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TWO UNIFORM BOUNDEDNESS THEOREMS

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A geodesically convex space is a metric space in which each two points can be connected by a unique segment (a path of minimal length). An affine transformation between two geodesically convex spaces is a map which takes segments into segments. It is shown that, if the domain is complete, a pointwise-bounded family of continuous affine transformations is uniformly bounded. Under a mild additional hypothesis, the following stronger theorem holds: if

$$\mathscr{F} = \{T_\sigma \mid A \in A\}$$

is a pointwise-bounded family of affine transformatons and T_a is continuous on a closed geodesically convex S_α with

$$\bigcap_{A \in \alpha} S_\alpha \neq \varnothing ,$$

then $\exists \alpha_1, \cdots, \alpha_n$ such that $\mathscr{F}$ is uniformly bounded on

$$\bigcap_{k=1}^{n} S_{\alpha_k} .$$

Let (X, d), (Y, d') be metric spaces, and $\mathscr{F} = \{T_\alpha \mid \alpha \in A\}$ a collection of maps from X to Y. We say $\mathscr{F}$ is pointwise-bounded if, for fixed $x, y \in X$, $\sup\{d'(T_\alpha x, T_\alpha y) \mid \alpha \in A\}$ is finite. If $x_0 \in S \subseteqq X$, we say $\mathscr{F}$ is uniformly bounded on S if $\sup\{d'(T_\alpha x, T_\alpha, x_0) \mid x \in S, \alpha \in A\}$ is finite. A uniform boundedness theorem is one in which uniform boundedness (for some family $\mathscr{F}$) is deduced from pointwise-boundedness.

Let $\gamma\colon [0, 1] \to X$ be continuous, $0 = t_0 < \cdots < t_n = 1$ a partition P of $[0, 1]$, define $\measuredangle(\gamma, P) = \sum_{k=1}^{n} d(\gamma(t_k), \gamma(t_{k-1}))$, and define $\measuredangle(\gamma)$ to be the supremum over all partitions P of the $\measuredangle(\gamma, P)$. For $x, y \in X$, define $d_g(x, y) = \inf\{\measuredangle(\gamma) \mid \gamma\colon [0, 1] \to X, \gamma(0) = x, \gamma(1) = y\}$; this is the geodesic or intrinsic distance between x and y. d_g is a generalized metric, and γ is said to be a segment from x to y if

$$\gamma(0) = x, \gamma(1) = y ,$$

and $\measuredangle(\gamma) = d_g(x, y) < \infty$.

DEFINITION 1. X is said to be geodesically convex if for any x, y in X there is a unique segment from x to y. We denote by $\Phi_g(x, y, t)$ the intrinsic parametrization of this segment (if $0 \leqq t \leqq s \leqq 1$, $d_g(\Phi_g(x, y, t), \Phi_g(x, y, s)) = (s - t)d_g(x, y)$; T is said to be an affine map between geodesically convex spaces if $T(\Phi_g(x, y, t)) = \Phi_g(Tx, Ty, t)$.

A term often used for a geodesically convex space is a space with unique segments. Throughout this paper we assume $d = d_g$.

Our first theorem is a generalization to geodesically convex spaces of the classical Banach-Steinhaus Theorem.

THEOREM 1. *Let (X, d) be a geodesically convex complete metric space and let (Y, d') be geodesically convex. Let $\mathscr{F} = \{T_\alpha \mid \alpha \in A\}$ be a pointwise-bounded family of geodesically affine maps from X to Y, each of which is continuous. Then for each $x_0 \in X$,*

$$\sup \{d'(T_\alpha x, T_\alpha x_0) \mid \alpha \in A, d(x, x_0) \leqq 1\}$$

is finite.

We shall need the following lemma.

LEMMA 1. *For each $\alpha \in A$, $z_0 \in X$ and $p > 0$,*

$$r(\alpha, z_0, p) = \sup \{d'(T_\alpha x, T_\alpha z_0) \mid d(x, z_0) \leqq p\}$$

is finite.

