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MIXED ARITHMETIC AND GEOMETRIC MEANS

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Consider all ordinary arithmetic and geometric means of n real nonnegative numbers taken k at a time. Let α_k be the geometric mean of all the arithmetic means and γ_k the arithmetic mean of all the geometric means. It is proved that α_k increases with k, γ_k decreases, and $\gamma_h \leqq \alpha_k$ if $h + k > n$. These results are generalized to mixed means of any real orders. Comparison of α_k and γ_k with elementary symmetric functions suggests a conjecture.

1. Introduction and summary. If x, y, z are real nonnegative numbers, then [2], [3]

$$(1.1) \qquad \frac{1}{3}[(xy)^{1/2} + (xz)^{1/2} + (yz)^{1/2}] \leqq \left[\frac{x+y}{2}\,\frac{x+z}{2}\,\frac{y+z}{2}\right]^{1/3}.$$

The inequality was proved originally to demonstrate that the right-hand side is the better approximation to the capacity of an ellipsoid with semiaxes x, y, z. To generalize (1.1) from three to n variables, $n \geqq 3$, consider the real nonnegative numbers $x_1, x_2, \cdots, x_n$ and form all ordinary (as opposed to weighted) arithmetic and geometric means of these numbers taken $n-1$ at a time. Then the arithmetic mean of the geometric means does not exceed the geometric mean of the arithmetic means [3], [4].

In the present note we consider arithmetic and geometric means of $x_1, \cdots, x_n$ taken k at a time, $k = 1, 2, \cdots, n$. Let α_k denote the geometric mean of the arithmetic means, and γ_k the arithmetic mean of the geometric means. For example, if $n = 4$, we have

$$(1.2) \qquad \begin{aligned} \alpha_2 &= \left[\frac{x_1+x_2}{2}\,\frac{x_1+x_3}{2}\,\frac{x_1+x_4}{2}\,\frac{x_2+x_3}{2}\,\frac{x_2+x_4}{2}\,\frac{x_3+x_4}{2}\right]^{1/6},\\ \gamma_3 &= \tfrac{1}{4}[(x_1x_2x_3)^{1/3} + (x_1x_2x_4)^{1/3} + (x_1x_3x_4)^{1/3} + (x_2x_3x_4)^{1/3}]\,. \end{aligned}$$

Note that $\alpha_n = \gamma_1$ is the arithmetic mean of $x_1, \cdots, x_n$, and $\alpha_1 = \gamma_n$ is the geometric mean. We shall show first (Theorem 1) that α_k increases with k while γ_k decreases. The second result (Theorem 2) states that $\gamma_h \leqq \alpha_k$ if $h + k > n$. If $n = 3$, for instance, we have $\gamma_2 \leqq \alpha_2$, which is (1.1), and if $n = 4$, as in (1.2), we have $\gamma_3 \leqq \alpha_2$. The case $h = k = n$ is the ordinary inequality of the arithmetic and geometric means, and the case $h = k = n - 1$, $n > 2$, is the inequality stated in [3].

More generally we shall define a mixed mean $M_{st}(k)$ of the real nonnegative numbers $x_1, \cdots, x_n$ taken k at a time. As k increases from 1 to n, $M_{st}(k)$ increases or decreases monotonically, according as $s < t$ or $s > t$, from the mean of order s of $x_1, \cdots x_n$ to the mean of order t. If $s \leqq t$ and $h + k > n$, then $M_{ts}(h) \leqq M_{st}(k)$.

Maclaurin's theorem for the elementary symmetric functions $p_1, \cdots, p_n$ suggests allowing s or t to depend on k, as in $M_{k0}(k) = p_k^{1/k}$. The inequality $\gamma_k \leqq p_k^{1/k}$ is very easy to prove, and it is conjectured that $p_h^{1/h} \leqq \alpha_k$ if $h + k > n$.

2. Monotony.

We first define α_k and γ_k and then show that each is monotonic in k.

