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WHITTAKER CONSTANTS FOR ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Let f be an entire function of a single complex variable. The *exponential type* of f is given by

$$\tau(f) = \limsup_{n\to\infty} | f^{(n)}(0) |^{1/n} .$$

The Whittaker constant W is defined to be the supremum of numbers c having the following property: if $\tau(f) < c$ and each of $f, f', f'', \cdots$ has a zero in the disc $|z| \leqq 1$, then $f \equiv 0$. The Whittaker constant is known to lie between .7259 and .7378.

The present paper provides a definition and characterization of the Whittaker constant $\mathscr{W}_n$ for n complex variables. The principle result of this characterization, which involves polynomial expansions of entire functions, is

$$W > \mathscr{W}_2 \geqq \mathscr{W}_3 \geqq \cdots .$$

To simplify notation, the presentation here is given for functions of two variables.

An exact determination of W was obtained by M. A. Evgrafov in 1954 [3]. The determination involves the Gončarov polynomials, defined recursively by

$$G_0(z) = 1 ,$$

$$(1.1)\quad G_n(z; z_0, z_1, \cdots, z_{n-1}) = \frac{z^n}{n!} - \sum_{k=0}^{n-1} \frac{z_k^{n-k}}{(n-k)!} G_k(z; z_0, z_1, \cdots, z_{k-1}) .$$

Let

$$H_n = \max | G_n(0; z_0, \cdots, z_{n-1}) | ,$$

where the maximum is taken over all sequences $\{z_k\}_{k=0}^{n-1}$ whose terms lie on $|z| = 1$. Evgrafov proved that

$$W = \left\{ \limsup_{n\to\infty} H_n^{1/n} \right\}^{-1} .$$

An improvement of this result and further characterizations of W were furnished by J. D. Buckholtz [1]. Using properties of the Gončarov polynomials, Buckholtz proved that

$$(1.2)\quad (.4)^{1/n} H_n^{-1/n} < W \leqq H_n^{-1/n} ,$$

for $n = 1, 2, 3, \cdots$. A consequence of these bounds is

$$W = \left\{\lim_{n\to\infty} H_n^{1/n}\right\}^{-1} = \left\{\sup_{1\leqq n<\infty} H_n^{1/n}\right\}^{-1} . \tag{1.3}$$

For an entire function f (of two complex variables) the exponential type $\tau(f)$ is given by

$$\tau(f) = \limsup_{m+n\to\infty} |f^{(m,n)}(0, 0)|^{1/(m+n)} .$$

We define the Whittaker constant $\mathscr{W}$ to be the supremum of positive numbers c having the following property: if $\tau(f) < c$ and each of $f^{(m,n)}$ $(0 \leqq m < \infty,\ 0 \leqq n < \infty)$ has a zero in the poly disc $\{(z_1, z_2): |z_1| \leqq 1, |z_2| \leqq 1\}$, then $f \equiv 0$. The bound $\mathscr{W} \geqq (\log 2)/2$ was obtained by M. M. Dzrbasjan in 1957 [2].

The estimate furnished by Džrbašjan depends on a system of polynomials defined as follows. Let $\alpha = (\alpha_{pq})$ and $\beta = (\beta_{pq})$ be infinite matrices of complex numbers. The polynomials $A_{m,n}(z_1, z_2; \alpha, \beta)$ are defined by the recursion formula

$$A_{0,0}(z_1, z_2) = 1 ,$$

$$A_{r,s}(z_1, z_2; \alpha, \beta) = \frac{z_1^r z_2^s}{r!s!} - \sum_{\substack{p=0\\ p+q<r+s}}^{r} \sum_{q=0}^{s} \frac{A_{p,q}(z_1, z_2; \alpha, \beta)\alpha_{pq}^{r-p}\beta_{pq}^{s-q}}{(r-p)!\,(s-q)!} \tag{1.4}$$

for $r, s = 0, 1, 2, \cdots$. Note that $A_{r,s}$ depends only on those parameters α_{pq} and β_{pq} for which $p + q < r + s$. Let

$$H_{r,s} = \max |A_{r,s}(0, 0; \alpha, \beta)| ,$$

where the maximum is taken over all matrices α and β whose entries lie on $|z| = 1$. We show that bound $H_{r,s} \leqq (2/\log 2)^{r+s}$ holds for all r and s. The justifies the definition

$$H = \sup_{1\leqq r,s<\infty} H_{r,s}^{1/(r+s)} .$$

We prove the following expansion theorem.