Proof. By continuity of T_α at z_0, $\exists \delta > 0$ such that

$$d(x, z_0) < \delta \Rightarrow d'(T_\alpha x, T_\alpha z_0) < 1 \; ;$$

we can clearly assume $\delta < p$. If $x \in X$, $d(x, z_0) \leqq p$, let $z = \Phi_g(z_0, x, \delta/2p)$, then $d(z_0, z) = \delta/2pd(z_0, x) < \delta$, so $d'(T_\alpha z_0, T_\alpha z) < 1$. But

$$T_\alpha(\Phi_g(z_0, x, \delta/2p)) = \Phi_g(T_\alpha z_0, T_\alpha x, \delta/2p) \; ,$$

and so $d'(T_\alpha z_0, T_\alpha z) = \delta/2pd'(T_\alpha z_0, T_\alpha x) < 1$, so $d'(T_\alpha z_0, T_\alpha x) < 2p/\delta$.

For purposes of simplicity, we prove the following lemma.

LEMMA 2. *Assume the conclusion of the theorem is false. Let $M > 0$, $x_1, \cdots, x_n \in X$ and $T_1, \cdots, T_n \in \mathscr{F}$ be given, with $d(x_0, x_k) < 1 (1 \leqq k \leqq n)$. Then $\exists x_{n+1} \in X$, $T_{n+1} \in \mathscr{F}$ with $d(x_0, x_{n+1}) < 1$, $d(x_n, x_{n+1}) < 1/2^{n+1}$, $d'(T_{n+1}x_{n+1}, T_{n+1}x_0) > M$, and*

$$d'(T_k x_n, T_k x_{n+1}) < 1/2^{n+1}$$

for $1 \leqq k \leqq n$.

Proof. For $x \in X$, let $S(x) = \sup \{d'(T_\alpha x, T_\alpha x_0) \mid \alpha \in A\}$. Let

$$\alpha = 1/3 \min (2^{-n-1} r(1, x_n, 2)^{-1}, \cdots, 2^{-n-1} r(n, x_n, 2)^{-1}, 2^{-n-1}, 1 - d(x_n, x_0)) \; ,$$

then $\alpha > 0$. If the theorem is false, then for any $K > 0$ there is a $z \in X$ with $d(x_0, z) < 1$ and a $T \in \mathscr{F}$ with $K < d'(Tx_0, Tz)$, consequently

$$K < d'(Tx_0, Tz) \leqq d'(Tx_0, Tx_n) + d'(Tx_n, Tz) \leqq S(x_n) + d'(Tx_n, Tz).$$

This means that we can always find a $z \in X$ and $T \in \mathscr{F}$ with

$$d(x_0, z) < 1$$

and $d'(Tx_n, Tz)$ arbitrarily large. Having defined α, choose $y \in X$, $T(=T_{n+1}) \in \mathscr{F}$ with $d(y, x_0) < 1$, $\alpha d'(Tx_n, Ty) - S(x_n) > M$, and let $x_{n+1} = \Phi_g(x_n, y, \alpha)$. Then $d(x_n, x_{n+1}) = \alpha d(x_n, y) \leqq 2^{-n-1}$. For $1 \leqq k \leqq n$, we have

$$\begin{aligned} d'(T_k x_n, T_k x_{n+1}) &= d'(T_k \Phi_g(x_n, y, 0), T_k \Phi_g(x_n, y, \alpha)) \\ &= d'(\Phi_g(T_k x_n, T_k y, 0), \Phi_g(T_k x_n, T_k y, \alpha)) \\ &= \alpha d'(T_k x_n, T_k y) \leqq \alpha r(k, x_n, 2) < 2^{-n-1} . \end{aligned}$$

We also have

$$\begin{aligned} d(x_0, x_{n+1}) &\leqq d(x_0, x_n) + d(x_n, x_{n+1}) \\ &\leqq d(x_0, x_n) + d(\Phi_g(x_n, y, 0), \Phi_g(x_n, y, \alpha)) \\ &= d(x_0, x_n) + \alpha d(x_n, y) < d(x_0, x_n) + 2\alpha \\ &< d(x_0, x_n) + 1 - d(x_0, x_n) \\ &= 1 . \end{aligned}$$

Finally,

$$\begin{aligned} \alpha d'(Tx_n, Ty) &= d'(Tx_n, Tx_{n+1}) \\ &\leqq d'(Tx_n, Tx_0) + d(Tx_0, Tx_{n+1}) \\ &\leqq S(x_n) + d'(Tx_0, Tx_{n+1}) \Rightarrow d'(Tx_0, Tx_{n+1}) \\ &\geqq \alpha d(Tx_n, Ty) - S(x_n) > M , \end{aligned}$$

completing the proof.