DEFINITION 1. If $V = \{y_1, y_2, \cdots, y_k\}$, where $k \geqq 1$ and $y_1, \cdots, y_k$ are real nonnegative numbers, we define $A(V)$ and $G(V)$ to be the arithmetic and geometric means of the elements of V:

$$A(V) = \frac{1}{k}(y_1 + \cdots + y_k)\ , \quad G(V) = (y_1 \cdots y_k)^{1/k}\ . \tag{2.1}$$

DEFINITION 2. Let $x_1, \cdots, x_n$ be real nonnegative numbers and let k be a positive integer not exceeding n. The subsets of $\{x_1, \cdots, x_n\}$ with cardinal number k will be denoted by $V_1, V_2, \cdots, V_v$, where $v = \binom{n}{k}$. We define

$$\alpha_k = \left[\prod_{i=1}^{v} A(V_i)\right]^{1/v}, \qquad \gamma_k = \frac{1}{v}\sum_{i=1}^{v} G(V_i)\ . \tag{2.2}$$

THEOREM 1. *Let $x_1, \cdots, x_n$ be real nonnegative numbers and let the means $\alpha_1, \cdots, \alpha_n$ and $\gamma_1, \cdots, \gamma_n$ be defined by Definition 2. Then $\alpha_{k-1} \leqq \alpha_k$, $k = 2, 3, \cdots, n$, with strict inequality unless either* (1) *$x_1 = x_2 = \cdots = x_n$ or* (2) *k of the x_i are zero. Also, $\gamma_{k-1} \geqq \gamma_k$, $k = 2, 3, \cdots, n$, with strict inequality unless either* (1) *$x_1 = x_2 = \cdots = x_n$ or* (2) *$n - k + 2$ of the x_i are zero.*

Proof. Let $W_1, W_2, \cdots, W_w$, $w = \binom{n}{k-1}$, be the subsets of $\{x_1, \cdots, x_n\}$ with cardinal number $k - 1$. Also, let V_i, $i = 1, \cdots, v$, be defined by Definition 2 and let the subsets of V_i with cardinal number $k - 1$ be denoted by $V_{i1}, \cdots, V_{ik}$. Then each set V_{ij} equals one of the sets $W_1, \cdots, W_w$, and each of the W_i occurs exactly $n - k + 1$ times among the sets $V_{11}, \cdots, V_{vk}$. Note that $w(n - k + 1) = vk$.

Using the inequality of the arithmetic and geometric means, we have

$$(2.3)\qquad A(V_i) = \frac{1}{k}\sum_{j=1}^{k} A(V_{ij}) \geqq \left[\prod_{j=1}^{k} A(V_{ij})\right]^{1/k}, \qquad i = 1, \cdots, v ,$$

with strict inequality unless all the elements of V_i are equal. Hence there is strict inequality for some value of i unless $x_1 = x_2 = \cdots = x_n$. It follows that

$$(2.4)\qquad \alpha_k = \left[\prod_{i=1}^{v} A(V_i)\right]^{1/v} \geqq \left[\prod_{i=1}^{v}\prod_{j=1}^{k} A(V_{ij})\right]^{1/vk},$$

with strict inequality unless either (1)$x_1 = x_2 = \cdots = x_n$ or (2) k of the x_i are zero (with the result that $A(V_i) = 0$ for some value of i). The third member of (2.4) equals

$$(2.5)\qquad \left[\prod_{j=1}^{w} A(W_i)\right]^{(n-k+1)/vk} = \left[\prod_{i=1}^{w} A(W_i)\right]^{1/w} = \alpha_{k-1} .$$

Similarly we have

$$(2.6)\qquad G(V_i) = \left[\prod_{j=1}^{k} G(V_{ij})\right]^{1/k} \leqq \frac{1}{k}\sum_{j=1}^{k} G(V_{ij}) , \qquad i = 1, \cdots, v ,$$

with strict inequality unless either (i) all the elements of V_i are equal or (ii) two elements of V_i are zero. Hence there is strict inequality for some value of i unless either (1) $x_1 = x_2 = \cdots = x_n$ or (2) $n - k + 2$ of the x_i are zero (with the result that two elements of V_i are zero for each value of i). It now follows that

$$(2.7)\qquad \gamma_k = \frac{1}{v}\sum_{i=1}^{v} G(V_i) \leqq \frac{1}{vk}\sum_{i=1}^{v}\sum_{j=1}^{k} G(V_{ij}) ,$$

with strict inequality unless (1) or (2) holds. The third member of (2.7) equals

$$(2.8)\qquad \frac{n-k+1}{vk}\sum_{i=1}^{w} G(W_i) = \frac{1}{w}\sum_{i=1}^{w} G(W_i) = \gamma_{k-1} .$$

