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SCHLICHT MAPPINGS AND INFINITELY DIVISIBLE KERNELS

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The purpose of this note is to give a simple condition which is sufficient for a function on a real interval to be the boundary value of a schlicht (univalent) analytic mapping of the upper half plane into itself. This condition leads to a simple transformation which takes (possibly) non-schlicht mappings into schlicht ones. The methods used have applications to probability theory as well; they yield an interesting class of infinitely divisible characteristic functions.

We shall require some facts about infinitely divisible kernels; for a detailed exposition see [5]. If I is a real interval we denote by $L_0(I)$ the set of all continuous complex valued functions which have compact support in I and whose integral over I vanishes. A continuous kernel $K(x, y)$ on $I \times I$ is said to be *conditionally positive definite on* I if

$$(1) \qquad \iint_{I\times I} K(x, y)\phi(x)\bar{\phi}(y)dxdy \geqq 0$$

for all functions $\phi \in L_0(I)$; it is said to be *positive definite on* I if (1) is satisfied for all continuous functions ϕ with compact support in I; it is said to be *infinitely divisible on* I if (for some fixed continuous determination of the argument) the kernel $K^\alpha(x, y)$ is positive definite for all $\alpha > 0$.

The connection among these concepts is that a continuous Hermitian kernel $K(x, y)$ with no zeroes is infinitely divisible on I if and only if (for some continuous determination of the argument) the kernel $\log K(x, y)$ is conditionally positive definite on I. If $K(x, y) > 0$ for all $x, y \in I$ there is, of course, no difficulty about determining the argument. Finally, the relevance of these notions to function theory is indicated by the following result [6], [4]. If f is a differentiable function we define $K_f(x, y) \equiv [f(x) - f(y)]/(x - y)$ and agree that $K_f(x, x) = f'(x)$.

THEOREM 1. *Let f be a continuously differentiable real valued function with positive derivative on a real interval I. The function f possesses an analytic continuation onto the upper half plane which maps the upper half plane into itself if and only if the kernel $K_f(x, y)$ is positive definite on I. This mapping is schlicht if and only if $K_f(x, y)$ is infinitely divisible on I.*

Although this result completely characterizes the boundary values of schlicht mappings, it is in practice much harder to verify that the kernel $K_f(x, y)$ is infinitely divisible than to test it for positive definiteness. By our remarks above, one must check whether $\log K_f(x, y)$ is conditionally positive definite, but the non-linearity of this expression in f often leads to computational difficulties. In the following, we shall derive a more linear, and hopefully more useful, sufficient condition. Recall that a C^∞ function ϕ defined on $(0, \infty)$ is *completely monotonic* if $(-1)^n\phi^{(n)}(x) \geqq 0$ for all $x > 0$ and all $n = 1, 2, 3, \cdots$.

LEMMA 2. *Let $H(x, y)$ be a continuous Hermitian kernel on a real interval I such that* $\operatorname{Re}\{H(x, y)\} > 0$ *and such that $-H(x, y)$ is conditionally positive definite. If ϕ is any completely monotonic function then the kernel $\phi(H(x, y))$ is positive definite on I.*

Proof. It is well known that a function ϕ is completely monotonic if and only if there exists a nonnegative measure $d\mu$ such that $\phi(x) = \int_0^\infty e^{-xs}d\mu(s)$ for all $x > 0$ ([8], p. 160); in this event ϕ is analytic in the whole right half plane. But since $\operatorname{Re}\{H(x, y)\} > 0$ and $\exp(-sH(x, y))$ is positive definite (even infinitely divisible) for all $s > 0$, it follows that $\phi(H(x, y)) = \int_0^\infty \exp(-sH(x, y))d\mu(s)$ is convergent and is a positive definite kernel.

An *infinitely divisible completely monotonic function* ϕ is a function such that ϕ^α is completely monotonic for all $\alpha > 0$; if $\phi \not\equiv 0$, a necessary and sufficient condition for this is that the derivative of $-ln\phi$ be completely monotonic ([3], p. 229). Using the lemma and the definition of an infinitely divisible kernel we obtain

COROLLARY 3. *Let ϕ be a positive differentiable function on $(0, \infty)$ such that $-\phi'/\phi$ is completely monotonic, and suppose $H(x, y)$ satisfies the hypotheses of Lemma 2. Then $\phi(H(x, y))$ is an infinitely divisible kernel.*

Since the function $\phi(x) = 1/x$ satisfies this condition, the following result is immediate.