We return to the proof of the theorem. Assume the theorem is false. Then $\exists x_1 \in X$, $T_1 \in \mathscr{F}$ with

$$d(x_0, x_1) < 1, d'(T_1 x_0, T_1 x_1) > 2 .$$

Having chosen $x_1, \cdots, x_n \in X$, $T_1, \cdots, T_n \in \mathscr{F}$ with

$$d(x_0, x_k) < 1 (1 \leqq k \leqq n) ,$$

by Lemma 2 choose $x_{n+1} \in X$, $T_{n+1} \in \mathscr{F}$ with $d(x_0, x_{n+1}) < 1$, $d(x_n, x_{n+1}) < 2^{-n-1}$, $d'(T_{n+1}x_0, T_{n+1}x_{n+1}) > n+2$ and $d'(T_k x_n, T_k x_{n+1}) < 2^{-n-1}$ for $1 \leqq k \leqq n$. Since $d(x_n, x_{n+1}) < 2^{-n-1}$, the sequence $\{x_n | n = 1, 2, \cdots\}$ is Cauchy ($n < m \Rightarrow d(x_n, x_m) < \sum_{k=n}^{m-1} 2^{-k-1}$); by completeness $x_n \to x \in X$. By continuity of T_n we have $\lim_{m\to\infty} d'(T_n x, T_n x_{m+1}) = 0$, so

$$d'(T_n x_0, T_n x_n) \leqq d'(T_n x_0, T_n x) + d'(T_n x, T_n x_n) \leqq \cdots$$
$$\leqq d'(T_n x_0, T_n x) + \sum_{k=n}^{m} d'(T_n x_k, T_n x_{k+1}) + d'(T_n x, T_n x_{m+1}) ;$$

letting $m \to \infty$ we obtain

$$d'(T_n x_0, T_n x_n) \leqq d'(T_n x_0, T_n x) + \sum_{k=n}^{\infty} d'(T_n x_k, T_n x_{k+1})$$
$$< d'(T_n x_0, T_n x) + \sum_{k=n}^{\infty} 2^{-k-1} < d'(T_n x_0, T_n x) + 1 ,$$

since $k \geqq n \Rightarrow d'(T_n x_k, T_n x_{k+1}) < 2^{-k-1}$. So

$$n + 1 < d'(T_n x_0, T_n x_n) \leqq d'(T_n x_0, T_n x) + 1 \Rightarrow d'(T_n x_0, T_n x) > n ,$$

contradicting the pointwise-boundedness of $\mathscr{F}$.

We now make an additional hypothesis, which will enable us to prove a stronger version of this theorem. Let $\Phi = \Phi_g$.

DEFINITION 2. If $0 < \alpha < 1$, define

$$M(\alpha) = \sup \{d(\Phi(x, y, \alpha), \Phi(x, z, \alpha))/d(y, z) \mid x, y, z \in X, y \neq z\} ,$$

and define $M'(\alpha)$ similarly in Y. Note that, if $M(\alpha) < \infty$, then

$$x, y, z \in X \Rightarrow d(\Phi(x, y, \alpha), \Phi(x, z, \alpha)) \leqq M(\alpha) d(y, z) .$$

For the remainder of this paper we shall make the following assumption: $\exists \alpha \in (0, 1)$ such that both $M(\alpha)$ and $M'(\alpha)$ are finite. This α will be fixed from now on.

DEFINITION 3. Let $\{x_n \mid n = 1, 2, \cdots\} \subseteqq X$, and let $x_0 \in X$. Define $z_1^{(n)} = \Phi(x_n, x_0, \alpha)$, and for $2 \leqq k \leqq n$ define $z_k^{(n)} = \Phi(x_{n+1-k}, z_{k-1}^{(n)}, \alpha)$. Now define $y_n = z_n^{(n)}$ for $n = 1, 2, \cdots$.

If X were a Banach space and $x_0 = 0$, then we would have

$$y_n = \sum_{k=1}^{n} (1 - \alpha)^k x_k .$$

In general, however, we have $y_n = \Phi(x_1, \Phi(x_2, \cdots, \Phi(x_n, x_0, \alpha), \cdots, \alpha)$, which will henceforth be abbreviated $\Phi(x_1, \cdots, \Phi(x_n, x_0, \alpha), \cdots, \alpha)$.