3. Inequality of mixed means. Let $V_1, V_2, \cdots, V_v, v = \binom{n}{k}$, be defined by Definition 2, and similarly denote the subsets of $\{x_1, \cdots, x_n\}$ with cardinal number h by $U_1, U_2, \cdots, U_u, u = \binom{n}{h}$.

LEMMA 1. *If $h + k > n$, then*

$$(3.1)\qquad \frac{1}{u}\sum_{i=1}^{u} A(U_i \cap V_j) = A(V_j) , \qquad j = 1, \cdots, v ,$$

$$(3.2)\qquad \left[\prod_{j=1}^{v} G(U_i \cap V_j)\right]^{1/v} = G(U_i) , \qquad i = 1, \cdots, u .$$

Proof. The condition $h + k > n$ implies that $U_i \cap V_j$ is nonvoid for all i and j. Each term on the left side of (3.1) is a linear combination of the elements of V_j, and so is the sum. Since the summation extends over all subsets U_i with cardinal number h, the sum is unchanged by permuting the elements of V_j and must therefore be a constant multiple of $A(V_j)$. The multiplier is determined to be unity by choosing all the x_i to be equal.

Each factor on the left side of (3.2) is a product of powers of the elements of U_i, and so is the product of the factors. Since there is one factor for every subset V_j with cardinal number k, the product is unchanged by permuting the elements of U_i and must therefore be a power of $G(U_i)$. The power is determined to be unity by choosing all the x_i to be equal.

THEOREM 2. *Let $x_1, \cdots, x_n$ be real nonnegative numbers and let the means $\alpha_1, \cdots, \alpha_n$ and $\gamma_1, \cdots, \gamma_n$ be defined by Definition 2. Then $\gamma_h \leqq \alpha_k$ if $h + k > n$. The inequality is strict unless either* (1) $(h-1)(k-1) = 0$ *or* (2) $x_1 = x_2 = \cdots = x_n$ *or* (3) k *of the x_i are zero.*

Proof. One form of Hölder's inequality states that if $(a), (b), \cdots, (s)$ are u-tuples of real nonnegative numbers and $\alpha, \beta, \cdots, \sigma$ are positive weights with $\alpha + \beta + \cdots + \sigma = 1$, then [5, p. 22]

$$\sum_{i=1}^{u} a_i^\alpha b_i^\beta \cdots s_i^\sigma \leqq \left(\sum_{i=1}^{u} a_i\right)^\alpha \left(\sum_{i=1}^{u} b_i\right)^\beta \cdots \left(\sum_{i=1}^{u} s_i\right)^\sigma . \tag{3.3}$$

(The proof of this inequality consists in dividing each term on the left side by the right side and using the inequality of the arithmetic and geometric means.) Let $U_1, \cdots, U_u$ and $V_1, \cdots, V_v$ have the same meaning as in Lemma 1, assume $h + k > n$, and introduce the abbreviations

$$G_i = G(U_i),\ A_j = A(V_j),\ G_{ij} = G(U_i \cap V_j),\ A_{ij} = A(U_i \cap V_j) . \tag{3.4}$$

