\Big| \\ + \Big|\int_{I_{nk}'} \exp[ih_1(t) + G(t)]dt\Big|\Big] \\ + o(n^{(1-p/q)/2}), \quad n \to \infty \end{aligned}$$

and if p is odd

$$\begin{aligned} \sum |a_{nk}^{(1)}| = \sum_{\gamma \in U(n)} n^{-1/q}\Big|\int_{I_{nk} \cup I_{nk}''} \exp[ih_2(t) + G(t)]dt\Big| \\ + o(n^{(1-p/q)/2}), \quad n \to \infty . \end{aligned}$$

If we let

$$C_{mj} = \sum_{\gamma \in U(n)} n^{-1/q}\Big|\int_{D_j} \exp[ih_m(t) + G(t)]dt\Big|$$

m $= 1, 2$, $j = 1, \cdots, 7$, by using (5) we can write, for p even,

$$\left| \sum_{k \in T(n)} |a_{nk}^{(1)}| - \sum_{\gamma \in U(n)} n^{-1/q} \left[\left| \int_{I_{nk}} \exp [ih_2(t) + G(t)] dt \right| + \left| \int_{I'_{nk}} \exp [ih_1(t) + G(t)] dt \right| \right] \right| \leqq \sum C_{1j} + \sum C_{2j} \tag{8}$$

where the first sum is for $j = 1, 2, 3, 4$ and the second is for $j = 1, 2, 5, 6$ and if p is odd

$$\left| \sum_{k \in T(n)} |a_{nk}^{(1)}| - \sum_{\gamma \in U(n)} n^{-1/q} \left| \int_{I_{nk} \cup I''_{nk}} \exp [ih_2(t) + G(t)] dt \right| \right| \leqq \sum C_{1j} + \sum C_{2j} \tag{9}$$

where the first sum is for 1, 2, 3, 5 and the second is for 1, 2, 6, 7.

We must show that each of these C_{ij} is $o(n^{(1-p/q)/2})$ as $n \to \infty$. First for $i = 1, 2$ and $j = 1, 2$, C_{ij} is bounded by

$$\sum_{\gamma \in U(n)} n^{-1/q} \int_{\lambda_n}^{\varepsilon_0 n^{1/q}} |\exp G(t)| \, dt \leqq \sum_{\gamma \in U(n)} n^{-1/q} \int_{\lambda_n}^{\infty} \exp \left(-\frac{1}{2} \beta t^q \right) dt$$

and since

$$\int_L^{\infty} \exp(-\eta t^q) dt \leqq C L^{1-q} \exp(-\eta L^q) \,, \quad L > 1$$

for some constant C and $\eta > 0$, the sum above is bounded by

$$C \sum_{\gamma \in U(n)} n^{-1/q} \lambda_n^{1-q} \exp [-(\log n^{\omega})^{q/(p-1)}] \leqq C n^c b_n n^{-1/q-\omega} \lambda_n^{1-q} = o(n^{(1-p/q)/2}), \text{ as } n \to \infty$$

because $\omega > (1 - p/q)/2$.

In order to estimate the remaining C_{ij} we apply Lemma 3. Consider those integrals which have one limit zero and the other either λ_n or $-\lambda_n$. We have $g(t) = n^{1-1/q} P'(t n^{-1/q})/h'(t)$ where $h(t)$ is either $h_1(t)$ or $h_2(t)$ and then $g(0) = 0$ and

$$|g(\pm\lambda_n)| \leqq 1/[1 - |\gamma/n^{1-1/q} P'(\pm\lambda_n n^{-1/q})|] \,.$$

But since $\lambda_n n^{-1/q} \to 0$ as $n \to \infty$, we have by (7)

$$|\gamma/n^{1-1/q} P'(\pm\lambda_n n^{-1/q})| < \left| \gamma/n^{1-p/q} \frac{1}{2} p b \lambda_n^{p-1} \right| \leqq 2b_n/p b \lambda_n^{p-1} \leqq 2\delta < 1 \,.$$

Thus, $|g(\pm\lambda_n)|$ is bounded and if h' has no zero in $[0, \lambda_n]$, $|h'(t)| \geqq \gamma$. Hence, $\max_{[0,\lambda_n]} |\gamma/h'(t)| \leqq 1$ and so we have

$$\left| \int_0^{\lambda_n} \exp [ih(t) + G(t)] dt \right| \leqq \frac{M}{\gamma}$$

for some constant M. A similar estimate for the integral over the interval $[-\lambda_n, 0]$ holds and thus we obtain a bound of order

$$(10)\qquad \sum_{\gamma \in U(n)} (1/\gamma) n^{-1/q} = \mathcal{O}(\log n) = o(n^{(1-p/q)/2}) \ , \quad n \to \infty \ ,$$

for C_{13}, C_{15} and C_{25}.

Consider the integral

$$\int_{D_6} \exp\,[ih_2(t) + G(t)]dt$$

and write $[0, \lambda_n] = [0, x_n) \cup I_{nk} \cup (y_n, \lambda_n]$. We will show that the conditions of Lemma 3 are satisfied on these two outside intervals. First, we have shown above that $|g(\lambda_n)|$ is bounded. Further, $\max_{[0,x_n]} 1/|h_2'(t)| = 1/|h_2'(x_n)|$ and

$$\begin{aligned} |h_2'(x_n)| &= |-\gamma + n^{1-1/q} P'[t_{nk}(1 - n^{-d})n^{-1/q}]| \\ &= \left| \sum_{k=p}^{q} k c_k t_{nk}^{k-1} n^{1-k/q} \sum_{m=1}^{k-1} \binom{k-1}{m} (-1)^m n^{-dm} \right| \end{aligned}$$

since $h_2'(t_{nk}) = 0$. This last sum is equal to

$$n^{1-p/q} p(p-1) b t_{nk}^{p-1} n^{-d}[1 + \mathcal{O}(n^{-1/q} \log n)] \ , \quad n \to \infty$$

and since

$$n^{1-p/q} p b t_{nk}^{p-1} = \gamma[1 + \mathcal{O}(n^{-1/q} \log n)] \ , \quad n \to \infty \ ,$$