LEMMA 3. *Given*

$$\{x_n \mid n = 1, 2, \cdots\} \subseteqq X ,$$

$x_0 \in X$, *define* $\{y_n \mid n = 1, 2, \cdots\}$ *as in Definition* 3. *Then*

$$d(y_n, y_{n-1}) \leqq M(\alpha)^{n-1}(1-\alpha)d(x_n, x_0)$$

if $n \geqq 2$.

Proof. Clearly, we have

$$\begin{aligned} & d(y_n, y_{n-1}) \\ &= d(\Phi(x_1, \cdots, \Phi(x_n, x_0, \alpha), \cdots, \alpha), \Phi(x_1, \cdots, \Phi(x_{n-1}, x_0, \alpha), \cdots, \alpha)) \\ &\leqq M(\alpha)d(\Phi(x_2, \cdots, \Phi(x_n, x_0, \alpha), \cdots, \alpha), \Phi(x_2, \cdots, \Phi(x_{n-1}, x_0, \alpha), \cdots, \alpha)) \\ &\leqq \cdots \leqq M(\alpha)^{n-2}d(\Phi(x_{n-1}, \Phi(x_n, x_0, \alpha), \alpha), \Phi(x_{n-1}, x_0, \alpha)) \\ &\leqq M(\alpha)^{n-1}d(\Phi(x_n, x_0, \alpha), x_0) \\ &= (1-\alpha)M(\alpha)^{n-1}d(x_n, x_0) . \end{aligned}$$

LEMMA 4. *Let S be a convex subset of X, $p > 0$, and let $x_0 \in S$, $\mathscr{F} = \{T_\lambda \mid \lambda \in \Lambda\}$ a collection of affine functions on X. If $\mathscr{F}$ is not uniformly bounded on $S \cap S(x_0, p)$, then given $M > 0$, $\varepsilon > 0$, we can find a $T \in \mathscr{F}$ and an $x \in S \cap S(x_0, p)$ such that $d(x_0, x) < \varepsilon$ and*

$$d'(Tx, Tx_0) > M .$$

Proof. We can assume without loss of generality that $\varepsilon < p$. Choose $T \in \mathscr{F}$, $y \in S \cap S(x_0, p)$ such that $d'(Ty, Tx_0) > Mp/\varepsilon$. Let $x = \Phi(x_0, y, \varepsilon/p)$; $x \in S$ by the convexity of S. Now

$$d(x, x_0) = (\varepsilon/p)d(y, x_0) < \varepsilon ,$$

and

$$\begin{aligned} d'(Tx, Tx_0) &= d'(T\Phi(x_0, y, \varepsilon/p), Tx_0) \\ &= d'(\Phi(Tx_0, Ty, \varepsilon/p), Tx_0) \\ &= \varepsilon/pd'(Tx_0, Ty) > M , \end{aligned}$$

completing the proof.

The next lemma will be critical in proving the desired theorem.

LEMMA 5. *Let $\{S_n \mid n = 1, 2, \cdots\}$ be a collection of closed convex subsets of X, and let $\{T_n \mid n = 1, 2, \cdots\}$ be a collection of affine functions on X such that $T_n \mid S_n$ is continuous for $n = 1, 2, , \cdots$. Assume that $x_{n+1} \in \bigcap_{k=1}^n S_k$ for $n = 1, 2, \cdots$, and that $d(x_n, x_0)$ is sufficiently small to make $\{y_n \mid n = 1, 2, \cdots\}$ (as defined in Definition 2) a Cauchy sequence (we do this by requiring $\sum_{n=1}^\infty M(\alpha)^{n-1}d(x_n, x_0)$ to converge). By completeness of X, let $y = \lim_{n\to\infty} y_n$. Then for each integer N, $T_N y = \lim_{n\to\infty} T_N y_n$.*

Proof. Observe first that, if $\lim_{n\to\infty} u_n = u$ (in either X or Y), then $\lim_{n\to\infty} \Phi(v, u_n, \alpha) = \Phi(v, u, \alpha)$, as

$$d(\Phi(v, u_n, \alpha), \Phi(v, u, \alpha)) \leqq M(\alpha)d(u_n, u) \to 0 .$$

If $n > N$, let $z_n = \Phi(x_{N+1}, \cdots, \Phi(x_n, x_0, \alpha), \cdots, \alpha)$. As in Lemma 1, we can show that $d(z_n, z_{n-1}) \leqq (1-\alpha)M(\alpha)^{n-N-1}d(x_n, x_0)$, and since $\sum_{n=1}^{\infty} M(\alpha)^{n-1}d(x_n, x_0)$ converges, we can define $z = \lim_{n\to\infty} z_n$. Note that $n > N \Rightarrow z_n \in S_N$, as $x_{N+1}, \cdots, x_n \in S_N$ and S_N is convex. Since S_N is closed, $z \in S_N$, and so $T_N z_n \to T_N z$ by the continuity of $T_N | S_N$. If $n > N$, we have