We now choose

$$\begin{bmatrix} a_1 \cdots s_1 \\ \vdots \quad \vdots \\ a_u \cdots s_u \end{bmatrix} = \begin{bmatrix} G_{11} \cdots G_{1v} \\ \vdots \quad \vdots \\ G_{u1} \cdots G_{uv} \end{bmatrix} \tag{3.5}$$

and put $\alpha = \beta = \cdots = \sigma = 1/v$. Hölder's inequality becomes

$$\sum_{i=1}^{u} G_i = \sum_{i=1}^{u} \prod_{j=1}^{v} G_{ij}^{1/v} \leqq \prod_{j=1}^{v} \left(\sum_{i=1}^{u} G_{ij}\right)^{1/v} , \tag{3.6}$$

where the equality of the first two members follows from Lemma 1. The inequality of the arithmetic and geometric means implies

$$\sum_{i=1}^{u} G_{ij} \leqq \sum_{i=1}^{u} A_{ij} = uA_j\,, \qquad j = 1, \cdots, v\,, \tag{3.7}$$

with strict inequality for some value of j unless either (1) $(h-1)(k-1)=0$ or (2) $x_1 = x_2 = \cdots = x_n$. The equality of the second and third members follows from Lemma 1. Hence

$$\prod_{j=1}^{v}\left(\sum_{i=1}^{u} G_{ij}\right)^{1/v} \leqq u \prod_{j=1}^{v} A_j^{1/v}\,, \tag{3.8}$$

with strict inequality unless either (1) $(h-1)(k-1)=0$ or (2) $x_1 = x_2 = \cdots = x_n$ or (3) k of the x_i are zero (with the result that $A_j = 0$ for some value of j). We have finally

$$\gamma_h = \frac{1}{u}\sum_{i=1}^{u} G_i \leqq \prod_{j=1}^{v} A_j^{1/v} = \alpha_k\,, \qquad h + k > n\,, \tag{3.9}$$

with obvious equality if (1) or (2) or (3) holds and strict inequality otherwise.

4. Mixed means of general orders. No extension of these results to weighted means is known. However, it is not difficult to generalize the inequalities for mixed arithmetic and geometric means to mixed means of any real orders, and we shall sketch briefly what modifications are necessary. Definition 1 is augmented by defining the mean of any real order t,

$$\begin{aligned} M_t(V) &= \left(\frac{1}{k}\sum_{i=1}^{k} y_i^t\right)^{1/t}, \qquad t \neq 0\,, \\ M_0(V) &= G(V)\,. \end{aligned} \tag{4.1}$$

Note that $A(V) = M_1(V)$. Similarly we augment Definition 2 by defining the mixed mean of orders s and t of the real nonnegative numbers $x_1, \cdots, x_n$ taken k at a time:

$$\begin{aligned} M_{st}(k) &= \left\{\frac{1}{v}\sum_{i=1}^{v}\Big[M_t(V_i)\Big]^s\right\}^{1/s}, \qquad s \neq 0\,, \\ M_{0t}(k) &= \left[\prod_{i=1}^{v} M_t(V_i)\right]^{1/v}. \end{aligned} \tag{4.2}$$

Note that $\alpha_k = M_{01}(k)$ and $\gamma_k = M_{10}(k)$. Also, $M_{st}(1)$ is the mean of order s of $x_1, \cdots, x_n$ and $M_{st}(n)$ is the mean of order t. For fixed k, $M_{st}(k)$ increases with s and with t because of a basic property of mean values [5, p. 26]. Finally, $M_{st}(k)$ is nonzero unless either (1) $s \leqq 0, t \leqq 0$, and one of the x_i is zero, or (2) $s \leqq 0, t > 0$, and k of the x_i are zero, or (3) $s > 0, t \leqq 0$, and $n - k + 1$ of the x_i are zero, or (4) $s > 0, t > 0$, and all the x_i are zero.

The generalization of Theorem 1 states that $M_{st}(k)$ increases or decreases with increasing k according as $s < t$ or $s > t$. In both cases $M_{st}(k)$ and $M_{st}(k-1)$ are unequal unless either (1) both are zero or (2) $x_1 = x_2 = \cdots = x_n$. If $s = t$ the mixed mean is independent of k:

$$M_{tt}(k) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^t\right)^{1/t}, \qquad k = 1, \cdots, n . \tag{4.3}$$

Here and subsequently, the cases in which s or t is zero are not written explicitly but may be included by agreeing that an appropriate geometric mean is intended or by taking the limit as $s \to 0$ or $t \to 0$. The proof of monotony in the general case is quite similar to the proof of Theorem 1, the extension of (2.3) being

$$M_t(V_i) = \left\{\frac{1}{k}\sum_{j=1}^{k} [M_t(V_{ij})]^t\right\}^{1/t} \geqq \left\{\frac{1}{k}\sum_{j=1}^{k} [M_t(V_{ij})]^s\right\}^{1/s}, \qquad t > s . \tag{4.4}$$

The inequality, which is reversed if $t < s$, holds because a mean of order s increases with s [5, p. 26].