THEOREM 1. *Suppose f is entire and $\tau(f) < 1/H$. If α and β are infinite complex matrices whose entries lie in $|z| \leqq 1$, then*

$$f(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f^{(m,n)}(\alpha_{mn}, \beta_{mn}) A_{m,n}(z_1, z_2; \alpha, \beta) \tag{1.5}$$

for all (z_1, z_2).

The following result shows that the expansion constant $1/H$ is as large as possible.

THEOREM 2. *There exists an entire function F, with $\tau(F) =$*

$1/H$, such that each of $F^{(m,n)}$ $(0 \leqq m < \infty, 0 \leqq n < \infty)$ has a zero in the polydisc $\{|z_1| \leqq 1, |z_2| \leqq 1\}$.

Theorem 1 and Theorem 2 will be proved in § 3. We note, however, that the following result is an easy consequence of Theorems 1 and 2.

COROLLARY 1. $\mathscr{W} = 1/H$.

Therefore, each of the numbers $H_{m,n}^{-1/(m+n)}$ is an upper bound for $\mathscr{W}$. In particular, $\mathscr{W} \leqq 1/\sqrt{H_{1,1}} = 1/\sqrt{3}$. In comparing this with the bound $W > .7259$, one sees that $\mathscr{W} < W$.

2. The Polynomials $A_{m,n}$. Let f be an entire function and let α and β be infinite complex matrices. Writing (1.4) in the form

$$\frac{z_1^r z_2^s}{r!\, s!} = \sum_{p=0}^{r} \sum_{q=0}^{s} \frac{A_{p,q}(z_1, z_2; \alpha, \beta)\alpha_{pq}^{r-p}\beta_{pq}^{s-q}}{(r-p)!\,(s-q)!}$$

we obtain the formal expansion

$$\begin{aligned} f(z_1, z_2) &= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} f^{(r,s)}(0, 0)\frac{z_1^r z_2^s}{r!\, s!} \\ &= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} f^{(r,s)}(0, 0)\left\{\sum_{p=0}^{r} \sum_{q=0}^{s} \frac{A_{p,q}(z_1, z_2; \alpha, \beta)\alpha_{pq}^{r-p}\beta_{pq}^{s-q}}{(r-p)!\,(s-q)!}\right\} \\ &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} A_{p,q}(z_1, z_2; \alpha, \beta)\left\{\sum_{r=p}^{\infty} \sum_{s=q}^{\infty} f^{(r,s)}(0, 0)\frac{\alpha_{pq}^{r-p}\beta_{pq}^{s-q}}{(r-p)!\,(s-q)!}\right\} \\ &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} f^{(p,q)}(\alpha_{pq}, \beta_{pq})A_{p,q}(z_1, z_2; \alpha, \beta)\ , \end{aligned} \tag{2.1}$$

which holds whenever the interchange in the order of summation can be justified. In particular, (2.1) holds if f is a polynomial and yields considerable information when f is taken to be one of the polynomials $A_{m,n}$.

LEMMA 1. *If λ is a complex number, then*

$$A_{m,n}(\lambda z_1, \lambda z_2; \lambda\alpha, \lambda\beta) = \lambda^{m+n}A_{m,n}(z_1, z_2; \alpha, \beta)\ , \tag{2.2}$$

where $\lambda\alpha$ denotes matrix scalar multiplication. Furthermore,

$$A_{m,n}(\alpha_{00}, \beta_{00}; \alpha, \beta) = 0 \qquad (m + n > 0)\ . \tag{2.3}$$

Proof. We will prove (2.2) using mathematical induction. The proof of (2.3) is similar. If $m + n = 0$, the result is clear. Suppose