$|h_2'(x_n)| \geqq (1/2)\gamma n^{-d}$. By replacing $1 - n^{-d}$ by $1 + n^{-d}$ in the above arguments it follows that $|h_2'(y_n)| \geqq (1/2)\gamma n^{-d}$. A similar calculation shows that $g(x_n)$ and $g(y_n)$ are both $\mathcal{O}(n^d)$ as $n \to \infty$. Thus, our integral is $\mathcal{O}(n^d)$ and we get by (10) that

$$C_{26} = \mathcal{O}(n^d) \sum_{\gamma \in U(n)} (1/\gamma) n^{-1/q} = o(n^{(1-p/q)/2})$$

as $n \to \infty$. A similar argument applies also to C_{14}, C_{27} and C_{26}.

Part 3

In this section we will obtain the asymptotic constant for the case when p is even and different from q. Indeed, if we let

$$\Sigma_n = \sum_{\gamma \in U(n)} n^{-1/q} \left| \int_{I_{nk}} \exp\,[ih_2(t) + G(t)]dt \right|$$

then

$$\Sigma_n \sim \frac{1}{2} L n^{(1-p/q)/2} \ , \quad n \to \infty \ ,$$

where

$$L = 2a\sqrt{\pi}\int_0^\infty F(x)dx$$

for

$$F(x) = x^{-(p-2)/2(p-1)} \exp(-\beta x^{q/(p-1)})$$

and

$$a = (2pb/(p-1))^{1/2} .$$

For simplicity write $h_2 = h$. Then $h^{(k)}(t_{nk}) = \mathcal{O}(n^{1-p/q}t_{nk}^{p-k})$ as $n \to \infty$ for $k \geqq 2$ and so for $t \in I_{nk}$

$$| h^{(k)}(t_{nk})(t - t_{nk})^k/k! | \leqq Mn^{(1-p/q)(1-7k/16)}t_{nk}^p ,$$

for some $M > 0$. Using $t_{nk}^p = \mathcal{O}[(\log n)^{p/(p-1)}]$, $n \to \infty$ and Taylor's formula we have for $t \in I_{nk}$

$$h(t) = h(t_{nk}) + \frac{1}{2}h''(t_{nk})(t - t_{nk})^2 + \mathcal{O}[n^{-5(1-p/q)/16}(\log n)^2]$$

as $n \to \infty$. A simple calculation also yields

$$G(t) = -\beta t_{nk}^q + \mathcal{O}[(\log n)^q n^{-d}]$$

for $t \in I_{nk}$ as $n \to \infty$ and so we can write

$$\left|\int_{I_{nk}} \exp[ih(t) + G(t)]dt\right|$$
$$= \exp(-\beta t_{nk}^q)\left|\int_{I_{nk}} \exp[ih''(t_{nk})(t - t_{nk})^2/2][1 + G^*(t)]dt\right|$$

where $G^*(t) = \mathcal{O}[(\log n)^2 n^{-5(1-p/q)/16}]$, $t \in I_{nk}$, as $n \to \infty$. A change of variable also yields

$$\int_{I_{nk}} \exp[ih''(t_{nk})(t - t_{nk})^2/2]dt = [2/h''(t_{nk})]^{1/2}\int_0^{w_{nk}} e^{iu}u^{-1/2}du$$

where $w_{nk} = (1/2)h''(t_{nk})t_{nk}^2 n^{-2d}$.

Let $\sigma_{nk} = \mu/n^c pb$ and $\varDelta\sigma_{nk} = 1/n^c pb$. Then we can write

$$\left| \Sigma_n n^{-(1-p/q)/2} - \frac{1}{2}L \right| \leqq \sum_{j=1}^4 K_j$$

where

$$K_1 = \left| \sum \left| \int_0^{w_{nk}} e^{iu}u^{-1/2}du \right| \right.$$
$$\left. \times [\exp(-\beta t_{nk}^q)(2/h''(t_{nk}))^{1/2} n^{-(1-p/q)/2-1/q} - aF(\sigma_{nk})\varDelta\sigma_{nk}] \right.$$

$$K_2 = \left| \sum aF(\sigma_{nk})\Delta\sigma_{nk}\left[\left| \int_0^{w_{nk}} e^{iu}u^{-1/2}du \right| - \sqrt{\pi} \right] \right| ,$$

$$K_3 = \left| a\sqrt{\pi} \sum F(\sigma_{nk})\Delta\sigma_{nk} - \frac{1}{2}L \right|$$

and

$$K_4 = n^{-(1-p/q)/2} \sum n^{-1/q} \exp(-\beta t_{nk}^q) \left| \int_{I_{nk}} G^*(t)dt \right| ,$$

where all sums are for $\gamma \in U(n)$.