COROLLARY 4. *If $H(x, y)$ satisfies the conditions of Lemma 2, then the kernel $1/H(x, y)$ is infinitely divisible.*

Now suppose that g is a continuously differentiable real valued function on a real interval I, so that $K_g(x, y)$ is a continuous symmetric kernel. If $g'(x) > 0$ on I then $K_g(x, y)$ is a positive kernel and the inverse function g^{-1} is defined on the interval $g(I)$. Thus, if we assume that $-K_g(x, y)$ is conditionally positive definite on I then we conclude from Corollary 4 that the kernel

$$\frac{1}{K_g(x, y)} = \frac{x - y}{g(x) - g(y)} = \frac{g^{-1}(g(x)) - g^{-1}(g(y))}{g(x) - g(y)}$$

is infinitely divisible. But this is equivalent to the kernel $K_{g^{-1}}(s, t)$ being infinitely divisible on $g(I)$ and so we may apply Theorem 1 to obtain the conclusion of the following

THEOREM 5. *Let g be a continuously differentiable real valued function with positive derivative on a real interval I and suppose that the kernel*

$$-K_g(x, y) = -\frac{g(x) - g(y)}{x - y}$$

is conditionally positive definite on I. Then the inverse function g^{-1} has an analytic continuation from $g(I)$ onto the upper half plane which is a schlicht mapping of the upper half plane into itself.

Thus, to ensure that a real function f on a real interval I is the boundary value of a schlicht self-mapping of the upper half plane it is sufficient to check that $f'(x) > 0$ and that $-K_{f^{-1}}(x, y)$ is conditionally positive definite on $f^{-1}(I)$.

The crucial condition in Theorem 5 is that the kernel $-K_g(x, y)$ be conditionally positive definite, and a great deal is known about functions which satisfy this condition. For example, they are real analytic and are analytically continuable onto the upper half plane, they have a simple integral representation, and they arise as the infinitesimal transformations of the pseudo-semigroup $\mathfrak{M}_\infty$ of self-mappings of the upper half plane which have real boundary values on I ([6] and [2], pp. 53-54). Furthermore, it is easy to find many non-trivial functions which satisfy this condition. Denote by $\mathfrak{M}_\infty(0)$ the class of functions f which are analytic in the upper half plane, map it into itself, are real valued on some open real interval containing zero, and are normalized by the condition $f(0) = 0$.

LEMMA 6. *Let a be a real number, let $b \geqq 0$ and let $f \in \mathfrak{M}_\infty(0)$. Then the functions $g_0(x) = a$, $g_1(x) = ax$, $g_2(x) = ax^2$, and $g_3(x) = bx^2f(x)$ are such that $K_{g_i}(x, y)$ is conditionally positive definite on some neighborhood of the origin, $i = 0, 1, 2, 3$.*

Proof. This follows from a direct computation for $i = 0, 1, 2$

but for $i = 3$ we need to know ([1], p. 63) that $f \in \mathfrak{M}_\infty(0)$ if and only if

$$f(x) = \int_{-\varepsilon}^{\varepsilon} \frac{x}{1 - tx} d\mu(t) \tag{2}$$

for some $\varepsilon > 0$ and some nonnegative bounded measure $d\mu$ on $[-\varepsilon, \varepsilon]$. Thus, since the assertion for $i = 3$ follows for the special case $f(x) = x/(1 - tx)$ by direct computation, it follows for all $f \in \mathfrak{M}_\infty(0)$ by linearity.

Using the four types of functions introduced in this lemma we can now use Theorem 5 to construct a wide class of schlicht mappings.

THEOREM 7. *Let $f \in \mathfrak{M}_\infty(0)$, let $a_1 > 0$, $a_3 \geqq 0$, and let a_0 and a_2 be real numbers. Then the function*

$$g(x) = a_0 + a_1 x + a_2 x^2 - a_3 x^2 f(x)$$

is such that the inverse function g^{-1} has an analytic continuation from a real neighborhood of a_0 onto the upper half plane which is a schlicht mapping of the upper half plane into itself.

Proof. The kernel $-K_g(x, y)$ is conditionally positive definite by Lemma 6 and $g'(x) > 0$ in some real neighborhood of zero. The result follows from Theorem 5.

Although this construction provides a wealth of schlicht mappings, it is far from exhaustive: the functions $f(z) = 3[\sqrt[3]{z + 1} - 1]$ and $f(z) = \log(z + 1)$ are schlicht mappings which are not of this form.

REMARK 1. Linear combinations of the four functions in Lemma 6 are in fact the *only* smooth functions g such that $K_g(x, y)$ is conditionally positive definite. In order to prove this we use the following criterion for a kernel to be conditionally positive definite.

LEMMA 8. *Let $H(x, y)$ be a continuous kernel on a real interval I and let $x_0 \in I$. Then $H(x, y)$ is conditionally positive definite on I if and only if the kernel*

$$H_{x_0}^*(x, y) \equiv H(x, y) - H(x, x_0) - H(x_0, y) + H(x_0, x_0)$$

is positive definite on I.