N is a positive integer and (2.2) holds for the polynomials $A_{p,q}$ with $p+q<N$. If r and s are nonnegative integers such that $r+s=N$, then

$$\begin{aligned} &A_{r,s}(\lambda z_1, \lambda z_2; \lambda\alpha, \lambda\beta) \\ &= \lambda^{r+s}\frac{z_1^r z_2^s}{r!\,s!} - \sum_{\substack{p=0\\p+q<r+s}}^{r}\sum_{q=0}^{s} \frac{A_{p,q}(\lambda z_1, \lambda z_2; \lambda\alpha, \lambda\beta)(\lambda\alpha_{pq})^{r-p}(\lambda\beta_{pq})^{s-q}}{(r-p)!\,(s-q)!} \\ &= \lambda^{r+s}\frac{z_1^r z_2^s}{r!\,s!} - \lambda^{r+s}\sum_{\substack{p=0\\p+q<r+s}}^{r}\sum_{q=0}^{s} \frac{A_{p\,q}(z_1, z_2; \alpha, \beta)\alpha_{pq}^{r-p}\beta_{pq}^{s-q}}{(r-p)!\,(s-q)!} \\ &= \lambda^{r+s}A_{r,s}(z_1, z_2; \alpha, \beta) \end{aligned}$$

and this completes the proof.

Let $\alpha = (\alpha_{pq})_{p,q=0}^{\infty}$ be an infinite complex matrix. If j and k are nonnegative integers, we denote by R_{jk} the operator which transforms α into

$$R_{jk}(\alpha) = (\alpha_{p+j\ q+k})_{p,q=0}^{\infty}\ .$$

LEMMA 2. *If* $m+n>0$, $j \leqq m$ *and* $k \leqq n$, *then*

$$A_{m,n}^{(j,k)}(z_1, z_2; \alpha, \beta) = A_{m-j\ n-k}(z_1, z_2; R_{jk}(\alpha), R_{jk}(\beta))\ . \tag{2.4}$$

Proof. By direct computation, $A_{1\,0}(z_1, z_2; \alpha, \beta) = z_1 - \alpha_{00}$ and

$$A_{0\,1}(z_1, z_2; \alpha, \beta) = z_2 - \beta_{00}\ ,$$

so the result is clear if $m+n=1$. Proceeding inductively, let N be a positive integer and suppose the proposition is true for the polynomials $A_{p\,q}$ with $p+q<N$. If r and s are nonnegative integers such that $r+s=N$, then for $j \leqq r$ and $k \leqq s$ we have

$$\begin{aligned} &A_{r,s}^{(j,k)}(z_1, z_2; \alpha, \beta) \\ &= \frac{z_1^{r-j}z_2^{s-k}}{(r-j)!\,(s-k)!} - \sum_{\substack{p=0\\p+q<r+s}}^{r}\sum_{q=0}^{s} \frac{A_{p,q}^{(j,k)}(z_1, z_2; \alpha, \beta)\alpha_{pq}^{r-p}\beta_{pq}^{s-q}}{(r-p)!\,(s-q)!} \\ &= \frac{z_1^{r-j}z_2^{s-k}}{(r-j)!\,(s-k)!} - \sum_{\substack{p=j\\p+q<r+s}}^{r}\sum_{q=k}^{s} \frac{A_{p-j\ q-k}(z_1, z_2; R_{jk}(\alpha), R_{jk}(\beta))\alpha_{pq}^{r-p}\beta_{pq}^{s-q}}{(r-p)!\,(s-q)!} \\ &= \frac{z_1^{r-j}z_2^{s-k}}{(r-j)!\,(s-k)!} - \sum_{\substack{p=0\\p+q<r-j+s-k}}^{r-j}\sum_{q=0}^{s-k} \frac{A_{p\,q}(z_1, z_2; R_{jk}(\alpha), R_{jk}(\beta))\alpha_{p+j,q+k}^{r-j-p}\beta_{p+j,q+k}^{s-k-q}}{(r-j-p)!\,(s-k-q)!} \\ &= A_{r-j,s-k}(z_1, z_2; R_{jk}(\alpha), R_{jk}(\beta))\ , \end{aligned}$$

and this completes the proof.

