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GAUSSIAN MARKOV EXPECTATIONS AND RELATED INTEGRAL EQUATIONS*

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Let $\{X(w), s \leqq w \leqq t\}$ be a Gaussian Markov stochastic process with continuous sample functions. Examples of such processes are the Wiener, Ornstein-Uhlenbeck, and Doob-Kac processes. An operator valued function space integral is defined for each process. This was done for the Wiener process by R. H. Cameron and D. A. Storvick. For functionals of the form $F(x) = \exp\left\{\int_s^t \theta(t-w, x(w))dw\right\}$ where $\theta(t, u)$ is bounded and almost everywhere continuous, the special integrals satisfy integral equations related to the generalized Schroedinger equations studied by the first author. For the Wiener process, a "backwards time" equation is coupled with the Cameron-Storvick equation to give a pair of integral equations.

In [12] R. H. Cameron and D. A. Storvick defined an operator valued function space integral based on the Wiener stochastic process. For an appropriate functional, such an integral solves an integral equation related to the Schroedinger equation. The purpose of this paper is to define such integrals for Gaussian Markov stochastic processes, and prove that for appropriate functionals they satisfy an integral equation related to the generalized Schroedinger equation discussed by the first author in [5], [6], [7], and [8]. Examples of Gaussian Markov processes are the Wiener, Ornstein-Uhlenbeck, and Doob-Kac processes. For the Wiener process we will give a "backwards time" equation which when coupled with the Cameron-Storvick "forwards time" equation will give a *pair* of integral equations. That a function space integral solves a pair of integral equations was first done in [14] by D. A. Darling and A. J. F. Siegert.

This area of research is motivated, in many respects, by R. P. Feynman's function space integral which he first discussed in 1948 [16]. Since then extensive work has been done to enlarge the class of functionals for which "Feynman integrals" exist. See, for example, the work of R. H. Cameron [9, 10, 11], Donald Babbitt [1, 2, 3, 4], Jacob Feldman [15], K. Itô [18, 19], Edward Nelson [22], and G. W. Johnson and D. L. Skoug [20, 21, 23]. In the papers by Cameron and Storvick [12, 13] the integral equation involved is related to the Schroedinger equation. A heuristic discussion of that relation is

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contained in the first author's paper [8].

2. Notation and definitions. Let $\{x(p), s \leqq p \leqq t\}$ be a Gaussian Markov process with mean function $m(p) = \xi v(p)/v(s)$, $s \leqq p \leqq t$ and covariance function

$$R(a, b) = \begin{cases} U(a)v(b) \ , & a \leqq b \\ U(b)v(a) \ , & b \leqq a \end{cases}$$

where $U(p) = u(p) - u(s)v(p)/v(s)$, $s \leqq p \leqq t$,

$$u(p) \geqq 0 \ , \quad v(p) > 0 \ , \quad s \leqq p \leqq t \tag{2.1}$$

$$u'(p) \text{ and } v'(p) \text{ continuous on } [s, t] \tag{2.2}$$

$$[v(p)u'(p) - u(p)v'(p)] > 0 \ , \quad s \leqq p \leqq t \ . \tag{2.3}$$

These processes have almost all sample functions continuous, and since $U(s) = 0$, $x(s) = \xi$ for almost all sample functions.

Let $C_\xi[s, t]$ be the set of continuous functions defined on $[s, t]$ with $x(s) = \xi$. The Gaussian Markov expectation of a functional $F[x]$ over $C_\xi[s, t]$ will be denoted by $\int_{C_\xi[s,t]} F[x]d_{m,R}x$. The "$m$" will be omitted if it is identically zero. We have the relation (see [5])

$$\int_{C_\xi[s,t]} F[x]d_{m,R}x = \int_{C[s,t]} F[y(\cdot) + \xi v(\cdot)/v(s)]d_R y \ . \tag{2.4}$$

Examples of the u and v functions are:

EXAMPLE 1. Wiener process: $u(p) = p$, $v(p) = 1$, $s \leqq p \leqq t$.

EXAMPLE 2. Doob-Kac process: $u(p) = p$, $v(p) = 1 - p$, $0 \leqq s \leqq p \leqq t < 1$.

EXAMPLE 3. Ornstein-Uhlenbeck family of processes: $u(p) = \sigma^2 e^{\alpha p}$, $v(p) = e^{-\alpha p}$, $\sigma^2 > 0$, $\alpha > 0$, $s \leqq p \leqq t$.

We will now introduce some ideas and notation from [12] extended to Gaussian Markov processes.

For $\lambda > 0$, let

$$\begin{aligned} &(I_\lambda(F)\psi)(\xi) \\ &\equiv \int_{C_0[s,t]} F(\lambda^{-1/2}x(\cdot) + \xi v(\cdot)/v(s))\psi(\lambda^{-1/2}x(t) + \xi v(t)/v(s))d_R x \end{aligned} \tag{2.5}$$

where F is a real or complex valued functional defined for all continuous functions on $[s, t]$ and ψ is a real or complex valued function defined almost everywhere on $(-\infty, \infty)$ and ξ is a real number;

and the λ, F, ψ, ξ are so chosen that the Gaussian Markov integral exists. Now assume that for a certain choice of the λ, F, ψ the integral (2.5) exists for all (or almost all) values of ξ in a set S of real numbers. Then

$$I_\lambda(F)\psi \tag{2.6}$$

denotes the function which maps ξ into $(I_\lambda(F)\psi)(\xi)$ for almost all values of ξ in S. Now assume that for a certain choice of λ, F, the function (2.6) exists for ψ in a class of functions D and belongs to a class of functions E. Then the operator

$$I_\lambda(F) \tag{2.7}$$

maps D into E. In this paper D and E will usually be $L_2(-\infty, \infty)$, so $I_\lambda(F)$ will usually be an operator that maps the Hilbert space L_2 into L_2. It can therefore be regarded as an operator valued function space integral.

If $F(x) = \exp\left\{\int_s^t \theta(p, x(p))dp\right\}$ and certain smoothness and order of growth conditions are placed on θ and ψ, the expression (2.5) (which now depends on s and ξ) is a solution of a partial differential equation (see [5]). In this case ψ is not necessarily in L_2.

We define

$$I_\lambda^{an}(F) \tag{2.8}$$

to be the operator valued analytic function of λ, if it exists, which agrees with $I_\lambda(F)$ for real λ and is analytic throughout $\operatorname{Re}\lambda > 0$.