$$\begin{aligned} T_N y_n &= T_N \Phi(x_1, \cdots, \Phi(x_N, z_n, \alpha), \cdots, \alpha) \\ &= \Phi(T_N x_1, \cdots, \Phi(T_N x_N, T_N z_n, \alpha), \cdots, \alpha) , \end{aligned}$$

and so

$$\begin{aligned} \lim_{n\to\infty} T_N y_n &= \Phi\Big(T_N x_1, \cdots, \lim_{n\to\infty} \Phi(T_N x_N, T_N z_n, \alpha), \cdots, \alpha\Big) \\ &= \Phi\Big(T_N x_1, \cdots, \Phi\Big(T_N x_N, \lim_{n\to\infty} T_N z_n, \alpha\Big), \cdots, \alpha\Big) \\ &= \Phi(T_N x_1, \cdots, \Phi(T_N x_N, T_N z, \alpha), \cdots, \alpha) . \end{aligned}$$

Since $y_n = \Phi(x_1, \cdots, \Phi(x_N, z_n, \alpha), \cdots, \alpha)$ and

$$\begin{aligned} y = \lim_{n\to\infty} y_n &= \Phi\Big(x_1, \cdots, \lim_{n\to\infty} \Phi(x_N, z_n, \alpha)\Big) \\ &= \Phi(x_1, \cdots, \Phi(x_N, z, \alpha), \cdots, \alpha) , \end{aligned}$$

we see that $T_N y = \Phi(T_N x_1, \cdots, \Phi(T_N x_N, T_N z, \alpha), \cdots, \alpha) = \lim_{n\to\infty} T_N y_n$.

It is now necessary to perform some calculations. Assume

$$\{x_n | n = 1, 2, \cdots\} \subseteqq X ,$$

$x_0 \in X$, and $\{y_n | n = 1, 2, \cdots\}$ is defined as in Definition 3. Now define

$$z_k = \Phi(x_k, \cdots, \Phi(x_n, x_0, \alpha), \cdots, \alpha) = \Phi(x_k, z_{k+1}, \alpha)$$

(for the purpose of these calculations, n will be assumed to be fixed) for $k \leqq n-1$, $z_n = \Phi(x_n, x_0, \alpha)$. We now have $d(x_0, \Phi(x_n, x_0, \alpha)) = d(x_0, z_n) \leqq d(x_0, y_n) + \sum_{k=1}^{n-1} (d(z_k, z_{k+1})$, as clearly $z_1 = y_n$. Observe further that

$$\begin{aligned} d(z_k, z_{k+1}) &= d(\Phi(x_k, z_{k+1}, \alpha), z_{k+1}) \\ &= (1-\alpha)d(x_k, z_{k+1}) \\ &\leqq (1-\alpha)[d(x_k, x_0) + d(x_0, z_{k+1})] \end{aligned}$$

for k $\leqq n-1$.

We now prove some computational lemmas.

LEMMA 6. *If* $k \leqq n-2$,

$$d(x_0, z_{k+1}) \leqq (1+\alpha)d(x_{k+1}, x_0) + \alpha d(x_0, z_{k+2}) .$$

Proof.

$$\begin{aligned} d(x_0, z_{k+1}) &\leqq d(z_{k+1}, x_{k+1}) + d(x_{k+1}, x_0) \\ &= d(\Phi(x_{k+1}, z_{k+2}, \alpha), x_{k+1}) + d(x_{k+1}, x_0) \\ &= \alpha d(x_{k+1}, z_{k+2}) + d(x_{k+1}, x_0) \\ &\leqq \alpha[d(x_{k+1}, x_0) + d(x_0, z_{k+2})] + d(x_{k+1}, x_0) \\ &= (1+\alpha)d(x_{k+1}, x_0) + \alpha d(x_0, z_{k+2}) . \end{aligned}$$

LEMMA 7. *If* $k \leqq n-2$, *then*

$$d(x_0, z_{k+1}) \leqq (1+\alpha)\sum_{j=0}^{n-k-2} \alpha^j d(x_0, x_{k+1+j}) + (1-\alpha)^{n-k-1} d(x_0, x_n) .$$