Lemma 2 and the expansion (2.1) provide a useful expression for the polynomials $A_{m,n}$. Replacing α and β by γ and δ, respectively, and applying (2.1) to the polynomial $A_{r,s}(z_1, z_2; \alpha, \beta)$, we have

$$A_{r,s}(z_1, z_2; \alpha, \beta) = \sum_{p=0}^{r}\sum_{q=0}^{s} A_{r,s}^{(p,q)}(\gamma_{pq}, \delta_{pq}; \alpha, \beta) A_{p,q}(z_1, z_2; \gamma, \delta) = \sum_{p=0}^{r}\sum_{q=0}^{s} A_{p,q}(z_1, z_2; \gamma, \delta) A_{r-p,s-q}(\gamma_{pq}, \delta_{pq}; R_{pq}(\alpha), R_{pq}(\beta)) . \tag{2.5}$$

If each of γ and δ is the zero matrix, it is easy to see that

$$A_{p,q}(z_1, z_2; \gamma, \delta) = \frac{z_1^p z_2^q}{p!\,q!} .$$

In this case (2.5) yields

$$A_{r,s}(z_1, z_2; \alpha, \beta) = \sum_{p=0}^{r}\sum_{q=0}^{s} A_{r-p,s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta)) \frac{z_1^p z_2^q}{p!\,q!} . \tag{2.6}$$

Let m and n be integers such that $0 \leqq m \leqq r$, $0 \leqq n \leqq s$, and $m + n > 0$. In (2.5) choose

$$\gamma_{pq} = \begin{cases} 0, & \text{if } p \geqq m \text{ and } q \geqq n \\ \alpha_{pq}, & \text{otherwise} \end{cases}$$

and

$$\delta_{pq} = \begin{cases} 0, & \text{if } p \geqq m \text{ and } q \geqq n \\ \beta_{pq}, & \text{otherwise .} \end{cases}$$

In view of (2.3) we have

$$A_{r,s}(z_1, z_2; \alpha, \beta) = \sum_{p=m}^{r}\sum_{q=n}^{s} A_{p,q}(z_1, z_2; \gamma, \delta) A_{r-p,s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta)) . \tag{2.7}$$

More generally, we define the operator P_{jk} as follows. If $j + k > 0$, then $P_{jk}(\alpha)$ is the matrix (a_{pq}), where

$$a_{pq} = \begin{cases} 0, & \text{if } p \geqq j \text{ and } q \geqq k \\ \alpha_{pq}, & \text{otherwise .} \end{cases}$$

Then (2.7) becomes

$$A_{r,s}(z_1, z_2; \alpha, \beta) = \sum_{p=m}^{r}\sum_{q=n}^{s} A_{p,q}(z_1, z_2; P_{mn}(\alpha), P_{mn}(\beta)) A_{r-p,s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta)) . \tag{2.8}$$

Equation (2.8) may be regarded as a separation of variables formula, in the following sense. If $p \geqq m$ and $q \geqq n$, then $R_{pq}(\alpha)$ depends on the parameters α_{jk}, where $j \geqq m$ and $k \geqq m$, and $P_{mn}(\alpha)$ depends on the parameters α_{jk}, where $j < m$ or $k < n$. The usefulness of (2.8) is seen in the next lemma.

LEMMA 3. *If* $0 \leqq m \leqq r$ *and* $0 \leqq n \leqq s$, *then*

$$H_{r,s} \geqq H_{m,n} H_{r-m,s-n} . \tag{2.9}$$

Proof. If $m + n = 0$, the result is trivial. Suppose $m + n > 0$ and choose matrices α and β, whose entries lie on $|z| = 1$, such that

$$H_{m,n} = | A_{m,n}(0, 0; P_{mn}(\alpha), P_{mn}(\beta)) |$$

and

$$H_{r-m,s-n} = | A_{r-m,s-n}(0, 0; R_{mn}(\alpha), R_{mn}(\beta)) .$$

For each complex number λ, define the matrices $\gamma = \gamma(\lambda)$ and $\delta = \delta(\lambda)$ by

$$\gamma_{pq} = \begin{cases} \alpha_{pq}, \text{ if } p \geqq m \text{ and } q \geqq n \\ \lambda\alpha_{pq}, \text{ otherwise} \end{cases}$$

and

$$\delta_{pq} = \begin{cases} \beta_{pq}, \text{ if } p \geqq m \text{ and } q \geqq n \\ \lambda\beta_{pq}, \text{ otherwise .} \end{cases}$$