We shall see that for $\operatorname{Re}\lambda > 0$, $I_\lambda(F)\psi = I_\lambda(K)\hat{\psi}$ where the expression on the right is given in terms of a Wiener integral, and the F and K, and ψ and $\hat{\psi}$ are suitably related.

To show the existence of $I_\lambda^{an}(F)$, $\operatorname{Re}\lambda > 0$, we shall follow the method of [12] and obtain $I_\lambda^{an}(F)$ as a weak limit of operators $I_\lambda^\sigma(K)$ which are defined in terms of finite dimensional integrals; thus

$$I_\lambda^{an}(F) \equiv w \lim_{|\sigma|\to 0} I_\lambda^\sigma(K) \tag{2.9}$$

where σ is a partition of $[s, t]$. See (0.8) of [12] for the definition of I_λ^σ.

Actually definition (2.9) can be made in terms of finite dimensional Gaussian Markov integrals. This would involve using the multivariate normal density

$$\lambda^{n/2}[(2\pi)^n A(a, t_1) \cdots A(t_{n-1}, t_n)]^{-1/2}$$
$$\times \exp\left\{-\sum_{j=1}^n \lambda[x_j - v(t_j)x_{j-1}/v(t_{j-1})]^2/(2A(t_{j-1}, t_j))\right\}$$

in the definition (0.8) of [12] where

$$A(t_{i-1}, t_i) = u(t_i)v(t_i) - u(t_{i-1})v^2(t_i)/v(t_{i-1})$$

and the u and v functions are subject to (2.1) through (2.3). However, our proofs will use $I_\lambda^{an}(F)$ as defined in (2.9).

Recently, Johnson and Skoug [21] have shown that the weak limit of (2.9) may be replaced by the strong limit.

Finally we shall use

$$J_p(F) \equiv w \lim_{\eta \to 0^+} I_{\eta - ip}^{an}(F) \tag{2.10}$$

to obtain the solution of the integral equation in the pure imaginary case—the Feynman case.

3. The integral equation for $\lambda > 0$.

THEOREM 1. *Let $\theta(t, u)$ be continuous almost everywhere in the strip R: $0 \leqq t \leqq t_0$, $-\infty < u < \infty$, and let $|\theta(t, u)| \leqq M$ for (t, u) in the strip. Let $\psi \in L_2(-\infty, \infty)$, let $\lambda > 0$, $t > s \geqq 0$, and ξ real. Let $G(s, t, \xi, \lambda)$ be defined by*

$$\begin{aligned} G(s, t, \xi, \lambda) \equiv \int_{C_0[s,t]} \exp\left\{\int_s^t \theta[t-\tau, \lambda^{-1/2}x(\tau) + \xi v(\tau)/v(s)]d\tau\right. \\ \times \psi[\lambda^{-1/2}x(t) + \xi v(t)/v(s)]d_{G.M.}x\,. \end{aligned} \tag{3.1}$$

Then $G(s, t, \xi, \lambda)$ satisfies the following integral equation.

$$\begin{aligned} G(s, t, \xi, \lambda) = {} & \lambda^{1/2}(2\pi A(s, t))^{-1/2}\int_{-\infty}^{\infty} \psi(x) \\ & \times \exp\{-\lambda[x - \xi v(t)/v(s)]^2/[2A(s, t)]\}dx \\ & + \lambda^{1/2}(2\pi)^{-1/2}\int_s^t [A(s, \tau)]^{-1/2}d\tau \int_{-\infty}^{\infty} \theta(t-\tau, x)G(\tau, t, x, \lambda) \\ & \times \exp\{-\lambda[x - \xi v(\tau)/v(s)]^2/[2A(s, \tau)]\}dx\,. \end{aligned} \tag{3.2}$$

Proof. Let $\theta^*(\tau, u) = \theta(t-\tau, u)$. Let $\tilde{\theta}(\tau, u) = \theta^*(\tau, \lambda^{-1/2}u)$ and $\tilde{\psi}(u) = \psi(\lambda^{-1/2}u)$. Then

$$\begin{aligned} G(s, t, \xi, \lambda) = \int_{C_0[s,t]} \exp\left\{\int_s^t \tilde{\theta}[\tau, x(\tau) + \lambda^{1/2}\xi v(\tau)/v(s)]d\tau\right\} \\ \times \tilde{\psi}[x(t) + \lambda^{1/2}\xi v(t)/v(s)]d_{G.M.}x = H(s, t, \lambda^{1/2}\xi) \end{aligned}$$

where $H(s, t, y) = G(s, t, y\lambda^{-1/2}, \lambda)$.

We now apply (3.3) and (3.6) of [5]. The hypotheses on $\tilde{\theta}$ required by Theorem 3 of [5] are not necessary as can be seen by consulting the Darling-Siegert paper [14]. The hypotheses of our present theorem are sufficient. Hence

$$\begin{aligned} H(s, t, y) = {} & (2\pi A(s, t))^{-1/2}\int_{-\infty}^{\infty} \tilde{\psi}(u) \exp\{-[u - yv(t)/v(s)]^2/[2A(s, t)]\}du \\ & + \int_s^t\int_{-\infty}^{\infty} \tilde{\theta}(\tau, \alpha)H(\tau, t, \alpha)[2\pi A(s, \tau)]^{-1/2} \\ & \times \exp\{-[\alpha - yv(\tau)/v(s)]^2/[2A(s, \tau)]\}d\alpha d\tau\ . \end{aligned}$$

We transform the first integral with the substitution $x = \lambda^{-1/2}u$ to obtain

$$\lambda^{1/2}(2\pi A(s, t))^{-1/2}\int_{-\infty}^{\infty} \psi(x) \exp\{-\lambda[x - \xi v(t)/v(s)]^2/(2A(s, t))\}dx\ .$$

We transform the second integral with the substitutions $H(\tau, t, \alpha) = G(\tau, t, \lambda^{-1/2}\alpha, \lambda)$ and $x = \lambda^{-1/2}\alpha$ to obtain

$$\begin{aligned} & \int_s^t\int_{-\infty}^{\infty} \theta(t - \tau, x)G(\tau, t, x, \lambda)\lambda^{1/2}[2\pi A(s, \lambda)]^{-1/2} \\ & \qquad \times \exp\{-\lambda[x - \xi v(\tau)/v(s)]^2/(2A(s, \tau))\}dx d\tau\ . \end{aligned}$$

This completes the derivation for $\lambda > 0$.