Proof. If $\phi \in L_0(I)$, then

$$\iint_{I \times I} H_{x_0}^*(x, y)\phi(x)\bar{\phi}(y)dxdy = \iint_{I \times I} H(x, y)\phi(x)\bar{\phi}(y)dxdy \ ,$$

and hence $H(x, y)$ is conditionally positive definite if $H^*_{x_0}(x, y)$ is positive definite. Conversely, suppose $H(x, y)$ is conditionally positive definite and let $\{f_n(x)\}$, $n = 1, 2, 3, \cdots$ be an approximate identity based at x_0, i.e., each f_n is a nonnegative continuous function with support in $I \cap [x_0 - n^{-1}, x_0 + n^{-1}]$ and $\int_I f_n(x)dx = 1$ for all n. If ϕ is any continuous function with compact support in I, let $\phi_n(x) \equiv \phi(x) - f_n(x)\int_I \phi(t)dt$ and observe that $\phi_n \in L_0(I)$ for all large n. Thus,

$$\begin{aligned}
0 &\leqq \iint_{I\times I} H(x, y)\phi_n(x)\bar{\phi}_n(y)dxdy \\
&= \iint_{I\times I}\Big\{H(x, y) - \int_I H(x, t)f_n(t)dt - \int_I H(s, y)f_n(s)ds \\
&\qquad\qquad + \iint_{I\times I} H(s, t)f_n(s)f_n(t)dsdt\Big\}\phi(x)\bar{\phi}(y)dxdy \\
&\to \iint_{I\times I}\{H(x, y) - H(x, x_0) - H(x_0, y) + H(x_0, x_0)\}\phi(x)\bar{\phi}(y)dxdy \\
&= \iint_{I\times I} H^*_{x_0}(x, y)\phi(x)\bar{\phi}(y)dxdy
\end{aligned}$$

as $n \to \infty$. Since ϕ is arbitrary, we conclude that the kernel $H^*_{x_0}(x, y)$ must be positive definite.

LEMMA 9. *Let $K(x, y)$ be a continuous kernel on a real interval I. Then $xyK(x, y)$ is positive definite kernel if and only if $K(x, y)$ is a positive definite kernel.*

Proof. If zero is not a point of I this is trivial, so suppose $0 \in I$, let $\varepsilon > 0$, and denote by f_ε the unique even function such that

$$f_\varepsilon(x) \equiv \begin{cases} 0 \text{ if } x \in [0, \varepsilon] \\ \varepsilon^{-1}(x - \varepsilon) \text{ if } x \in [\varepsilon, 2\varepsilon] \\ 1 \text{ if } x \geqq \varepsilon\,. \end{cases}$$

Let $M \equiv \sup_{I\times I} |K(x, y)|$, let ϕ be a continuous function with compact support in I, and assume that $xyK(x, y)$ is positive definite on I. Then

$$\begin{aligned}
&\iint_{I\times I} K(x, y)\phi(x)\bar{\phi}(y)dxdy \\
&= \iint_{I\times I} K(x, y)\phi(x)\bar{\phi}(y)(1 - f_\varepsilon(y))dxdy \\
&\quad + \iint_{I\times I} K(x, y)\phi(x)\bar{\phi}(y)f_\varepsilon(y)(1 - f_\varepsilon(x))dxdy \\
&\quad + \iint_{I\times I} K(x, y)\phi(x)f_\varepsilon(x)\bar{\phi}(y)f_\varepsilon(y)dxdy
\end{aligned}$$

$$\geqq -\iint_{I\times I} |K(x, y)\phi(x)\bar{\phi}(y)(1 - f_\varepsilon(y))|\, dxdy$$
$$- \iint_{I\times I} |K(x, y)\bar{\phi}(y)f_\varepsilon(y)\phi(x)(1 - f_\varepsilon(x))|\, dxdy$$
$$+ \iint_{I\times I} xyK(x, y)x^{-1}\phi(x)f_\varepsilon(x)y^{-1}\bar{\phi}(y)f_\varepsilon(y)dxdy$$
$$\geqq -6M\varepsilon \sup_I |\phi(x)| \int_I |\phi(x)|\, dx \; .$$

For the last inequality we have used the hypothesis that $xyK(x, y)$ is positive definite and the fact that the function $x^{-1}\phi(x)f_\varepsilon(x)$ is a continuous function with compact support in I. Since $\varepsilon > 0$ is arbitrary we conclude that

$$\iint_{I\times I} K(x, y)\phi(x)\bar{\phi}(y)dxdy \geqq 0 \; ,$$

i.e., $K(x, y)$ is positive definite. The converse is trivial.