By (2.8) and (2.2),

$$\begin{aligned} &A_{r,s}(0, 0; \gamma, \delta) \\ &= \sum_{p=m}^{r} \sum_{q=n}^{s} A_{p,q}(0, 0; P_{mn}(\gamma), P_{mn}(\delta)) A_{r-p,s-q}(0, 0; R_{pq}(\gamma), R_{pq}(\delta)) \\ &= \sum_{p=m}^{r} \sum_{q=n}^{s} \lambda^{p+q} A_{p,q}(0, 0; P_{mn}(\alpha), P_{mn}(\beta)) A_{r-p,s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta)) \\ &= \lambda^{m+n} Q(\lambda) , \end{aligned}$$

where $Q(\lambda)$ is a polynomial in λ. Since

$$H_{r,s} \geqq \max_{|\lambda|=1} | A_{r,s}(0, 0; \gamma, \delta) | = \max_{|\lambda|=1} | Q(\lambda) | \geqq | Q(0) |$$

and

$$\begin{aligned} | Q(0) | &= | A_{m,n}(0, 0; P_{mn}(\alpha), P_{mn}(\beta)) | \, | A_{r-m,s-n}(0, 0; R_{mn}(\alpha), R_{mn}(\beta)) | \\ &= H_{m,n} H_{r-m,s-n} , \end{aligned}$$

we have

$$H_{r,s} \geqq H_{m,n} H_{r-m,s-n} .$$

LEMMA 4. *There is an infinite subsequence* $S = \{(m_j, n_j)\colon j = 1, 2, 3, \cdots\}$ *such that*

$$\text{(i)} \qquad H = \lim_{j\to\infty} H_{m_j,n_j}^{1/(m_j+n_j)}$$

and

$$\text{(ii)} \qquad H_{m_j,n_j}^{1/(m_j+n_j)} \geqq H_{p,q}^{1/(p+q)}$$

for all p *and* q *such that* $p + q \leqq m_j + n_j$.

Proof. If there is a pair (r, s) such that $H_{r,s}^{1/(r+s)} = H$, then (2.9) implies

$$H \geqq H_{jr,js}^{1/j(r+s)} \geqq (H_{r,s}^j)^{1/j(r+s)} = H_{r,s}^{1/(r+s)} = H$$

for $j = 1, 2, 3, \cdots$. In this case we take $S = \{(jr, js)\colon j = 1, 2, 3, \cdots\}$.

Suppose, on the other hand, that $H > H_{r,s}^{1/(r+s)}$ for all r and s. For each positive integer k, let

$$T_k = \max_{p+q=k} H_{p,q}^{1/(p+q)} .$$

Then $T_k < H (1 \leqq k < \infty)$ and $\sup_{1\leqq k<\infty} T_k = H$. We can therefore find a subsequence $\{T_{k_j}\}_{j=1}^{\infty}$ with the properties that

$$\lim_{j\to\infty} T_{k_j} = H$$

and

$$T_{k_j} > T_n$$

for $n < k_j$. For each j, choose integers m_j and n_j such that $m_j + n_j = k_j$ and $T_{k_j} = H_{m_j,n_j}^{1/(m_j+n_j)}$, and let $S = \{(m_j, n_j)\colon j = 1, 2, 3, \cdots\}$. This completes the proof of the lemma.

COROLLARY 2. $H = \limsup_{m+n\to\infty} H_{m,n}^{1/(m+n)}$.

LEMMA 5. *For each pair of nonnegative integers* (m, n) *we have*

$$\text{(2.10)} \qquad H_{m,n} \leqq (2/\log 2)^{m+n} .$$

Proof. The result is trivial if $m + n = 0$. Let N be a positive integer and suppose (2.10) holds whenever $m + n < N$. Let r and s be nonnegative integers such that $r + s = N$. The defining relations (1.4) imply

$$
\begin{aligned}
H_{r,s} &\leqq \sum_{\substack{p=0 \\ p+q<r+s}}^{r} \sum_{q=0}^{s} \frac{H_{p\,q}}{(r-p)!\,(s-q)!} = \sum_{\substack{j=0 \\ j+k>0}}^{r} \sum_{k=0}^{s} \frac{H_{r-j,s-k}}{j!\,k!} \\
&\leqq \sum_{\substack{j=0 \\ j+k>0}}^{r} \sum_{k=0}^{s} \frac{(2/\log 2)^{r-j+s-k}}{j!\,k!} \\
&= (2/\log 2)^{r+s}\left\{\sum_{j=0}^{r} \sum_{k=0}^{s} \frac{((\log 2)/2)^{j+k}}{j!\,k!} - 1\right\} \\
&< (2/\log 2)^{r+s}\left\{\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{((\log 2)/2)^{j+k}}{j!\,k!} - 1\right\} \\
&= (2/\log 2)^{r+s}\{e^{(2\log 2)/2} - 1\} = (2/\log 2)^{r+s} .
\end{aligned}
$$

COROLLARY 3. $H \leqq (2/\log 2)$.