COROLLARY TO THEOREM 1. *Assume that $\theta, \psi, \lambda, t, s,$ and ξ are as in the theorem. For the Wiener process, $G(s, t, \xi, \lambda)$ satisfies a* pair *of integral equations*:

$$\begin{aligned} G(s, t, \xi, \lambda) = {} & \lambda^{1/2}(2\pi(t - s))^{-1/2}\int_{-\infty}^{\infty} \psi(u) \exp\{-\lambda(\xi - u)^2/(2(t - s))\}du \\ & + \lambda^{1/2}(2\pi)^{-1/2}\int_s^t (t - w)^{-1/2}dw\int_{-\infty}^{\infty} \theta(w - s, u)G(s, w, u, \lambda) \\ & \times \exp\{-\lambda(\xi - u)^2/(2(t - w))\}du\ , \end{aligned} \tag{3.3}$$

$$\begin{aligned} G(s, t, \xi, \lambda) = {} & \lambda^{1/2}(2\pi(t - s))^{-1/2}\int_{-\infty}^{\infty} \psi(u) \exp\{-\lambda(\xi - u)^2/(2(t - s))\}du \\ & + \lambda^{1/2}(2\pi)^{-1/2}\int_s^t (w - s)^{-1/2}dw\int_{-\infty}^{\infty} \theta(t - w, u)G(w, t, u, \lambda) \\ & \times \exp\{-\lambda(\xi - u)^2/(2(w - s))\}du\ . \end{aligned} \tag{3.4}$$

To prove this, we will need the following lemma.

LEMMA. *Let*

$$\begin{aligned} p(w, a; z, b) = \frac{\partial}{\partial z}P[X(b) \leqq z \mid X(a) = w] = {} & [2\pi A(a, b)]^{-1/2} \\ & \times \exp\left\{-\frac{[z - wv(b)/v(a)]^2}{2A(a, b)}\right\} \end{aligned}$$

be the transition density function for the process. Assume that it is *stationary*; i.e. *assume that* $A(a, b) = A(a + h, b + h)$ *and* $v(b)/v(a) = v(b + h)/v(a + h)$ *for* $h > 0$. *Then*

$$\begin{aligned}&\int_{C_\zeta[t-p,t]} \exp\left\{\int_{t-p}^{t} \theta[t - \tau, \lambda^{-1/2}x(\tau) + \eta v(\tau)/v(t - p)]d\tau\right\}\\ &\quad \times \psi[\lambda^{-1/2}x(t) + \eta v(t)/v(t - p)]d_{G.M.}x\\ &\overset{e}{=} \int_{C_\zeta[0,p]} \exp\left\{\int_0^p \theta[p - \tau, \lambda^{-1/2}x(\tau) + \eta v(\tau)/v(0)]d\tau\right\}\\ &\quad \times \psi[\lambda^{-1/2}x(p) + \eta v(p)/v(0)]d_{G.M.}x\end{aligned}$$

where $\overset{e}{=}$ means if one side exists so does the other and they are equal.

REMARK. It is easy to verify that the transition density functions for the Wiener and Ornstein-Uhlenbeck processes are stationary. This is not true for the Doob-Kac process.

Proof of lemma. Assume that the left hand side exists. Call it $G(t - p, t, \eta, \lambda)$. Then by Lemma 2 of [5],

$$\begin{aligned}G(t - p, t, \eta, \lambda) = \int_{C\eta[t-p,t]} &\exp\left\{\int_{t-p}^{t} \theta[t - \tau, \lambda^{-1/2}x(\tau)]d\tau\right\}\\ &\times \psi[\lambda^{-1/2}x(t)]d_{G.M.}x\ .\end{aligned}$$

Using a definition from [8] and a mild extension of Theorem 1 of [7],

$$\begin{aligned}G(t - p, t, \eta, \lambda) = \lim_{\|\tau\|\to 0}\int_{R_n} &\prod_{i=1}^{n} p^*(\xi_{i-1}, \tau_{i-1}; \xi_i, \tau_i)\\ &\times \exp\int_{t-p}^{t} \theta[t - \tau, \lambda^{-1/2}\Gamma_{\tau,\vec{\xi}}]d\tau\Big\}\psi[\lambda^{-1/2}\xi_n]d\vec{\xi}\end{aligned}$$

where $\tau_0 \equiv t - p < \tau_1 < \tau_2 < \cdots < \tau_n = t$, $\xi_0 = \eta$, $\Gamma_{\tau,\xi}(\tau_i) = \xi_i$, $i = 0, 1, \cdots, n$, and $\Gamma_{\tau,\xi}$ is linear on each $[\tau_{i-1}, \tau_i]$; also $p^*(w, a; z, b)$ equals $p(w, a; z, b)$ with $A(a, b)$ replaced by $A(a, b)/\lambda$.

From the hypotheses on $A(a, b)$ and $v(b)/v(a)$ we have

$$\begin{aligned}&p^*(\xi_{i-1}, \tau_{i-1}; \xi_i, \tau_i)\\ &= [2\pi A(\tau_{i-1}, \tau_i)/\lambda]^{-1/2} \exp\{-\lambda[\xi_i - \xi_{i-1}v(\tau_i)/v(\tau_{i-1})]^2/(2A(\tau_{i-1}, \tau_i))\}\\ &= p^*(\xi_{i-1}, \tau_{i-1} - (t - p); \xi_i, \tau_i - (t - p)).\end{aligned}$$

Also, as in [7],

$$\begin{aligned}\|\tau\| &= \max_{j=1,\cdots,n} (\tau_j - \tau_{j-1})\\ &= \max_{j=1,\cdots,n} ([\tau_j - (t - p)] - [\tau_{j-1} - (t - p)]) = \|\tau - (t - p)\|\ .\end{aligned}$$

Thus

$$\begin{aligned}
&G(t-p, t, \eta, \lambda) \\
&= \lim_{\|\tau-(t-p)\|\to 0} \int_{R_n} \prod_{i=1}^{n} p^*(\xi_{i-1}, \tau_{i-1}-(t-p); \xi_i, \tau_i-(t-p)) \\
&\quad \times \exp\left\{\int_{t-p}^{t} \theta[t-\tau, \lambda^{-1/2}\Gamma_{\tau,\vec{\xi}}]d\tau\right\}\psi[\lambda^{-1/2}\xi_n]d\vec{\xi} .
\end{aligned}$$