Now assume that the kernel $H(x, y)$ is of the special form $H(x, y) = K_g(x, y)$, where g is a real valued function which is three times continuously differentiable on an open real interval containing zero. Assume that $g(0) = g'(0) = g''(0) = 0$. Then $g(x)/x^2$ is continuously differentiable and

$$H_0^*(x, y) = K_g(x, y) - K_g(x, 0) - K_g(0, y) + K_g(0, 0)$$
$$= xy\frac{\dfrac{g(x)}{x^2} - \dfrac{g(y)}{y^2}}{x - y} = xyK_h(x, y) \; ,$$

where we set $h(x) \equiv g(x)/x^2$. Thus, Lemma 8 says that $K_g(x, y)$ is conditionally positive definite if and only if $xyK_h(x, y)$ is positive definite, and Lemma 9 says this is equivalent to the kernel $K_h(x, y)$ being positive definite. We conclude that $K_g(x, y)$ is conditionally positive definite if and only if $K_h(x, y)$ is positive definite. But this means that $h(x) = g(x)/x^2 \in \mathfrak{M}_\infty(0)$ and hence h has the integral representation (2). The normalization we assumed for g can always be attained by subtracting a suitable quadratic polynomial, since Lemma 6 shows that every such polynomial has a conditionally positive definite difference quotient kernel. We summarize our results as

THEOREM 10. *Let g be real valued function on an open real interval I containing zero. The following are equivalent:*

(a) *The function g is three times continuously differentiable and the kernel $K_g(x, y) = [g(x) - g(y)]/(x - y)$ is conditionally positive* *definite on I.*

(b) *The function g has the form*

$$g(x) = a_0 + a_1x + a_2x^2 + a_3x^2f(x) ,$$

where a_0, a_1, a_2 *are real numbers,* $a_3 \geqq 0$ *and* $f \in \mathfrak{M}_\infty(0)$.

(c) *The function g has the form*

$$g(x) = a_0 + a_1 + a_2x^2 + \int_{-\varepsilon}^{\varepsilon} \frac{x^3 d\mu}{1 - xt}$$

where a_0, a_1, a_2 *are real numbers,* $\varepsilon \geqq 0$ *and* $d\mu$ *is a nonnegative bounded measure.*

It should be noted that it is sufficient in (a) to assume only that g is continuously differentiable; the condition on the kernel then implies that g is analytic [6]. This characterization of the functions g such that $K_g(x, y)$ is conditionally positive definite was obtained first by C. FitzGerald [2] using less elementary results on analytic kernels.

REMARK 2. Lemma 2 and its corollaries are also useful in probability theory where one is interested in continuous Hermitian kernels of the form $K(x, y) = f(x - y)$, $f(0) = 1$. Such a kernel is positive definite if and only if $f(x)$ is the Fourier transform of a (unique) probability measure on the line, i.e., $f(x)$ is a *characteristic function*; this kernel is infinitely divisible if and only if the measure is infinitely divisible. If $f(x)$ is the characteristic function of an infinitely divisible probability measure, then the kernel $H(x, y) = lnf(x - y)$ is conditionally positive definite and has nonpositive real part since $|f(x)| \leqq 1$ for all real x. Thus, the kernel $H(x, y) = \lambda - lnf(x - y)$ satisfies the hypotheses of Lemma 2 if $\lambda > 0$ and hence the kernel

$$\frac{\lambda}{\lambda - lnf(x - y)}$$

is infinitely divisible by Corollary 4. But this means that the function $\phi(x) = \lambda/(\lambda - lnf(x))$ is an infinitely divisible characteristic function whenever f is an infinitely divisible characteristic function and $\lambda > 0$. This result was obtained by F. W. Steutel [7] from very different considerations.

REFERENCES

1. J. Bendat and S. Sherman, *Monotone and convex matrix functions*, Trans. Amer. Math. Soc., **79** (1965), 58-71.

2. C. FitzGerald, *Topics in Geometric Function Theory*, Stanford University PhD Thesis, 1967.
3. R. A. Horn, *On infinitely divisible matrices, kernels, and functions*, Z. Wahrscheinlichkeitstheorie verw. Geb., **8** (1967), 219-230.
4. ———, *On boundary values of a Schlicht mapping*, Proc. Amer. Math. Soc., **18** (1967), 782-787.
5. ———, *The theory of infinitely divisible matrices and kernels*, Trans. Amer. Math. Soc., **136** (1969), 269-286.
6. C. Loewner, *On Schlicht-monotonic functions of higher order*, J. Math. Anal. Appl., **14** (1966), 320-325.
7. F. W. Steutel, *A class of infinitely divisible mixtures*, Ann. Math. Stat., **39** (1968), 1153-1157.
8. D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, N. J., 1946.

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