To estimate K_4 we let $s_{nk} = \sigma_{nk}^{1/(p-1)}$ and observe that

$$\begin{aligned} \sum_{\gamma \in U(n)} \exp(-\beta s_{nk}^q) &= pbn^c \sum_{\gamma \in U(n)} \exp(-\beta s_{nk}^q)(pbn^c)^{-1} \\ &= \mathcal{O}(n^c)\int_0^\infty \exp[-\beta x^{q/(p-1)}]dx \\ &= \mathcal{O}(n^c) , \quad n \to \infty . \end{aligned}$$

Then, since $t_{nk} = s_{nk}[1 + \mathcal{O}(n^{-1/q}\log n)]$ and $s_{nk} = \mathcal{O}(\log n)$ as $n \to \infty$,

$$t_{nk}^q = s_{nk}^q + \mathcal{O}[n^{-1/q}(\log n)^{q+1}] , \quad n \to \infty$$

and so

$$\begin{aligned} K_4 &\leqq 2n^{-(1-p/q)/2-1/q} \sum_{\gamma \in U(n)} \exp(-\beta s_{nk}^q)\int_{I_{nk}} |G^*(t)|dt \\ &= \mathcal{O}[n^{c-5(1-p/q)/4-1/q}(\log n)^3] = o(1) , \quad n \to \infty , \end{aligned}$$

since for $t \in I_{nk}$, $G^*(t) = \mathcal{O}[(\log n)^2 n^{5(1-p/q)/16}]$, $n \to \infty$. To estimate K_3 we define a sequence of functions as follows: let

$$F_n(x) = \begin{cases} \sum F(\sigma_{nk})\chi_{nk}(x), & x \in [b_n(n^c pb)^{-1}, b_n(pb)^{-1}] \\ 0 & \text{elsewhere} \end{cases}$$

where χ_{nk} is the indicator function of $[\sigma_{nk}, \sigma_{nk-1})$ and the sum is for $\gamma \in U(n)$. Then, the sum in K_3 is equal to $\int_0^\infty F_n(x)dx$ and

$$K_3 \leqq a\sqrt{\pi}\left[\int_0^\varepsilon (F_n + F)dx + \int_R^\infty (F_n + F)dx + \int_\varepsilon^R |F_n - F|dx\right]$$

for $0 < \varepsilon < R$. For any $\delta < 0$, a straightforward calculation shows that there exist ε and R so that the sum of the first two integral expressions is bounded by $\delta/2a\sqrt{\pi}$ and since $F_n(x)$ converges uniformly to $F(x)$ on $[\varepsilon, R]$ there is an n_0 such that for $n > n_0$, the third integral is bounded by $\delta/2a\sqrt{\pi}$. Thus, $K_3 = o(1)$ as $n \to \infty$.

Also for $\varepsilon > 0$ and some $M > 0$, since $\int_0^\infty e^{iu}u^{-1/2}du = \sqrt{\pi}e^{i\pi/4}$,

$$K_2 \leqq a \sum_{\gamma \in U(n)} F(\sigma_{nk}) \left| \int_{w_{nk}}^{\infty} e^{iu} u^{-1/2} du \right| \Delta\sigma_{nk}$$
$$\leqq M \sum_{\mu \leqq n^c p b \varepsilon} F(\sigma_{nk}) \Delta\sigma_{nk}$$
$$+ a \sup_{\mu > n^c p b \varepsilon} \left| \int_{w_{nk}}^{\infty} e^{iu} u^{-1/2} du \right| \int_{\varepsilon}^{\infty} F_n(x) dx .$$

The first of these terms is bounded by $M'\varepsilon^{p/2(p-1)}$ for some constant M' and the second is bounded by

$$M'' \sup_{\mu > n^c p b \varepsilon} \left| \int_{w_{nk}}^{\infty} e^{iu} u^{-1/2} du \right|$$

for some constant M''. Further, integration by parts twice gives

$$\left| \int_{w_{nk}}^{\infty} e^{iu} u^{-1/2} du \right| < 2/\sqrt{w_{nk}}$$

and since $t_{nk}^p = \sigma_{nk}^{p/(p-1)}[1 + \mathcal{O}(n^{-1/q} \log n)] \geqq (1/2)\varepsilon^{p/(p-1)}$ for sufficiently large n, we have

$$w_{nk} \geqq \frac{1}{4} n^{1-p/q} p(p-1) b \varepsilon^{p/(p-1)} n^{-2\lambda} = \frac{1}{4} n^{(1-p/q)/8} p(p-1) b \varepsilon^{p/(p-1)} .$$

Thus,

$$\sup_{\mu > n^c p b \varepsilon} \left| \int_{w_{nk}}^{\infty} e^{iu} u^{-1/2} du \right| = \mathcal{O}(n^{-(1-p/q)/16})$$

as $n \to \infty$ and so for $\delta > 0$ we can find an $\varepsilon > 0$ and n_0 such that $n > n_0$ implies that $K_2 < \delta$. To show that K_1 tends to zero as $n \to \infty$ we need the following fact:

Let Ψ, ϕ, θ be complex valued functions of a real variable and let (t_{nk}), (s_{nk}) be double sequences. If $\omega_n \to 0$ uniformly in k as $n \to \infty$ and $\Psi(t_{nk})/\phi(s_{nk}) = 1 + \mathcal{O}(\omega_n)$, $n \to \infty$ and

$$\sum_k | \phi(s_{nk}) \theta(t_{nk}) | = \mathcal{O}(1) , \quad n \to \infty ,$$

then,

$$\left| \sum_k \Psi(t_{nk}) \theta(t_{nk}) - \sum \phi(s_{nk}) \theta(t_{nk}) \right| = \mathcal{O}(\omega_n) , \quad n \to \infty$$

To apply this it will be sufficient to estimate the behavior of

(11) $$\exp(-\beta t_{nk}^q)[h''(t_{nk})]^{-1/2} \cdot [n^{1-p/q} p(p-1) b s_{nk}{}^{p-2}]^{1/2} \exp(\beta s_{nk}^q).$$