Let $\delta_i = \tau_i - (t-p)$, $i = 0, 1, \cdots, n$. Then

$$\begin{aligned}
&G(t-p, t, \eta, \lambda) \\
&= \lim_{\|\delta\|\to 0} \int_{R_n} \prod_{i=1}^{n} p^*(\xi_{i-1}, \delta_{i-1}; \xi_i, \delta_i) \exp\left\{\int_0^p \theta[p-\delta, \lambda^{-1/2}\Gamma_{\delta,\vec{\xi}}]d\delta\right\}\psi[\lambda^{-1/2}\xi_n]d\vec{\xi} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{since } \Gamma_{\delta+t-p} = \Gamma_\delta) \\
&= \int_{C_\eta[0,p]} \exp\left\{\int_0^p \theta[p-\delta, \lambda^{-1/2}x(\delta)]d\delta\right\}\psi[\lambda^{-1/2}x(p)]d_{G.M.}x \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{by Theorem 1 of [7]}) \\
&= G(0, p, \eta, \lambda) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by Lemma 2 of [5]}.
\end{aligned}$$

By assuming that the right side exists, the proof follows in reverse order. Proof of Corollary: Since the Wiener transition density function is stationary, by the Lemma

$$\begin{aligned}
G(s, t, \xi, \lambda) = \int_{C_w[0,t-s]} &\exp\left\{\int_0^{t-s} \theta[t-s-\tau, \lambda^{-1/2}x(\tau)+\xi]d\tau\right\} \\
&\times \psi[\lambda^{-1/2}x(t-s)+\xi]d_w x = G(0, t-s, \xi, \lambda) .
\end{aligned}$$

Hence by (9.1) of [12]

$$\begin{aligned}
&G(0, t-s, \xi, \lambda) \\
&= \lambda^{1/2}(2\pi(t-s))^{-1/2}\int_{-\infty}^{\infty} \psi(u) \exp\left[-\lambda(\xi-u)^2/(2(t-s)\right)du \\
&\quad + \lambda^{1/2}(2\pi)^{-1/2}\int_0^{t-s}(t-s-p)^{-1/2}dp\int_{-\infty}^{\infty}\theta(p, u)G(0, p, u, \lambda) \\
&\quad \times \exp\{-\lambda(\xi-u)^2/(2(t-s-p))\}du .
\end{aligned}$$

Now let $w - s = p$ and note that $G(0, w-s, u, \lambda) = G(s, w, u, \lambda)$ by the Lemma. Hence the second term becomes

$$\begin{aligned}
\lambda^{1/2}(2\pi)^{-1/2}&\int_s^t (t-w)^{-1/2}dw\int_{-\infty}^{\infty}\theta(w-s, u)G(s, w, u, \lambda) \\
&\times \exp\{-\lambda(\xi-u)^2/(2(t-w))\}du .
\end{aligned}$$

Thus (3.3) is verified.

Integral equation (3.4) is obtained from Theorem 1 since $v(\tau) = 1$ and $A(s, t) = t - s$ in the Wiener case.

REMARK. The concept of a pair of integral equations representing forward and backward time equations was suggested by the Darling-Siegert paper [14].

4. The analyticity of G and the integral equation for $Re\lambda > 0$.

THEOREM 2. *Under the assumptions of Theorem* 1, $G(s, t, \xi, \lambda)$ *has an analytic extension to* $Re\lambda > 0$ *and this extension satisfies the integral equation* (3.2). *Furthermore this analytic extension satisfies*

$$\|G(s, t, \cdot, \lambda)\| \leqq \|\psi\| (v(s)/v(t))^{1/2} \exp [MU(t_0)/v(t_0)] \; . \tag{4.1}$$

Proof. First we show G has the analytic extension. By a formula in [5, p. 792] we can write

$$\begin{aligned} &G(s, t, \xi, \lambda) \\ &= \int_{C_0[0,U(t)/v(t)]} \exp\left\{\int_s^t \theta(t-\tau, \lambda^{-1/2}v(\tau)x(U(\tau)/v(\tau)) + \xi v(\tau)/v(s))d\tau\right\} \\ &\quad \times \psi[\lambda^{-1/2}v(t)x(U(t)/v(t)) + \xi v(t)/v(s)]d_w x \end{aligned}$$

where the right hand integral is a Wiener integral.

Let $\alpha(\tau) \equiv U(\tau)/v(\tau)$. Our hypotheses on u, v (see § 2) insure that α' is positive and continuous on $[s, t]$. Thus there exists C such that $0 < 1/\alpha'(t) \leqq C$ on $[s, t]$. Next we transform the inner integral in $G(s, t, \xi, \lambda)$ by $\tau' = \alpha(\tau)$. Then

$$\begin{aligned} &G(s, t, \xi, \lambda) \\ &= \int_{C[0,\alpha(t)]} \exp\left\{\int_0^{\alpha(t)} \theta(t-\alpha^{-1}(\tau'), \lambda^{-1/2}v(\alpha^{-1}(\tau'))x(\tau') + \xi v(\alpha^{-1}(\tau'))/v(s)) \right. \\ &\quad \left. \times \frac{d}{d\tau'}(\alpha^{-1}(\tau'))d\tau'\right\}\psi[\lambda^{-1/2}v(t)x(\alpha(t)) + \xi v(t)/v(s)]d_w x \\ &= \int_{C[0,t']} \exp\left\{\int_0^{t'} \hat{\theta}(\tau', \lambda^{-1/2}x(\tau') + \xi/v(s)d\tau'\right\}\hat{\psi}[\lambda^{-1/2}x(\tau') + \xi/v(s)]d_w x \end{aligned}$$

where $t' = \alpha(t)$, $\hat{\theta}(\tau', u) = \theta(t - \alpha^{-1}(\tau'), uv(\alpha^{-1}(\tau')))(d/d\tau')(\alpha^{-1}(\tau'))$, $\hat{\psi}(u) = \psi(v(t)u)$. Since $1/\alpha'(\tau) \leqq C$ on $[s, t]$ and v, α are continuous, there exists M such that $|\hat{\theta}(\tau, u)| \leqq M$, and $\hat{\theta}$ is continuous on $[0, t'] \otimes (-\infty, \infty)$. Let

$$K(x) = \exp\left\{\int_0^{t'} \hat{\theta}(\tau', x(\tau'))d\tau'\right\} . \tag{4.2}$$