Proof. If $j \leqq n-k-2$, we shall verify the inequality

$$d(x_0, z_{k+1}) \leqq (1+\alpha)\sum_{i=0}^{j} \alpha^i d(x_0, x_{k+1+i}) + \alpha^{j+1} d(x_0, z_{k+j+2}) .$$

If $j=0$, this inequality is the conclusion of Lemma 4. Inductively, assume it is true for j. By Lemma 6, we have

$$\alpha^{j+1} d(x_0, z_{k+j+2}) \leqq \alpha^{j+1}[(1+\alpha)d(x_0, x_{k+j+2}) + \alpha d(x_0, z_{k+j+3})] ;$$

adding this term to the j^{th} inequality yields the inequality for $j+1$. When $j = n-k-2$, we therefore have

$$\begin{aligned} d(x_0, z_{k+1}) &\leqq \sum_{j=0}^{n-k-2} (1+\alpha)\alpha^j d(x_0, x_{k+1+j}) + \alpha^{n-k-1} d(x_0, z_n) \\ &= (1+\alpha)\sum_{j=0}^{n-k-2} \alpha^j d(x_0, x_{k+1+j}) + (1-\alpha)\alpha^{n-k-1} d(x_0, x_n) . \end{aligned}$$

A consequence of Lemma 7 and a previous observation is that

$$\begin{aligned} d(z_k, z_{k+1}) &\leqq (1-\alpha)[d(x_k, x_0) + d(x_0, z_{k+1})] \\ &\leqq (1-\alpha)[d(x_k, x_0) + (1+\alpha)\sum_{j=0}^{n-k-2} \alpha^j d(x_0, x_{k+1+j}) \\ &\quad + (1-\alpha)\alpha^{n-k-1} d(x_0, x_n)] . \end{aligned}$$

Now let $1 \leqq k \leqq n-1$. We make the following definition for $k \leqq j \leqq n$.

$$\begin{aligned} \mu_j^{(k)} &= 1-\alpha && \text{if } j = k \\ &= (1-\alpha^2)\alpha^{j-k-1} && \text{if } k < j < n \\ &= (1-\alpha)^2 \alpha^{n-k-1} && \text{if } j = n . \end{aligned}$$

Then $d(z_k, z_{k+1}) \leqq \sum_{j=k}^{n} \mu_j^{(k)} d(x_j, x_0)$, and so

$$\begin{aligned}(1-\alpha)d(x_0, x_n) &= d(x_0, \Phi(x_n, x_0, \alpha)) \\ &\leqq d(x_0, y_n) + \sum_{k=1}^{n-1} d(z_k, z_{k+1}) \\ &\leqq d(x_0, y_n) + \sum_{k=1}^{n-1}\Big(\sum_{j=k}^{n} \mu_j^{(k)} d(x_j, x_0)\Big) \\ &= d(x_0, y_n) + \sum_{k=1}^{n-1}\Big(\sum_{j=1}^{k} \mu_k^{(j)}\Big) d(x_k, x_0) + \sum_{j=1}^{n-1} \mu_n^{(j)} d(x_n, x_0) .\end{aligned}$$

If $1 \leqq k \leqq n-1$, let $\beta_k = \sum_{j=1}^{k} \mu_k^{(j)}$, and let

$$\beta_n = \sum_{j=1}^{n-1} \mu_n^{(j)} - (1-\alpha) .$$

Obviously $\beta_k > 0$ if $1 \leqq k \leqq n-1$, and also

$$\begin{aligned}\sum_{j=1}^{n-1} \mu_n^{(j)} &= (1-\alpha)^2 \sum_{j=1}^{n-1} \alpha^{n-j-1} \\ &= (1-\alpha)^2 \sum_{j=0}^{n-2} \alpha^j \\ &= (1-\alpha)^2[(1-\alpha^{n-1})/(1-\alpha)] \\ &= (1-\alpha)(1-\alpha^{n-1}) < 1-\alpha ,\end{aligned}$$

and so $\beta_n < 0$. Since this calculation has been performed for the integer n, we shall relabel the constants just obtained $\beta_1^{(n)}, \cdots, \beta_n^{(n)}$.