Note that this result, together with Corollary 1, implies Džrbašjan's estimate $\mathscr{W} \geqq (\log 2)/2$.

3. Main Results.

Let

$$M(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{H_{p,q}} \frac{z_1^p z_2^q}{p!\,q!} .$$

Note that $M(z_1, z_2)$ is an entire function of exponential type 1 or less. Suppose α and β have entries lying in $|z| \leqq 1$. By (2.6),

$$A_{r,s}(z_1, z_2; \alpha, \beta) = \sum_{p=0}^{r} \sum_{q=0}^{s} A_{r-p,s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta)) \frac{z_1^p z_2^q}{p!\,q!} .$$

Since

$$| A_{r-p,s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta)) | \leqq H_{r-p,s-q} \leqq H_{r,s}/H_{p,q} ,$$

it follows that the coefficients of $A_{r\,s}$ are bounded by the respective coefficients of $H_{r,s}M(z_1, z_2)$; i.e., $A_{r\,s}$ is *majorized* by $H_{r,s}M(z_1, z_2)$. In particular,

$$| A_{r,s}(z_1, z_2; \alpha, \beta) | \leqq H_{r,s}M(| z_1 |, | z_2 |) . \tag{3.1}$$

We are now ready to prove Theorem 1.

Suppose f is an entire function, with $\tau(f) < 1/H$, and suppose α and β are matrices whose entries lie in $|z| \leqq 1$. In order to justify the expansion (2.1) we show that the series

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} | f^{(r,s)}(0, 0) | \sum_{p=0}^{r} \sum_{q=0}^{s} \frac{| A_{p,q}(z_1, z_2; \alpha, \beta) |}{(r-p)!\,(s-q)!} \tag{3.2}$$

is convergent. Equation (3.1) implies

$$| A_{p,q}(z_1, z_2; \alpha, \beta) | \leqq H_{p\ q}M(| z_1 |, | z_2 |) \leqq H_{r\ s}M(| z_1 |, | z_2 |)/H_{r-p\ s-q} ;$$

therefore

$$\begin{aligned} &\sum_{p=0}^{r}\sum_{q=0}^{s} \frac{| A_{p\ q}(z_1, z_2; \alpha, \beta) |}{(r-p)!\,(s-q)!} \\ &\leqq H_{r\ s}M(| z_1 |, | z_2 |) \sum_{p=0}^{r}\sum_{q=0}^{s} \frac{1}{H_{r-p\ s-q}(r-p)!\,(s-q)!} \\ &< H_{r\ s}M(| z_1 |, | z_2 |)M(1, 1) . \end{aligned}$$

The series (3.2) is therefore convergent provided that

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} | f^{(r,s)}(0, 0) | H_{r\ s} \tag{3.3}$$

converges. Choose $\varepsilon > 0$ such that $\tau(f) + \varepsilon < 1/H$ and let N be a positive integer such that $r + s \geqq N$ implies

$$| f^{(r,s)}(0, 0) |^{1/(r+s)} < \tau(f) + \varepsilon .$$

Then

$$\sum_{r+s\geqq N}\sum | f^{(r,s)}(0, 0) | H_{r,s} \leqq \sum_{r+s\geqq N}\sum [H(\tau(f) + \varepsilon)]^{r+s} .$$

Let $\rho = H(\tau(f) + \varepsilon)$ and $K = \sum\sum_{r+s<N} | f^{(r,s)}(0, 0) | H_{r,s}$. Then (3.3) is less than

$$K + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \rho^{r+s} = K + \frac{1}{(1-\rho)^2}$$

and the convergence of (3.2) follows.