Let I_λ, I_λ^{an} be as in (2.7), (2.8) for the Wiener process with $v(\tau) \equiv 1$. Then $I_\lambda K\hat{\psi}(\xi/v(s)) = G(s, t, \xi, \lambda)$. Then because of the hypotheses on $\hat{\theta}$ and since $\hat{\psi} \in L_2$, by Theorem 4 of [12], $I_\lambda^{an}K = I_\lambda^{seq}K$ is an analytic extension of $I_\lambda K$; thus G has an analytic extension

$$G(s, t, \xi, \lambda) = I_\lambda^{an}K\hat{\psi}(\xi/v(s)) = I_\lambda^{an}K\psi(\xi v(t)/v(s)) \; . \tag{4.3}$$

Furthermore by Theorem 4 of [12] (see also line 19, p. 542 of [12])

$\| G(t, \cdot, \lambda) \| \leqq \| \psi((\cdot)v(t)/v(s)) \| \exp (M\alpha(t_0))$. But

$$\begin{aligned}\| \psi((\cdot)v(t)/v(s)) \|^2 &= \int_{-\infty}^{\infty} | \psi^2(\xi v(t)/v(s)) | d\xi \\ &= (v(s)/v(t)) \int_{-\infty}^{\infty} | \psi^2(\xi) | d\xi = (v(s)/v(t)) \| \psi \|^2 .\end{aligned}$$

Thus (4.1) holds.

Finally, Morera's Theorem can be used to show the right hand side of the integral equation (3.2) is analytic. Thus (3.2) is valid for $0 \leqq s < t \leqq t_0$, $\operatorname{Re} \lambda > 0$, ξ real.

Next, we note that as in the preceding proof, we can extend the Corollary to Theorem 1. We embody this remark in a

COROLLARY TO THEOREM 2. *Under the assumptions of Corollary to Theorem* 1, *G has an analytic extension to* $\operatorname{Re} \lambda > 0$ *which then satisfies equations* (3.3) *and* (3.4).

5. The integral equation for $\operatorname{Re} \lambda = 0$—The Feynman case.

THEOREM 3. *Let* $\theta(t, u)$ *be continuous almost everywhere in the strip* R: $0 \leqq t \leqq t_0$, $-\infty < u < \infty$, $| \theta(t, u) | \leqq M$ *for* $(t, u) \in R$. *Let* $\psi \in L_2(-\infty, \infty)$; *then* $\Gamma(s, t, \cdot, q) \equiv w \lim_{\eta \to 0^+} G(s, t, \cdot, \eta - iq)$ *exists for* $(t, \xi) \in R$, $t > s \geqq 0$, *and almost all real* q. *Then for each* $s \in [0, t)$ *and almost every real* q,

$$\begin{aligned}&\Gamma(s, t, \xi, q) \\ &= \underset{\substack{B \to \infty \\ \xi}}{\text{l.i.m.}}\, q^{1/2}(2\pi i A(s, t))^{-1/2} \int_{-B}^{B} \psi(x) \exp \{iq(x - \xi v(t)/v(s))^2/(2A(s, t))\} dx \\ &\quad + \underset{\substack{B \to \infty \\ \xi}}{\text{l.i.m.}}\, q^{1/2}(2\pi i)^{-1/2} \int_s^t (A(s, \tau))^{-1/2} d\tau \\ &\quad \times \int_{-B}^{B} \theta(t - \tau, x) \Gamma(\tau, t, x, q) \exp \{iq(x - \xi v(\tau)/v(s))^2/(2A(s, \tau))\} dx .\end{aligned} \tag{5.1}$$

Proof. Let θ and ψ satisfy the hypotheses. Then

$$w \lim_{\eta \to 0^+} G(s, t, \cdot, \eta - iq)$$

exists for $(t, \xi) \in R$, $0 \leqq s < t$ and almost all real q. To see this observe that from (4.3) $G(s, t, \xi, \lambda) = I_\lambda^{an} K\psi(\xi v(t)/v(s))$ where K is given by (4.2). Now from our hypotheses on θ and by Theorem 5 of [12, p. 534], $w \lim_{\lambda \to -iq} I_\lambda^{an} K\psi$ exists, for almost all q, with limit denoted $J_q K\psi$. Since $v(t)/v(s)$ is bounded away from zero our conclusion follows. This weak limit, denoted $\Gamma(s, t, \cdot, q)$ can be chosen to be measurable.

Next let $h(\tau, x) = \theta(t - \tau, x)G(\tau, t, x, \lambda)$ and

$$g(\tau, \xi) = \lambda^{1/2}(2\pi A(s, \tau))^{-1/2}\int_{-\infty}^{\infty} h(\tau, x) \times \exp\{-\lambda(x - \xi v(\tau)/v(s))^2/(2A(s, \tau))\}dx .$$

Since $|\theta(t - \tau, x)| < M$ and using the bound on $\|G(\tau, t, \cdot, \lambda)\|$ given by (4.1) means there exists B such that

$$\|h(\tau, \cdot)\| \leqq B < \infty \quad \text{for all} \quad \tau \in [s, t] . \tag{5.2}$$

By Lemma 1 of [12, p. 522], $\|g(\tau, (\cdot)v(s)/v(\tau)\| \leqq \|h(\tau, \cdot)\| \leqq B$. Thus $\|g(\tau, \cdot)\| \leqq |v(s)/v(\tau)|^{1/2}B \leqq B'$ for some B' since $v(\tau)$ is positive and continuous on $[s, t]$. Then $\left\|\int_s^t g(\tau, \cdot)d\tau\right\| \leqq B'(t - s)$. Thus if $\varphi \in L_2$, $\int_{-\infty}^{\infty}\left|\varphi(\xi)\int_s^t g(\tau, \xi)d\tau\right|d\xi \leqq \|\varphi\|B'(t - s)$ so by Fubini's Theorem

$$\int_{-\infty}^{\infty}\varphi(\xi)\int_s^t g(\tau, \xi)d\tau d\xi = \int_s^t\int_{-\infty}^{\infty}\varphi(\xi)g(\tau, \xi)d\xi d\tau .$$