The generalization of Lemma 1 states that

$$\left\{\frac{1}{u}\sum_{i=1}^{u} [M_t(U_i \cap V_j)]^t\right\}^{1/t} = M_t(V_j) , \qquad h + k > n . \tag{4.5}$$

The same sequence of steps as in the proof of Theorem 2 can now be followed in the general case, with Hölder's inequality replaced by Jessen's form of Minkowski's inequality [5, p. 31] and with the inequality $G \leqq A$ in (3.7) replaced by $M_s \leqq M_t$ if $s \leqq t$. The result is

$$M_{ts}(h) \leqq M_{st}(k) , \qquad s \leqq t, h + k > n . \tag{4.6}$$

The inequality is strict unless either (1) $s = t$ or (2) $(h-1)(k-1) = 0$ or (3) $x_1 = x_2 = \cdots = x_n$ or (4) $s \leqq t \leqq 0$ and one of the x_i is zero or (5) $s \leqq 0, t > 0$, and k of the x_i are zero.

5. Comparison with elementary symmetric functions. Let p_k denote the elementary symmetric function of degree k of the real nonnegative numbers $x_1, \cdots, x_n$, multiplied by a normalizing constant to make $p_k = 1$ if $x_1 = \cdots = x_n = 1$. For example, if $n = 3$ and $(x_1, x_2, x_3) = (x, y, z)$, we have

$$p_1 = \tfrac{1}{3}(x + y + z) , \qquad p_2 = \tfrac{1}{3}(xy + yz + zx) , \qquad p_2 = xyz , \tag{5.1}$$

and for any n it follows from (4.2) that

$$p_k^{1/k} = M_{k0}(k) . \tag{5.2}$$

The counterpart of Theorem 1 is Maclaurin's theorem [5, p. 52], which states that $p_k^{1/k}$ is a decreasing function of k. Let us try to compare three increasing sequences of mean values, each beginning with the geometric mean and ending with the arithmetic mean:

$$\begin{aligned} &\alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_{n+1-k} \leqq \cdots \leqq \alpha_{n-1} \leqq \alpha_n \,, \\ &p_n^{1/n} \leqq p_{n-1}^{1/(n-1)} \leqq \cdots \leqq p_k^{1/k} \leqq \cdots \leqq p_2^{1/2} \leqq p_1 \,, \\ &\gamma_n \leqq \gamma_{n-1} \leqq \cdots \leqq \gamma_k \leqq \cdots \leqq \gamma_2 \leqq \gamma_1 \,. \end{aligned} \tag{5.3}$$

If $n = 3$ we have $\gamma_2 \leqq p_2^{1/2} \leqq \alpha_2$. The inequality of the first and third members is (1.1), the inequality of the first and second members follows at once from the inequality of the arithmetic mean and the root-mean-square, and the inequality of the second and third members is proved in several steps in [2].

For all n and k, Theorem 2 implies $\gamma_k \leqq \alpha_{n+1-k}$. Also, since a mean value increases with its order [5, p. 26] we have $M_{10}(k) \leqq M_{k0}(k)$ and hence

$$\gamma_k \leqq p_k^{1/k} \,, \qquad k = 1, \cdots, n \,. \tag{5.4}$$

The known results for $n = 3$ suggest the conjecture that $p_k^{1/k} \leqq \alpha_{n+1-k}$. In view of (5.3) it is equivalent to conjecture that $p_h^{1/h} \leqq \alpha_k$ if $h + k > n$. Were this proved true, Theorem 2 would follow as a corollary because of (5.4).

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