We have

$$\exp[-\beta(t_{nk}^q - s_{nk}^q)] = 1 + \mathcal{O}[n^{-1/q} (\log n)^{q+1}]$$

as $n \to \infty$ and since $t_{nk} = \mathcal{O}(\log n)$, $h''(t_{nk}) = n^{1-p/q}p(p-1)bt_{nk}^{p-2}\cdot [1 + \mathcal{O}(n^{-1/q}\log n)]$. This together with $t_{nk}^{p-2} = s_{nk}^{p-2}[1 + \mathcal{O}(n^{-1/q}\log n)]$ yields

$$[n^{1-p/q}p(p-1)bs_{nk}^{p-2}]^{1/2}/[h''(t_{nk})]^{1/2} = 1 + \mathcal{O}(n^{-1/q}\log n)$$

and so (11) is of order $1 + \mathcal{O}(n^{-1/q}(\log n)^{q+1})$ and hence $K_1 = \mathcal{O}[n^{-1/q}(\log n)^{q+1}] = o(1)$ as $n \to \infty$. Thus, we have

$$(12) \qquad \sum_{\gamma \in U(n)} n^{-1/q}\left|\int_{I_{nk}} \exp[h_2(t) + G(t)]dt\right| = n^{(1-p/q)/2}\left[\frac{1}{2}L + o(1)\right], \quad n \to \infty .$$

In order to treat

$$\sum_{\gamma \in U(n)} n^{-1/q}\left|\int_{I_{nk}} \exp[ih_1(t) + G(t)]dt\right|$$

we set $u = -t$ and notice that $h_1(-t)$ and $G(-t)$ are of the same form respectively as $h_2(t)$ and $G(t)$ since both p ane q are even; therefore, by precisely the same argument

$$(13) \qquad \sum_{\gamma \in U(n)} n^{-1/q}\left|\int_{I_{nk}} \exp[ih_1(t) + G(t)]dt\right| = n^{(1-p/q)/2}\left[\frac{1}{2}L + o(1)\right], \quad n \to \infty ,$$

and then by combining (4), (8), (12) and (13) we obtain (a) in the statement of the theorem since by computing the integral

$$(14) \qquad L = 2[2pb\pi/(p-1)]^{1/2}[(p-1)/q](1/\beta)^{p/2q}\Gamma(p/2q) .$$

PART 4

We consider the case when p is odd and different from q. As before we can write

$$\begin{aligned}\int_{I_{nk}} &\exp[ih_2(t) + G(t)]dt \\ &= \exp[ih_2(t_{nk}) - \beta t_{nk}^q][2/h_2''(t_{nk})]^{1/2}\int_0^{\omega_{nk}} e^{iu}u^{-1/2}du \\ &\quad + \exp[ih_2(t_{nk}) - \beta t_{nk}^q]\int_{I_{nk}} G^*(t)dt\end{aligned}$$

where $G^*(t) = \mathcal{O}[n^{-5(1-p/q)/16}(\log n)^2]$, $t \in I_{nk}$, as $n \to \infty$. But we must also take into account this same integrand for the interval I''_{nk}. Let

$$h_3(t) = -h_2(-t) = -\gamma t + n^{1-p/q}bt^p + n\sum_{k=p+1}^{q} c_k(-1)^{k+1}n^{-k/q}t^k .$$

Then, a change of variable and conjugation produce

$$\int_{I''_{nk}} \exp[ih_2(t) + G(t)]dt = \overline{\int_{-I''_{nk}} \exp[ih_3(t) + \overline{G(-t)}]dt}$$

$$= \exp[-ih_3(\tau_{nk}) - \beta\tau^q_{nk}][2/h''_3(\tau_{nk})]^{1/2}\int_0^{\omega'_{nk}} e^{-iu}u^{-1/2}du$$

$$+ \exp[-ih_3(\tau_{nk}) - \beta\tau^q_{nk}]\int_{-I''_{nk}} G^{**}(t)dt$$

where

$$G^{**}(t) = \mathcal{O}(n^{-5(1-p/q)/16}(\log n)^2),\ t \in -I''_{nk}\ ,\quad n \to \infty\ ,$$

$\omega'_{nk} = h''_3(\tau_{nk})\tau^2_{nk}n^{-2d}/2$ and $-I''_{nk} = [\tau_{nk}(1 - n^{-d}),\ \tau_{nk}(1 + n^{-d})]$. Let

$$\Sigma_n = \sum_{\gamma \in U(n)} n^{-1/q} \left| \int_{I_{nk} \cup I''_{nk}} \exp[ih_2(t) + G(t)]dt \right| . \tag{15}$$

We shall show that

$$\Sigma_n \sim (2/\pi)Ln^{(1-p/q)/2}\ ,\quad \text{as}\quad n \to \infty\ .$$

To do this we write

$$\left| \Sigma_n n^{-(1-p/q)/2} - \frac{2}{\pi}L \right| \leqq \sum_{m=5}^{13} K_m \tag{16}$$

where $K_5 = K_4$, $K_7 = K_1$,

$$K_6 = n^{-(1-p/q)/2} \sum_{\gamma \in U(n)} n^{-1/q} \exp[-\beta\tau^q_{nk}] \int_{-I''_{nk}} |G^{**}(t)|dt\ ,$$

K_8 is K_1 with t_{nk} replaced by τ_{nk} and w_{nk} replaced by w'_{nk},

$$K_9 = \left| \sum F(\sigma_{nk})\Delta\sigma_{nk} \left| \exp[ih_2(t_{nk})]\int_0^{\omega_{nk}} e^{iu}u^{-1/2}du \right.\right.$$

$$\left.\left. + \exp[-ih_3(\tau_{nk})]\int_0^{\omega'_{nk}} e^{-iu}u^{-1/2}du \right| - \frac{2}{\pi}\cdot a\sqrt{\pi}\int_\varepsilon^R F(x)dx \right|\ ,$$

where the sum is for $\varepsilon n^c pb \leqq \mu \leqq Rn^c pb$, K_{10} is the sum term of K_9 with the range of summation $b_n \leqq \mu < \varepsilon n^c pb$, K_{11} is the sum term in K_9 with the range of summation $Rn^c pb < \mu \leqq b_n n^c$,

$$K_{12} = (2a/\sqrt{\pi})\int_0^\varepsilon F(x)dx\ ,$$

$$K_{13} = (2a/\sqrt{\pi})\int_R^\infty F(x)dx$$

and a and $F(x)$ are as defined in Part 3.