Now $\int_{-\infty}^{\infty}|\varphi(\xi)g(\tau, \xi)|d\xi \leqq \|\varphi\|\cdot\|g(\tau, \cdot)\| \leqq \|\varphi\|B'$ so by Fubini's Theorem

$$\begin{aligned}\int_{-\infty}^{\infty}\varphi(\xi)g(\tau, \xi)d\xi &= \int_{-\infty}^{\infty}\varphi(\xi)\lambda^{1/2}(2\pi A(s, \tau))^{-1/2}\int_{-\infty}^{\infty}h(\tau, x)\\ &\qquad \times \exp\{-\lambda(x - \xi v(\tau)/v(s))^2/(2A(s, \tau))\}dx d\xi\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(\xi)\lambda^{1/2}(2\pi A(s, \tau))^{-1/2}h(\tau, x)\\ &\qquad \times \exp\{-\lambda(x - \xi(v(\tau)/v(s))^2/(2A(s, \tau))\}d\xi dx .\end{aligned} \tag{5.3}$$

Thus

$$\begin{aligned}\int_{-\infty}^{\infty}\varphi(\xi)d\xi\int_s^t g(\tau, \xi)d\tau &= \int_s^t \lambda^{1/2}(2\pi A(s, \tau))^{-1/2}\int_{-\infty}^{\infty}h(\tau, x)\int_{-\infty}^{\infty}\varphi(\xi)\\ &\qquad \times \exp[-\Lambda(x - \xi v(s)/v(\tau))^2/2]d\xi dx d\tau\end{aligned} \tag{5.4}$$

where $\Lambda = \lambda/A(s, \tau)$.

Now let H_n be the nth degree Hermite polynomial. Then

$$\begin{aligned}&\int_{-\infty}^{\infty}H_n(\xi)\exp(-\xi^2/2)d\xi\int_s^t g(\tau, \xi)d\tau\\ &= \int_s^t \lambda^{1/2}(2\pi A(s, \tau))^{-1/2}d\tau\int_{-\infty}^{\infty}h(\tau, x)\tilde{\Phi}_n(x, \Lambda)dx\end{aligned}$$

where $\tilde{\Phi}_n(x, \Lambda) = \int_{-\infty}^{\infty}H_n(\xi)\exp(-\xi^2/2)\exp[-\Lambda(\xi v(\tau)/v(s) - x)^2/2]d\xi$. Now $\tilde{\Phi}_n(x, \Lambda) = \Phi_n(xv(s)/v(\tau), \Lambda(v(\tau)/v(s))^2)$ where

$$\Phi_n(n, \Lambda) = \int_{-\infty}^{\infty}H_n(\xi)\exp(-\xi^2/2)\exp(-\Lambda(\xi - u)^2/2)d\xi .$$

In [12, p. 547] it was shown

$$\| \Phi_n(\cdot, \Lambda) \| \leqq (2\pi)^{1/2} |\Lambda|^{-1/2} \| H_n(\cdot) \exp[-(\cdot)^2] \| ;$$

thus $\| \tilde{\Phi}_n(\cdot, \Lambda) \| \leqq (2\pi v(\tau)/(|\Lambda| v(s)))^{1/2} \| H_n(\cdot) \exp[-(\cdot)^2] \|$.

Now the proof in [12, p. 548] can be generalized, slightly, to show that

$$\underset{\substack{\lambda \to -iq \\ \mathrm{Re}\lambda > 0}}{\mathrm{l.i.m.}} \Phi_n(\cdot, \lambda/c) = \Phi_n(\cdot, -iq/c)$$

where c is independent of λ and $c > 0$. Thus

$$\underset{\substack{\lambda \to -iq \\ \mathrm{Re}\lambda > 0}}{\mathrm{l.i.m.}} \Phi_n(\cdot, \lambda v(\tau)/(v(s)A(s, \tau))) = \Phi_n(\cdot, -iqv(\tau)/(v(s)A(s, \tau))) .$$

But then

$$\begin{aligned} &\| \tilde{\Phi}_n(\cdot, \lambda/A(s, \tau)) - \tilde{\Phi}_n(\cdot, -iq/A(s, \tau)) \| \\ &= (v(\tau)/v(s))^{1/2} \| \Phi_n(\cdot, \lambda v(\tau)/(v(s)A(s, \tau))) \\ &\quad - \Phi_n(\cdot, -iqv(\tau)/(v(s)A(s, \tau))) \| \to 0 \end{aligned}$$

as $\lambda \to -iq^+$. Thus

$$\underset{\lambda \to -iq^+}{\mathrm{l.i.m.}} \tilde{\Phi}_n(\cdot, \lambda/A(s, \tau)) = \tilde{\Phi}_n(\cdot, -iq/A(s, \tau)) . \tag{5.5}$$

But $|\theta| \leqq M$ and $\| h(\tau, \cdot) \| \leqq B$ on $[s, t]$ by (5.2) so

$$\begin{aligned} |\mathscr{I}_1| &\equiv \left| \int_s^t (A(s, \tau))^{-1/2} d\tau \int_{-\infty}^{\infty} \theta(t - \tau, x) G((\tau, t, x, \lambda)[\tilde{\Phi}_n(x, \lambda/A(s, \tau)) \right. \\ &\quad \left. - \tilde{\Phi}_n(x, -iq/A(s, \tau))] dx \right| \\ &\leqq \int_s^t (A(s, \tau))^{-1/2} B \| \tilde{\Phi}_n(x, \lambda/A(s, \tau)) - \tilde{\Phi}_n(, x - iq/A(s, \tau) \| d\tau . \end{aligned}$$

By (5.5) the limit of the integrand is zero. Also, from above,

$$\| \tilde{\Phi}_n(\cdot, \lambda/A(s, \tau)) \| \leqq [2\pi A(s, \tau) v(\tau)/(|\lambda| v(s))]^{1/2} \| H_n(\cdot) \exp[-(\cdot)^2] \|$$

so a bound for this integrand is

$$B \| H_n(\cdot) \exp[-(\cdot)^2] \| (2\pi v(\tau)/v(s))^{1/2} [|\lambda|^{-1/2} + |q|^{-1/2}] \leqq C$$

where C is chosen such that the bound holds uniformly for all $\tau \in [s, t]$. Such a C exists since $v(\tau)$ is continuous and, we may assume, $|\lambda| \geqq |q|/2$. Thus by bounded convergence, $\mathscr{I}_1 \to 0$ as $\lambda \to -iq$.