The last inequality proved shows that

$$0 \leqq d(x_0, y_n) + \sum_{k=1}^{n} \beta_k^{(n)} d(x_0, x_k) ,$$

which implies that $d(x_0, y_n) \geqq (-\beta_n^{(n)}) d(x_0, x_n) - \sum_{k=1}^{n-1} \beta_k^{(n)} d(x_0, x_k)$. A reexamination of the work done subsequent to Lemma 3 shows that, if $T: X \rightarrow Y$ is affine, then

$$d'(Tx_0, Ty_n) \geqq (-\beta_n^{(n)}) d'(Tx_0, Tx_n) - \sum_{k=1}^{n-1} \beta_k^{(n)} d'(Tx_0, Tx_k) .$$

We have therefore proved the following:

LEMMA 8. *Let $\mathscr{F} = \{T_\lambda \mid \lambda \in \Lambda\}$ be a pointwise-bounded family of affine functions from X into Y, and let $\{x_n \mid n = 1, 2, \cdots\}$ be given in X, $\{y_n \mid n = 1, 2, \cdots\}$ as in Definition 2. If*

$$S(x) = \sup d'(Tx, Tx_0) \mid T \in \mathscr{F}\} ,$$

then $d'(Tx_0, Ty_n) \geqq (-\beta_n^{(n)}) d'(Tx_0, Tx_n) - \sum_{k=1}^{n-1} \beta_k^{(n)} S(x_k)$ for any $T \in \mathscr{F}$.

Proof. Immediate from previous work and the fact that

$$d'(Tx_0,\ Tx_k) \leqq S(x_k)$$

for all $T \in \mathscr{F}$.

We come now to the desired theorem.

THEOREM 2. *Let (X, d), (Y, d') be spaces with unique segments, let X be complete, and assume there is an $\alpha \in (0, 1)$ such that $M(\alpha)$, $M'(\alpha)$ are finite. Let $\mathscr{F} = \{T_\lambda \mid \lambda \in \Lambda\}$ be a pointwise-bounded family of affine maps from X into Y, and let S_λ be a closed convex subset of X such that $\bigcap_{\lambda \in \Lambda} S_\lambda \neq \varphi$ and $T_\lambda \mid S_\lambda$ is continuous for each $\lambda \in \Lambda$. Then $\exists \lambda_1, \cdots, \lambda_n \in \Lambda$ such that $\mathscr{F}$ is uniformly bounded on $\bigcap_{k=1}^{n} S_{\lambda_k}$.*

Proof. Let $x_0 \in \bigcap_{\lambda \in \Lambda} S_\lambda$, $p > 0$, and assume that $\mathscr{F}$ is not uniformly bounded on the intersection of $S(x_0, p)$ and any finite intersection of the $\{S_\lambda \mid \lambda \in \Lambda\}$. We assert that we can prove the following: given $x_1, \cdots, x_n \in X$, $T_1, \cdots, T_n \in \mathscr{F}$ with $T_k \mid S_k$ continuous, $1 \leqq k \leqq n$ and $x_k \in \bigcap_{j=1}^{k-1} S_j$ for $2 \leqq k \leqq n$, and given $M > 0$, let $y_1, \cdots, y_n$ be derived from $x_1, \cdots, x_n$ as in Definition 3. Then we can find $x_{n+1} \in \bigcap_{k=1}^{n} S_k$ and $T_{n+1} \in \mathscr{F}$ such that, if we let y_{n+1} be derived from $x_1, \cdots, x_{n+1}$ as in Definition 3,

$$d(x_0,\ y_{n+1}) < p,\ d(y_n,\ y_{n+1}) < 1/2^{n+1},\ d'(T_{n+1}y_{n+1},\ T_{n+1}x_0) > M\ ,$$

and $d'(T_k y_n,\ T_k y_{n+1}) < 1/2^{n+1}$ for $1 \leqq k \leqq n$.