Again, we must show that each K_m is $o(1)$ as $n \to \infty$. K_6 is treated in the same fashion as K_4, K_8 as K_1. We remark that there is a constant M such that

$$\left|\int_0^Y e^{\pm iu}u^{-1/2}du\right| \leqq M$$

for all Y so that

$$K_{10} = \mathcal{O}\left[\int_0^\varepsilon F_n(x)dx\right] = o(1)\ , \quad \varepsilon \to 0$$

and

$$K_{11} = \mathcal{O}\left[\int_R^\infty F_n(x)dx\right] = o(1)\ , \quad \mathrm{R} \to \infty\ .$$

That $K_{12} = o(1)$, $\varepsilon \to \infty$ and $K_{13} = o(1)$, $R \to \infty$ follows easily from the definition of $F(x)$. To estimate K_9 we write

$$K_9 \leqq a \sum_\mu F(\tau_{nk})\left|\int_0^{\omega_{nk}} e^{iu}u^{-1/2}du - e^{i\pi/4}\sqrt{\pi}\right|\Delta\sigma_{nk}$$
$$+ a \sum_\mu F(\sigma_{nk})\left|\int_0^{\omega'_{nk}} e^{-iu}u^{-1/2}du - e^{-i\pi/4}\sqrt{\pi}\right|\Delta\sigma_{nk}$$

$$\text{(17)}\quad \begin{cases} +\left|a\sqrt{\pi}\sum_\mu F(\sigma_{nk})\left|\exp\left[ih_2(t_{nk}) + i\pi/4\right] + \exp\left[-ih_3(\tau_{nk}) - i\pi/4\right]\right|\right. \\ \left. -\ a\sqrt{\pi}\cdot(2/\pi)\int_\varepsilon^R F(x)dx\right| \end{cases}$$

where the sum in each of these terms is over the range $\varepsilon n^c pb \leqq \mu \leqq Rn^c pb$. The first and second sums above are treated in the same fashion as K_2 and both are $\mathcal{O}(n^{-(1-p/q)/16})$, $n \to \infty$. The sum in (17) can be written as

$$a\sqrt{\pi}\sum_\mu F(\sigma_{nk})\left|\cos\frac{1}{2}[h_2(t_{nk}) + h_3(\tau_{nk}) + \pi/2]\right|\Delta\sigma_{nk}\ .$$

Thus, it suffices to show that the limit of the sum in the last expression is equal to

$$(2/\pi)\int_\varepsilon^R F(x)dx\ .$$

This will be accomplished by the following steps:

(i) For each $n \geqq n_0$, for some n_0, there is a function $H_n(x)$ such that

$$-\frac{1}{2}(h_2(t_{nk}) + h_3(\tau_{nk})) = n^{1-p/q}H_n(\sigma_{nk})\ ;$$

thus, the sum above will take the form

$$s_n = \sum_\mu F(\sigma_{nk})\left|\cos\left(n^{1-p/q}H_n(\sigma_{nk}) - \frac{1}{4}\pi\right)\right|\Delta\sigma_{nk}\ .$$

(ii)

$$\left| s_n - \int_\varepsilon^R F(x) \left| \cos\left(n^{1-p/q} H_n(x) - \frac{1}{4}\pi \right) \right| dx \right| \to 0 , \quad n \to \infty .$$

(iii) The limit of the integral in ii) is precisely

$$(2/\pi)\int_\varepsilon^R F(x)dx .$$

After we construct the sequence $(H_n(x))$ iii) will follow directly as an application of Lemma 4. To construct the sequence $(H_n(x))$ first recall that

$$h_2(t) = -\gamma t + n^{1-p/q}bt^p + n\sum_{k=p+1}^{q} c_k n^{-k/q}t^k$$

and

$$h_3(t) = -\gamma t + n^{1-p/q}bt^p + n\sum_{k=p+1}^{q} c_k(-1)^k n^{-k/q}t^k$$

and that $h_2'(t_{nk}) = 0$ and $h_3'(\tau_{nk}) = 0$. Let

$$\phi_n(t) = t^{p-1} + \sum_{k=p+1}^{q} kc_k(pb)^{-1}n^{-(k-p)/q}t^{k-1}$$

and

$$\psi_n(t) = t^{p-1} + \sum_{k=p+1}^{q} kc_k(pb)^{-1}(-1)^k n^{-(k-p)/q}t^{k-1} .$$

Then, t_{nk} and τ_{nk} are respectively the unique solutions to $n^{1-p/q}pb\phi_n(t) = \gamma$ and $n^{1-p/q}pb\psi_n(t) = \gamma$ in the interval $[0, \lambda_n]$. That is, $\phi_n(t_{nk}) = \sigma_{nk}$ and $\psi_n(\tau_{nk}) = \sigma_{nk}$. Now, σ_{nk} is contained in the interval $[\varepsilon, R]$ since $\varepsilon n^c pb \leqq \mu \leqq Rn^c pb$. We will consider ϕ_n and ψ_n over an interval $[r, s]$ where $r > 0$. First,

$$\phi_n(t) = t^{p-1}(1 + \mathcal{O}(n^{-1/q})) = t^{p-1} + \mathcal{O}(n^{-1/q})$$

on $[r, s]$ as $n \to \infty$, and then

$$\begin{aligned}\phi_n'(t) &= (p-1)t^{p-2}(1 + \mathcal{O}(n^{-1/q})) \\ &\geqq \frac{1}{2}(p-1)r^{p-2} > 0\end{aligned}$$