Now $w \lim_{\lambda \to -iq} \theta(t - \tau, \cdot) G(\tau, t, \cdot, \lambda) = \theta(t - \tau, \cdot) \Gamma(\tau, t, \cdot, q)$ and $\tilde{\Phi}_n(\cdot, -iq/A(s, \tau)) \in L_2$ so

$$\int_{-\infty}^{\infty}\tilde{\Phi}_n(x, -iq/A(s, \tau))\theta(t-\tau, x)[G(\tau, t, x, \lambda) - \Gamma(\tau, t, x, q)]dx \to 0$$

as $\lambda \to -iq^+$. Now as a function of τ, this is bounded by

$$2B(2\pi A(s, \tau)/|q|)^{1/2}\,||H_n(\cdot)\exp[-(\cdot)^2]||\,.$$

Thus by bounded convergence

$$\begin{aligned}\mathscr{I}_2 \equiv \int_s^t (A(s, \tau))^{-1/2}d\tau\int_{-\infty}^{\infty}\theta(t-\tau, x)\tilde{\Phi}_n(x, -iq/A(s, \tau))[G(\tau, t, x, \lambda)\\ - \Gamma(\tau, t, x, q)]dx \to 0\end{aligned}$$

as $\lambda \to -iq^+$.

Thus $\mathscr{I}_1 + \mathscr{I}_2 \to 0$ as $\lambda \to -iq^+$ and using (5.4) we have

$$\begin{aligned} J_1 &\equiv \lim_{\lambda\to -iq^+}\int_{-\infty}^{\infty}H_n(\xi)\exp(-\xi^2/2)d\xi\int_s^t[\lambda/(2\pi A(s, \tau))]^{1/2}d\tau\\ &\quad\times\int_{-\infty}^{\infty}\theta(t-\tau, x)G(\tau, t, x, \lambda)\exp[-A(x-\xi v(s)/v(\tau))^2/2]dx\\ &= \lim_{\lambda\to -iq^+}\int_s^t[\lambda/(2\pi A(s, \tau))]^{1/2}d\tau\int_{-\infty}^{\infty}\theta(t-\tau, x)G(\tau, t, x, \lambda)\tilde{\Phi}_n(x, \lambda/A(s, \tau))dx\\ &= \int_s^t[-iq/(2\pi A(s, \tau))]^{1/2}d\tau\int_{-\infty}^{\infty}\theta(t-\tau, x)\Gamma(\tau, t, x, q)\tilde{\Phi}_n(x, -iq/A(s, \tau))dx\\ &= \int_s^t[-iq/(2\pi A(s, \tau))]^{1/2}d\tau\int_{-\infty}^{\infty}\theta(t-\tau, x)\Gamma(\tau, t, x, q)\int_{-\infty}^{\infty}H_n(\xi)\exp(-\xi^2/2)\\ &\quad\times\exp[iq(\xi v(\tau)/v(s) - x)^2/(2A(s, \tau))]d\xi dx\,. \end{aligned}$$

Then, using the notation ${}^{(\xi)}\!\int_{-\infty}^{\infty}f(u, \xi)du \equiv \underset{(\xi)}{\mathrm{l.i.m.}}_{B\to+\infty}\int_{-B}^{B}f(u, \xi)du$, a slight generalization of Lemma 10 of [12, p. 542] shows the expression becomes

$$\begin{aligned}(5.6)\quad J_1 = \int_s^t(q/(2\pi iA(s, \tau)))^{1/2}d\tau\int_{-\infty}^{\infty}H_n(\xi)\exp[-\xi^2/2]d\xi\,{}^{(\xi)}\!\int_{-\infty}^{\infty}\theta(t-\tau, x)\\ \times\Gamma(\tau, t, x, q)\exp[iq(x-\xi v(\tau)/v(s))^2/(2A(s, \tau))]dx\,.\end{aligned}$$

Since $w\lim_{\lambda\to -iq}G(\tau, t, \cdot, \lambda) = \Gamma(\tau, t, \cdot, q)$, we have

$$||\Gamma|| \leqq \liminf_{\lambda\to -iq}||G(\tau, t, \cdot, \lambda)|| \leqq C$$

where C is the bound on $||G||$ given in (4.1). Then by Lemma 1 of [12, p. 522], $\Gamma(\tau, t, \cdot, q)\in L_2$, $|\theta| \leqq M$ imply

$$\begin{aligned}&\left\|{}^{(\cdot)}\!\int_{-\infty}^{\infty}\theta(t-\tau, x)\Gamma(\tau, t, x, q)\exp[iq(x-(\cdot)v(\tau)/v(s))^2/(2A(s, \tau))]dx\right\|\\ &\leqq ||\Gamma(\tau, t, \cdot, q)||\,M(A(s, \tau)v(s)/(2\pi v(\tau))^{1/2} \leqq B''(A(s, \tau))^{1/2}\end{aligned}$$

where B'' is chosen to be independent of τ since $v(\tau)$ is positive and continuous on $[s, t]$. Thus by Schwarz's Lemma, since $H_n(\cdot)\exp[-(\cdot)^2/2] \in L_2$,

$$\left|\int_{-\infty}^{\infty} H_n(\xi)\exp[-\xi^2/2]d\xi \overset{(\xi)}{\ } \int_{-\infty}^{\infty} \theta(t-\tau, x)\Gamma(\tau, t, x, q) \times \exp[iq(x-\xi v(\tau)/v(s))^2/(2A(s,\tau))]dx\right|$$
$$\leqq \| H_n(\cdot)\exp[-(\cdot)^2/2] \| B''(A(s,\tau))^{1/2} = B'''(A(s,\tau))^{1/2}$$

for the appropriate B'''. Thus the integrand for J_1 in (5.6) is bounded by $(|q|/(2\pi))^{1/2}B'''$ for $s \leqq \tau \leqq t$, $-\infty < \xi < \infty$. Thus by Fubini's Theorem

$$J_1 = (q/2\pi i)^{1/2}\int_{-\infty}^{\infty} H_n(\xi)\exp[-\xi^2/2]d\xi\int_s^t (A(s,\tau))^{-1/2}d\tau \overset{(\xi)}{\ } \int_{-\infty}^{\infty}\theta(t-\tau, x)$$
$$\times \Gamma(\tau, t, x, q)\exp[iq(x-\xi v(\tau)/v(s))^2/2A(s,\tau)]dx\ .$$