Since $x_0 \in \bigcap_{k=1}^{n} S_k$, choose δ_k $(1 \leqq k \leqq n)$ such that $x \in S_k$,

$$d(x,\ x_0) < \delta_k \Rightarrow d'(T_k x,\ T_k x_0) < 1/2^{n+1}(1 - \alpha)M'(\alpha)^n\ ;$$

then if we define $y = \Phi(x_1, \cdots, \Phi(x_n, \Phi(x, x_0, \alpha), \alpha), \cdots, \alpha)$, by Lemma 3 we have $x \in S_k$, $d(x, x_0) < \delta_k \Rightarrow d'(T_k y_n, T_k y) < 1/2^{n+1}$. Now let

$$\gamma = 2^{-1} \min (p, \delta_1, \cdots, \delta_n, (p - d(x_0, y_n))/(1 - \alpha)M(\alpha)^n, 1/(1 - \alpha)M(\alpha)^n 2^{n+1})\ .$$

Finally, by Lemma 4 choose $x_{n+1} \in \bigcap_{k=1}^{n} S_k$ and $T(= T_{n+1}) \in \mathscr{F}$ with $d(x_0, x_{n+1}) < \gamma$ and $(-\beta_{n+1}^{(n+1)})d'(Tx_0, Tx_{n+1}) > M + \sum_{k=1}^{n} \beta_k^{(n+1)} S(x_k)$. Define $y_{n+1} = \Phi(x_1, \cdots, \Phi(x_{n+1}, x_0, \alpha), \cdots, \alpha)$. We have already observed that $1 \leqq k \leqq n \Rightarrow d'(T_k y_n, T_k y_{n+1}) < 1/2^{n+1}$. Now by Lemma 3

$$d(y_n,\ y_{n+1}) \leqq (1 - \alpha)M(\alpha)^n d(x_0,\ x_{n+1}) < 1/2^{n+1}\ ,$$

and also

$$\begin{aligned} d(x_0,\ y_{n+1}) &\leqq d(x_0,\ y_n) + d(y_n,\ y_{n+1}) \\ &\leqq d(x_0,\ y_n) + (1 - \alpha)M(\alpha)^n d(x_0,\ x_{n+1}) \\ &< d(x_0,\ y_n) + (p - d(x_0,\ y_n)) \\ &= p\ . \end{aligned}$$

By Lemma 8 we see that

$$d'(Ty_{n+1}, Tx_0) \geqq (-\beta_{n+1}^{(n+1)})d'(Tx_0, Tx_{n+1}) - \sum_{k=1}^{n} \beta_k^{(n+1)} S(x_k) > M .$$

Construct $\{y_n | n = 1, 2, \cdots\}$ by this procedure to insure that

$$d(x_0, y_{n+1}) < p,\ d(y_n, y_{n+1}) < 1/2^{n+1}$$

and choose $\{T_n | n = 1, 2, \cdots\} \subseteqq \mathscr{F}$ with $d'(T_{n+1}y_{n+1}, T_{n+1}x_0) > n + 2$ and $d'(T_k y_n, T_k y_{n+1}) < 1/2^{n+1}$ for $1 \leqq k \leqq n$. Now $\{y_n | n = 1, 2, \cdots\}$ is Cauchy, so let $y = \lim_{k\to\infty} y_n$. By Lemma 5, for each integer n we have

$$T_n y = \lim_{n\to\infty} T_n y_k ,$$

and so far any n we have $\lim_{m\to\infty} d'(T_n y, T_n y_{m+1}) = 0$. So

$$\begin{aligned} d'(T_n x_0, T_n y_n) &\leqq d'(T_n x_0, T_n y) + d'(T_n y, T_n y_n) \leqq \cdots \\ &\leqq d'(T_n x_0, T_n y) + \sum_{k=n}^{m} d'(T_n y_k, T_n y_{k+1}) + d'(T_n y, T_n y_{m+1}) ; \end{aligned}$$

as $m \to \infty$ we obtain

$$\begin{aligned} d'(T_n x_0, T_n y_n) &\leqq d'(T_n x_0, T_n y) + \sum_{k=n}^{\infty} d'(T_n y_k, T_n y_{k+1}) \\ &< d'(T_n x_0, T_n y) + \sum_{k=n}^{\infty} 2^{-k-1} \\ &< d'(T_n x_0, T_n y) + 1 , \end{aligned}$$

since $k \geqq n \Rightarrow d'(T_n y_k, T_n y_{k+1}) < 1/2^{k+1}$. So

$$n + 1 < d'(T_n x_0, T_n y_n) \leqq d'(T_n x_0, T_n y) + 1 \Rightarrow d'(T_n x_0, T_n y) > n ,$$

contradicting the pointwise-boundedness of $\mathscr{F}$ and completing the proof.

In conclusion, although spaces such that $M(\alpha)$ is infinite for every $\alpha \in (0, 1)$ are highly pathological, it would be nice to know whether or not the restriction that some $M(\alpha)$ and $M'(\alpha)$ be finite can be removed.

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