Proof of Theorem 2. Let $S = \{(m_j, n_j): j = 1, 2, 3, \cdots\}$ be an infinite sequence such that

$$H = \lim_{j\to\infty} H_{m_j,n_j}^{1/(m_j+n_j)}$$

and

$$H_{m_j,n_j}^{1/(m_j+n_j)} \geqq H_{p,q}^{1/(p+q)}$$

for all p and q such that $p + q \leqq m_j + n_j$. For each $(r, s) \in S$, let $\alpha = \alpha(r, s)$ and $\beta = \beta(r, s)$ be matrices with entries on $| z | = 1$ such that

$$| A_{r,s}(0, 0; \alpha, \beta) | = H_{r,s} .$$

Let

$$P_{r,s}(z_1, z_2) = \frac{A_{r,s}(z_1, z_2; \alpha, \beta)}{A_{r,s}(0, 0; \alpha, \beta)}$$

and

$$Q_{r,s}(z_1, z_2) = P_{r,s}\left(\frac{z_1 H_{r,s}^{1/(r+s)}}{H}, \frac{z_2 H_{r,s}^{1/(r+s)}}{H}\right).$$

Then $Q_{r,s}(0, 0) = P_{r,s}(0, 0) = 1$, and

$$Q_{r,s}^{(j,k)}\left(\frac{H\alpha_{jk}}{H_{r,s}^{1/(r+s)}}, \frac{H\beta_{jk}}{H_{r,s}^{1/(r+s)}}\right) = 0 \qquad (j < r, k < s), \tag{3.4}$$

Moreover, (2.6) implies

$$Q_{r,s}(z_1, z_2) = \sum_{p=0}^{r}\sum_{q=0}^{r} \frac{A_{r-p,s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta))H_{r,s}^{(p+q)/(r+s)}}{A_{r,s}(0, 0; \alpha, \beta)H^{p+q}} \frac{z_1^p z_2^q}{p!\,q!}$$

and

$$\left|\frac{A_{r-p,s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta))H_{r,s}^{(p+q)/(r+s)}}{A_{r,s}(0, 0; \alpha, \beta)H^{p+q}}\right|$$
$$\leqq \frac{H_{r-p,s-q}H_{r,s}^{(p+q)/(r+s)}}{H_{r,s}H^{p+q}} \leqq \frac{H_{r,s}^{(r-p+s-q)/(r+s)}H_{r,s}^{(p+q)/(r+s)}}{H_{r,s}H^{p+q}} = \frac{1}{H^{p+q}},$$

since $(r, s) \in S$. Therefore $Q_{r,s}$ is majorized by

$$\varphi(z_1, z_2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{1}{H^{p+q}} \frac{z_1^p z_2^q}{p!\,q!};$$

$\varphi(z_1, z_2)$ is an entire function of exponential type $1/H$. The sequence $\{Q_{m_j,n_j}\}$ is therefore uniformly bounded on compact sets. Extract a uniformly convergent subsequence from $\{Q_{m_j,n_j}\}$ and let F denote the limit function. Then F is entire, $F(0, 0) = 1$, and $\tau(F) \leqq 1/H$. Since $F^{(j,k)}$ is the uniform limit of a subsequence of $\{Q_{m_j,n_j}^{(j,k)}\}$, then (3.4) implies that $F^{(j,k)}$ has a zero in $\{|z_1| = 1, |z_2| = 1\}$. The expansion (1.5) implies that F has exponential type exactly $1/H$, and this completes the proof.

4. The Whittaker Constants W and $\mathscr{W}$. We have already seen that $\mathscr{W} < W$. The following result provides a precise relationship between $\mathscr{W}$ and W, and a determination of W different from [3] and [1].

THEOREM 3.
$$\limsup_{m+n\to\infty} H_{m,n}^{1/(m+n)} = 1/\mathscr{W},$$
$$\liminf_{m+n\to\infty} H_{m,n}^{1/(m+n)} = 1/W.$$

Proof. The first equation is a consequence of Corollary 1 and Corollary 2. To prove the second, we require the use of the Gončarov polynomials $G_n(z; z_0, \cdots, z_{n-1})$ and the sequence

$$H_n = \max | G_n(0; z_0, \cdots, z_{n-1}) | .$$

If m is a positive integer, the defining relation (1.4) implies

$$A_{m,0}(0, 0; \alpha, \beta) = -\sum_{p=0}^{m-1} \frac{A_{p,0}(0, 0; \alpha, \beta)\alpha_{p,0}^{m-p}}{(m-p)!} . \tag{4.1}$$