Next we consider

$$J_2 \equiv \lim_{\lambda\to -iq^+}\int_{-\infty}^{\infty} H_n(\xi)\exp[-\xi^2/2]d\xi(\lambda/2\pi A(s,t)))^{1/2}\int_{-\infty}^{\infty}\psi(x)$$
$$\times \exp\{-\lambda[x-\xi v(t)/v(s)]^2/(2A(s,t))\}dx\ .$$

As in (5.3) above we can interchange integration so

$$J_2 = \lim_{\lambda\to -iq^+}(\lambda/(2\pi A(s,t)))^{1/2}\int_{-\infty}^{\infty}\psi(x)dx\int_{-\infty}^{\infty} H_n(\xi)\exp[-\xi^2/2]$$
$$\times \exp\{-\lambda[x-\xi v(t)/v(s)]^2/(2A(s,t))\}d\xi\ .$$

Since $H_n(\xi)\exp[-\xi^2/2] \in L_1$, by dominated convergence

$$\lim_{\lambda\to -iq^+}\int_{-\infty}^{\infty} H_n(\xi)\exp[-\xi^2/2]\exp\{-\lambda(x-\xi v(t)/v(s))^2/(2A(s,t))\}d\xi$$
$$= \int_{-\infty}^{\infty} H_n(\xi)\exp[-\xi^2/2]\exp\{iq(x-\xi v(t)/v(s))^2/(2A(s,t))\}d\xi\ .$$

By Lemma 1 of [12], the L_2 norm of the left integrand as a function of ξ is bounded uniformly in λ (assume $|\lambda| > |q|/2$) so by 13.44 of [17] we have the weak limit

$$J_2 = (q/(2\pi i A(s,t)))^{1/2}\int_{-\infty}^{\infty}\psi(x)dx\int_{-\infty}^{\infty} H_n(\xi)\exp[-\xi^2/2]$$
$$\times \exp\{iq(x-\xi v(t)/v(s))^2/(2A(s,t))\}d\xi\ .$$

As with J_1 in (5.6) we have

$$J_2 = (q/(2\pi i A(s,t)))^{1/2}\int_{-\infty}^{\infty} H_n(\xi)\exp[-\xi^2/2]d\xi \overset{(\xi)}{\ } \int_{-\infty}^{\infty}\psi(x)$$
$$\times \exp\{iq(x-\xi v(t)/v(s))^2/(2A(s,t))\}dx\ .$$

Finally, multiply both sides of (3.2), valid for $\operatorname{Re}\lambda > 0$ by Theorem 2, by $H_n(\xi)\exp[-\xi^2/2]$ and integrate with respect to ξ. Taking limits of both sides as $\lambda \to -iq^+$ gives

$$\lim_{\lambda\to -iq^+}\int_{-\infty}^{\infty} H_n(\xi)\exp[-\xi^2/2]G(s, t, \xi, \lambda)d\xi = J_1 + J_2$$

or using the definition of Γ,

$$\begin{aligned}
&\int_{-\infty}^{\infty} H_n(\xi)\exp[-\xi^2/2]\Gamma(s, t, \xi, q)d\xi \\
&= \int_{-\infty}^{\infty} H_n(\xi)\exp[-\xi^2/2]d\xi\Big\{(q/(2\pi i A(s, t)))^{1/2} \\
&\quad\times \overset{(\xi)}{\int_{-\infty}^{\infty}} \psi(x)\exp[iq(x - \xi v(t)/v(s))^2/(2A(s, t))]dx \\
&\quad+ \int_s^t (q/(2\pi i A(s, \tau)))^{1/2} \overset{(\xi)}{\int_{-\infty}^{\infty}} \theta(t - \tau, x)\Gamma(\tau, t, x, q) \\
&\quad\times \exp[iq(x - \xi v(\tau)/v(s))^2/(2A(s, \tau))]dx\Big\}
\end{aligned}$$

for almost all q. Since the $H_n(\xi)\exp[-\xi^2/2]$ span L_2, the desired equation (5.1) results.

As in the preceding proof we can extend the Corollaries to Theorems 1 and 2 to obtain a

COROLLARY TO THEOREM 3. *Assume that θ, ψ, q, t, s and ξ are as in Theorem 3. For the Wiener process, $\Gamma(s, t, \xi, q)$ satisfies a pair of integral equations*:

$$\begin{aligned}
&\Gamma(s, t, \xi, q) \\
&= \underset{\substack{A\to\infty \\ (\xi)}}{\text{l.i.m.}}\ (q/(2\pi i(t - s)))^{1/2}\int_{-A}^{A} \psi(x)\exp[iq(x - \xi)^2/(2(t - s))]dx \\
&\quad+ \underset{\substack{A\to\infty \\ (\xi)}}{\text{l.i.m.}}\ (q/(2\pi i))^{1/2}\int_s^t (t - w)^{-1/2}dw\int_{-A}^{A} \theta(w - s, x)\Gamma(s, w, x, q) \\
&\quad\times \exp[iq(x - \xi)^2/(2(t - w))]dx\,,
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
&\Gamma(s, t, \xi, q) \\
&= \underset{\substack{A\to\infty \\ (\xi)}}{\text{l.i.m.}}\ (q/(2\pi i(t - s)))^{1/2}\int_{-A}^{A} \psi(x)\exp[iq(x - \xi)^2/(2(t - s))]dx \\
&\quad+ \underset{\substack{A\to\infty \\ (\xi)}}{\text{l.i.m.}}\ (q/(2\pi i))^{1/2}\int_s^t (w - s)^{-1/2}dw\int_{-A}^{A} \theta(t - w, x)\Gamma(w, t, x, q) \\
&\quad\times \exp[iq(x - \xi)^2/(2(w - s))]dx\,.
\end{aligned} \tag{5.8}$$

REMARK. In [13] Cameron and Storvick extend their results from *almost all* points of the imaginary axis, iq, to *all* points except $q = 0$. The extension of these results to Gaussian Markov processes is presently under investigation.

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