for n sufficiently large. Thus, $\phi_n(t)$ is strictly increasing for n greater than some n_0. Similarly,

$$\psi_n(t) = t^{p-1}(1 + \mathcal{O}(n^{-1/q}))$$

and so $\psi_n(t)$ is strictly increasing on $[r, s]$ and there

$$\phi_n^{-1}(x) = x^{1/(p-1)}(1 + \mathcal{O}(n^{-1/q})) = x^{1/(p-1)} + \mathcal{O}(n^{-1/q})$$

as $n \to \infty$. Also, $[\phi_n^{-1}(x)]' = 1/\phi_n'(t)$ where $x = \phi_n(t)$, and so there is a constant M such that $[\phi_n^{-1}(x)]' \leqq M$ for $x \in [\phi_n(r), \phi_n(s)]$. Similar statements hold for $\psi_n^{-1}(x)$. We rewrite

$$\begin{aligned}
&-\frac{1}{2}(h_2(t_{nk}) + h_3(\tau_{nk})) \\
&= \frac{1}{2}n^{1-p/q}\Big[pb\sigma_{nk}\phi_n^{-1}(\sigma_{nk}) - b[\phi_n^{-1}(\sigma_{nk})]^p - \sum_{k=p+1}^{q} c_k n^{-(k-p)/q}[\phi_n^{-1}(\sigma_{nk})]^k\Big] \\
&\quad + \frac{1}{2}n^{1-p/q}\Big[pb\sigma_{nk}\psi_n^{-1}(\sigma_{nk}) - b[\psi_n^{-1}(\sigma_{nk})]^p \\
&\quad - \sum_{k=p+1}^{q}(-1)^k c_k n^{-(k-p)/q}[\psi_n^{-1}(\sigma_{nk})]^k\Big]
\end{aligned}$$

for $n > n_0$. Thus, for $n > n_0$ we define

$$\begin{aligned}
H_n(x) = &\frac{1}{2}\Big[pbx\phi_n^{-1}(x) - b[\phi_n^{-1}(x)]^p - \sum_{k=p+1}^{q} c_k n^{-(k-p)/q}[\phi_n^{-1}(x)]^k\Big] \\
&+ \frac{1}{2}\Big[pbx\psi_n^{-1}(x) - b[\psi_n^{-1}(x)]^p - \sum_{k=p+1}^{q}(-1)^k c_k n^{-(k-p)/q}[\psi_n^{-1}(x)]^k\Big]
\end{aligned}$$

for $x \in [\varepsilon, R]$, and for $n > n_0$ the following conditions hold:

(a) $H_n(x)$ and $H_n'(x)$ are continuous

(b) $\lim_{n\to\infty} H_n(x) = (p-1)bx^{p/(p-1)}$ and
$\lim_{n\to\infty} H_n'(x) = pbx^{1/(p-1)}$ uniformly

(c) $|H_n(x) - H_n(y)| \leqq M|x - y|$ where M is a constant depending on p, b, ε and R but independent of n.

If we let $H(x) = (p-1)bx^{p/(p-1)}$, then $H'(x)$ is bounded away from 0 on $[\varepsilon, R]$ and hence the conditions of Lemma 4 are satisfied and so (iii) is proved. Let

$$F_n(x) = \begin{cases} \displaystyle\sum_{\mu=\varepsilon n^c pb}^{Rn^c pb} F(\sigma_{nk})\left|\cos\left(n^{1-p/q}H_n(\sigma_{nk}) - \frac{1}{4}\pi\right)\right|\chi_{nk}(x) \\ \quad \text{for} \quad x \in \bigcup_k [\sigma_{nk}, \sigma_{nk-1}) \\ 0, \quad \text{elsewhere.} \end{cases}$$

Then,

$$s_n = \int_\varepsilon^R F_n(x)dx$$

and we need only show that

$$(18) \qquad \left|F_n(x) - F(x)\left|\cos\left(n^{1-p/q}H_n(x) - \frac{1}{4}\pi\right)\right|\right|$$

tends uniformly to zero on $[\varepsilon, R]$ as $n \to \infty$ in order to prove (ii). Let $x \in [\varepsilon, R]$. Then, $x \in [\sigma_{nk_0}, \sigma_{nk_0-1}]$ and thus (18) is bounded above by

$$F(\sigma_{nk_0})\left|\cos\left(n^{1-p/q}H_n(\sigma_{nk_0}) - \frac{1}{4}\pi\right) - \cos\left(n^{1-p/q}H_n(x) - \frac{1}{4}\pi\right)\right| \\ + |F(\sigma_{nk_0}) - F(x)| .$$

Since $F'(x)$ is bounded on $[\varepsilon, R]$, we have $|F(\sigma_{nk_0}) - F(x)| \leqq M|\sigma_{nk_0} - x| \leqq Mn^c/pb$ for some constant M and so the second term tends uniformly to zero as $n \to \infty$. Also on $[\varepsilon, R]$, $F(x) \leqq M'$, for some M', so that the first term is majorized by (using condition c) above)

$$M'\left|\cos\left(n^{1-p/q}H_n(\sigma_{nk_0}) - \frac{1}{4}\pi\right) - \cos\left(n^{1-p/q}H_n(x) - \frac{1}{4}\pi\right)\right| \\ \leqq M'n^{1-p/q}|H_n(\sigma_{nk_0}) - H_n(x)| \leqq M''n^{1-p/q}|\sigma_{nk_0} - x| \leqq M''/n^{1/q}pb .$$

Thus, (18) tends uniformly to zero on $[\varepsilon, R]$ and so we have shown that $K_9 = o(1)$, as $n \to \infty$. Finally, by combining (4), (9), (15) and (16) we have the result and this completes the proof for p odd.