In comparing (4.1) with (1.1), one sees that

$$A_{m,0}(0, 0; \alpha, \beta) = G_m(0; \alpha_{00}, \alpha_{10}, \cdots, \alpha_{m-1,0}) .$$

It follows that $H_{m,0} = H_m$ and, similarly, $H_{0,m} = H_m$. By Lemma 3 and (1.2), we have

$$\begin{aligned} H_{m,n}^{1/(m+n)} &\geqq (H_{m,0}H_{0,n})^{1/(m+n)} = (H_m H_n)^{1/(m+n)} \\ &> \left(\frac{.16}{W^{m+n}}\right)^{1/(m+n)} = \frac{(.16)^{1/(m+n)}}{W} . \end{aligned}$$

Therefore

$$\liminf_{m+n\to\infty} H_{m,n}^{1/(m+n)} \geqq 1/W .$$

In the other direction,

$$\liminf_{m+n\to\infty} H_{m,n}^{1/(m+n)} \leqq \liminf_{m+0\to\infty} H_{m,0}^{1/(m+0)} = \lim_{m\to\infty} H_m^{1/m} = 1/W ,$$

and this completes the proof.

Using (2.10) and the estimate $W < .7378$, one easily obtains an interesting bound on $\mathscr{W}$. For all r and s, we have

$$H_{r,s} \leqq (2/\log 2)^{r+s} < \left(\frac{2}{\log 2}\frac{.7378}{W}\right)^{r+s} < \left(\frac{2.13}{W}\right)^{r+s}$$

and therefore

$$W > \mathscr{W} \geqq \frac{W}{2.13} .$$

Some remarks should be made relative to stating the above results in terms of k complex variables, $k > 2$. For $j = 1, 2, \cdots, k$, let $\alpha^{(j)} = (\alpha^{(j)}_{n_1, n_2 \cdots, n_k})$ denote a k-parameter sequence of complex numbers. The recursion relation corresponding to (1.4) is

$$A_{0,0,\cdots 0}(z_1, z_2, \cdots, z_k) = 1$$

and

$$A_{n_1,n_2,\cdots,n_k}(z_1, z_2, \cdots, z_k)$$
$$= \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1! \cdots n_k!} - \sum_{p_1=0}^{n_1} \cdots \sum_{p_k=0}^{n_k}$$
$$\times \frac{A_{p_1,\cdots,p_k}(z_1, \cdots, z_k)[\alpha^{(1)}_{p_1,\cdots,p_k}]^{n_1-p_1} \cdots [\alpha^{(k)}_{p_1,\cdots,p_k}]^{n_k-p_k}}{(n_1 - p_1)! \cdots (n_k - p_k)!}$$

where $p_1 + \cdots + p_k < n_1 + \cdots + n_k$.

The numbers $H_{n_1,\cdots,n_k}$ are also defined in the obvious way and we have

$$H_{n_1,\cdots,n_k} \geqq H_{m_1,\cdots,m_k} H_{n_1-m_1,\cdots,n_k-m_k} ,$$
$$H_{n_1,\cdots,n_l,0,\cdots,0} = H_{n_1,\cdots,n_l} .$$

The definition of $\mathscr{W}_k$, the Whittaker constant in k complex variables, is analogous to the definition of $\mathscr{W}$ in § 1. Apart from notational difficulties, it is a direct extension of the above results to see that

$$\limsup H_{n_1,\cdots,n_k}^{1/(n_1+\cdots+n_k)} = 1/ \mathscr{W}_k$$

and

$$\liminf H_{n_1,\cdots,n_k}^{1/(n_1+\cdots+n_k)} = 1/W .$$

If $1 \leqq l \leqq k$, we also have

$$\limsup H_{n_1,\cdots,n_l,0,\cdots,0}^{1/(n_1+\cdots+n_l)} = 1/ \mathscr{W}_l$$

and

$$\liminf H_{n_1,\cdots,n_l,0,\cdots,0}^{1/(n_1+\cdots+n_l)} = 1/W ,$$

and it follows that $\mathscr{W} = \mathscr{W}_2 \geqq \mathscr{W}_3 \geqq \mathscr{W}_4 \geqq \cdots$.

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