PART 5

We now consider the last case, namely $p = q$. By (4)

$$2\pi \sum |a_{nk}| = \sum_{k \in T(n)} |a_{nk}^{(1)}| + o(1) , \quad n \to \infty$$

and since $p = q$, $f(t) = \exp[i\alpha t + At^p + G_2(t)]$ where $ReA = -\beta$, $G_2(t) = \mathcal{O}(t^{p+1})$, $t \to 0$ and $Re[G_2(t)] \leqq (1/2)(ReA)t^p$, $|t| \leqq \varepsilon_0$. That is, the polynomial $p(t)$ is $(ImA)t^p$. Thus, by a change of variable

$$\sum |a_{nk}^{(1)}| = \sum n^{-1/q}\left|\int_{-\varepsilon_0 n^{1/q}}^{\varepsilon_0 n^{1/q}} \exp[i\gamma t + at^p + G(t)]dt\right|$$

where the sums are for $k \in T(n)$ and where $G(t) = n^{-1/p}\mathcal{O}(t^{p+1})$, $t/n^{1/p} \to 0$, as $n \to \infty$.

Further for $|t| < \lambda_n$ we can write $\exp G(t) = 1 + G^*(t)$ where $G^*(t) = \mathcal{O}[n^{-1/p}(\log n)^{p+1}]$ as $n \to \infty$.

If we let

$$s_n = \sum n^{-1/p}\left|\int_{-\varepsilon_0 n^{1/p}}^{\varepsilon_0 n^{1/p}} \exp[i\gamma t + At^p + G(t)]dt\right|$$

where the sum is for $\gamma \in U(n)$ and let

$$F(x) = \exp(At^p), \quad \hat{F}(x) = \int_{-\infty}^{\infty} F(t)\exp(ixt)dt ,$$

the Fourier transform of F, then

$$s_n - \int_0^\infty |\hat{F}|dx = o(1)\ , \quad n \rightarrow \infty\ . \tag{19}$$

Indeed, we can bound (19) in absolute value above by $\sum_{m=1}^4 L_m$ for

$$L_1 = \left| \sum n^{-1/p} \left| \int_{-\infty}^\infty \exp[i\gamma t + At^p]dt \right| - \int_0^\infty |\hat{F}|dx \right|\ ,$$

$$L_2 = \sum n^{-1/p} \left| \int_{D_1 \cup D_2} \exp[i\gamma t + At^p + G(t)]dt \right|\ ,$$

$$L_3 = \sum n^{-1/p} \left| \int_{D_3 \cup D_5} \exp[i\gamma t + At^p]G^*(t)dt \right|\ ,$$

$$L_4 = \sum n^{-1/p} \left| \int_D \exp[i\gamma t + At^p]dt \right|\ ,$$

where the D_m are defined in Part 2, $D = (-\infty, \lambda_n) \cup (\lambda_n, \infty)$ and all the sums are for $\gamma \in U(n)$. First, we remark that L_2 and L_4 are bounded above by

$$2 \sum_{\gamma \in U(n)} n^{-1/p} \int_{\lambda_n}^\infty \exp\left(-\frac{1}{2}\beta t^p\right)dt$$

which is $o(1)$ as $n \rightarrow \infty$. Next, since

$$L_3 = \mathcal{O}[b_n n^{-1/p}(\log n)^{p+1}] \int_{-\infty}^\infty \exp(ReAt^p)dt = o(1)\ , \quad n \rightarrow \infty\ ,$$

it remains to estimate L_1. As before we define a sequence of approximating functions

$$F_n(x) = \begin{cases} \sum_{\mu=b_n}^{n^{1/p}b_n} |\hat{F}(\mu n^{-1/p})\chi_{nk}(x), & b_n n^{-1/p} \leqq x < b_n \\ |\hat{F}(0)| & 0 \leqq x < b_n n^{-1/p} \\ 0 & \text{elsewhere}\ . \end{cases}$$

The sum in L_1 is then $\int_0^\infty F_n(x)dx + o(1)$, $n \rightarrow \infty$ and so we have reduced the problem to showing that

$$\left| \int_0^\infty [F_n(x) - |\hat{F}(x)|]dx \right| = o(1)\ , \quad n \rightarrow \infty\ .$$

Let R be a fixed positive number to be chosen below. The expression above is bounded by

$$\int_0^R |F_n(x) - |\hat{F}(x)||\, dx + \int_R^\infty F_n(x)dx + \int_R^\infty |\hat{F}(x)|\, dx$$

and by using the standard estimate: $|\hat{F}(y)| \leqq K/y^2$, $y \neq 0$, for some constant K, we have

$$\int_R^\infty F_n(x)dx = \sum_{\mu=n^{1/p}R}^{n^{1/p}b_n} |\hat{F}(\mu n^{-1/p})| n^{-1/p} < n^{1/p} \sum_{\mu=n^{1/p}R}^{\infty} 1/\mu^2 < 1/(R-2),$$

and so we can pick R so large that for $\delta > 0$ the sum of the last two integrals above is bounded by $2\delta/3$. Then, for this fixed value of R there is an n_0 such that for all $n > n_0$, the first integral is bounded by $\delta/3$ since $F_n(x)$ converges uniformly to $|\hat{F}(x)|$ on $[0, R]$.

Similarly, we can show that

$$\lim_{n\to\infty} \sum |a_{nk}^{(1)}| = \int_{-\infty}^0 |\hat{F}| \, dx$$

where the sum is for $k \in T(n)$, $k > n\alpha$, so that if $p = q$ we have

$$\lim_{n\to\infty} \sum |a_{nk}| = (2\pi)^{-1} \|\hat{F}\|_1$$

and this completes the proof of the theorem.

4. Several maximum points. The results we have obtained can be extended partially to the case when $|f(t)| = 1$ at several points in the interval $[-\pi, \pi]$. Assume that $f(t)$ is absolutely continuous and $f'(t)$ is of bounded variation; $|f(t)| < 1$, $t \neq t_j$, $f(t_j) = 1$ and $f(t)$ is analytic at $t = t_j$, $j = 1, \cdots, m$. For each of the points t_j we can define parameters α_j, p_j, q_j corresponding to the parameters α, p, q defined above. We let $T_j(n)$ and $S_j(n)$ be defined by replacing α, p, q in the definitions of $T(n)$ and $S(n)$ by α_j, p_j and q_j respectively.

Let Q_j, $j = 1, \cdots, m$ be sufficiently small intervals centered about each of the points t_j and set $I_j = \int_{Q_j} f^n(t)e^{-ikt}dt$, $I_j' = \int_{-\pi}^{\pi} f^n(t)e^{-ikt}dt - I_j$. If we assume that $\alpha_j \neq \alpha_i$; $i \neq j$, then $T_j(n) \cap T_i(n) = \phi$ for n sufficiently large and it follows by a straightforward application of the previous arguments that, for $j = 1, \cdots, m$,

$$\sum |I_j| = o(n^{(1-p_j/q_j)/2}), \quad \sum |I_j'| = o(n^{(1-p_j/q_j)/2})$$

as $n \to \infty$, where the first sum is for k in the complement of $T_j(n)$ and the second sum is for $k \in T_j(n)$. Thus, we have

$$2\pi \sum |a_{nk}| = \sum_{j=1}^m \sum_{k\in T_j} |I_j| + o(n^{(1-s)/2}), \quad n \to \infty$$

where $s = \min_j (p_j/q_j)$. We can also show that $\sum_{k\in T_j} |I_j| = n^{(1-p_j/q_j)/2} [C_j + o(1)]$, $n \to \infty$ for a constant C_j depending upon the parameters associated with each of the points t_j. Thus, we obtain

$$\lim_{n\to\infty} 2\pi n^{-(1-s)/2} \sum_{-\infty}^{\infty} |a_{nk}| = \sum C_j$$

where the sum extends over all those j such that $p_j/q_j = s$. This "additivity" does not, however, extend to the general case and, as an example by Newman [7, p. 40] shows, the asymptotic limit may fail to exist.

5. A stronger result. In our proof the condition of analyticity is used only in Part 1 to show that

$$\sum | a_{nk}^{(1)} | = o(n^{(1-p/q)/2}) , \quad n \to \infty$$

where the sum is for $k \in s(n)$. Here we outline a proof of this statement using only the smoothness conditions assumed in Hedstrom's paper.

We first remark that the proof of our theorem from Part 2 onward is not affected if we take $b_n = \tau \log n$ for any fixed $\tau \geqq 1 + (1 - p/q)/2$. Define

$$a_{nk}^{(3)} = \int_{\mathscr{I}_n} f^n(t) e^{-ikt} dt$$

where

$$\mathscr{I}_n = [-\lambda'_n n^{-1/q}, \lambda'_n n^{-1/q}] , \quad \lambda'_n = \left(\frac{2}{\beta}\right)^{1/q} (\omega' \log n)^{1/(p-1)}$$

$\omega' > (1/2)(1 + (p + 2)/q)$. For n sufficiently large

$$| a_{nk}^{(1)} - a_{nk}^{(3)} | \leqq 2\varepsilon_0 \left[\exp \left[-\frac{1}{2} n\beta(\lambda'_n n^{-1/q}) \right]^q \right]$$

and so

$$\sum | a_{nk}^{(1)} - a_{nk}^{(3)} | = \mathscr{O}[n^{1+1/q-\omega'}] = o(n^{(1-p/q)/2}) , \quad n \to \infty$$

where the sum is for $k \in S(n)$. In order to show that $\sum | a_{nk}^{(3)} | = o(n^{(1-p/q)/2})$, $n \to \infty$ we treat separately the cases $p \neq q$ and $p = q$.

In the case $p \neq q$ we fix a value for $\tau \geqq 1 + (1 - p/q)/2$ in $b_n = \tau \log n$ so that

$$pb(\lambda'_n)^{p-1}/b_n < 1/2 .$$

A change of variable followed by an application of Lemma 3 yields (as before)

$$\left| \int_{-\lambda'_n}^{\lambda'_n} \exp [ih(t) + G(t)] dt \right| \leqq \frac{M}{|\gamma|}$$

for some constant M; thus

$$\sum_{k \in S(n)} | a_{nk}^{(3)} | = \mathcal{O}\left[\sum \frac{1}{|\gamma|} n^{-1/q}\right] = \mathcal{O}[\log n] = o(n^{(1-p/q)/2}) , \quad n \to \infty .$$

If $p = q$, $a_{nk}^{(3)}$ takes the form

$$\int_{\mathcal{J}_n} [\exp i\mu t + ng(t)] dt .$$

Integration by parts twice, followed by majorization and use of the estimate

$$\operatorname*{Var}_{t \in \mathcal{J}_n} [ng'(t)e^{ng(t)}] = \mathcal{O}(n^{1/q}) , \quad n \to \infty$$

yields

$$\sum_{k \in S(n)} | a_{nk}^{(3)} | = \mathcal{O}\left(\frac{1}{\log n}\right) = o(1) , \quad n \to \infty .$$

Thus,

$$\sum | a_{nk}^{(1)} | \leqq \sum | a_{nk}^{(1)} - a_{nk}^{(3)} | + \sum | a_{nk}^{(3)} | = o(n^{(1-p/q)/2}) , \quad n \to \infty$$

in both cases.

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