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INDICES FOR THE ORLICZ SPACES

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The determination of the function spaces X which are intermediate in the weak sense between L^p and L^q has been shown, by the author, to depend on a pair of numbers (α, β) called the indices of the space. The indices depend on the function norm of X and on the properties of the underlying measure space: whether it has finite or infinite measure, is non-atomic or atomic. In this paper, formulas are given for the indices of an Orlicz space in case the measure space is non-atomic with finite or infinite measure, or else is purely atomic with atoms of equal measure. The indices for an Orlicz space over a non-atomic finite measure space turn out to be the reciprocals of the exponents of the space as introduced by Matuszewska and Orlicz, and generalized by Shimogaki. Some new results concerning submultiplicative functions are used in the proof of the main result.

DEFINITIONS AND MAIN RESULT. We suppose that $(\Omega, \mathscr{T}, \mu)$ is a measure space which is measure-theoretically isomorphic to one of the following three possibilities: the positive reals $\boldsymbol{R}^+$ with Lebesgue measure, the interval $I = [0, 1]$ with Lebesgue measure or the positive integers $\boldsymbol{Z}^+$ with counting measure. We shall denote these standard spaces by S_i with $i = 0, 1, 2$ for $\boldsymbol{R}^+$, I and $\boldsymbol{Z}^+$ respectively. We shall write $\Omega^* = S_i$ if $(\Omega, \mathscr{T}, \mu)$ is isomorphic to S_i.

Suppose that ρ is a rearrangement-invariant function norm on the measurable functions $\mathscr{M}(\Omega, \mathscr{T}, \mu)$ on $(\Omega, \mathscr{T}, \mu)$, (see [2]). Then for each measurable f, $\rho(f) = \sigma(f^*)$ where f^* is the decreasing rearrangement of f onto $\boldsymbol{R}^+$, and σ is a rearrangement invariant norm on $\boldsymbol{R}^+$. We define $E_s f^*$ by $E_s f^*(t) = f^*(st)$ for $0 < s < \infty$, and then let

$$h(s) = \sup\{\sigma(E_s f^*): f \in \mathscr{M}(\Omega, \mathscr{T}, \mu), \sigma(f^*) \leqq 1\} .$$

It is known that the following limits exist (see Lemma 1, below)

$$\alpha = \lim_{s \to 0+} \frac{\log h(s)}{-\log s}, \ \beta = \lim_{s \to \infty} \frac{-\log h(s)}{\log s}$$

and these are called the indices of the space $L^\rho(\Omega, \mathscr{T}, \mu)$. Note that in fact α depends only on σ and on Ω^* so we will write $\alpha = \alpha(\sigma, \Omega^*)$ and $\beta = \beta(\sigma, \Omega^*)$. (A slightly different definition was used in case $\Omega^* = \boldsymbol{Z}^+$ when the indices were introduced in [2], but it is equivalent to the definition given here as Lemma 3 will show). It was shown in [2] that X is intermediate between L^p and L^q in the weak sense if and only if $1/p > \alpha$ and $\beta > 1/q$.

We use the definition of Orlicz space given in [5]. Thus, let φ be a right-continuous nondecreasing function on $[0, \infty]$ and let ψ be its left-continuous inverse. Then the functions Φ and Ψ defined by

$$\Phi(u) = \int_0^u \varphi(t)dt, \quad \Psi(u) = \int_0^u \Psi(t)dt$$

are called *complementary Young's functions*. The general form of Φ (or Ψ) is that there are two numbers a and b with $0 \leqq a \leqq b \leqq \infty$ such that $\Phi(u) = 0$ for $0 \leqq u \leqq a$, Φ is convex and finite valued in $]a, b[$, and $\Phi(u) = \infty$ for $u > b$. If $b < \infty$ we say "Φ jumps" and if $a > 0$ we say "Φ is level". Otherwise we say Φ is strictly increasing.

Define the Minkowski functional by

$$M_\Phi(f^*) = \int_0^\infty \Phi(f^*(t))dt ,$$

and then $\rho_\Phi(f) = \inf\{c\colon M_\Phi(c^{-1}f^*) \leqq 1\} = \sigma_\Phi(f^*)$. The space $L_\Phi(\Omega, \mathscr{T}, \mu)$ is the subspace of $\mathscr{M}(\Omega, \mathscr{T}, \mu)$ determined by the function norm ρ_Φ, and its dual space (in the sense of function spaces) is isomorphic to L^Ψ.

We shall write $\alpha_i(\Phi) = \alpha(\sigma_\Phi, S_i)$ $(i = 0, 1, 2)$. And similarly for β. Note that the indices of a space X are invariant under isomorphism and that if (α', β') are the indices of the dual space of L^ρ, then $(\alpha', \beta') = (1 - \beta, 1 - \alpha)$ (see [2]). The case $i = 0$ of the following theorem was proved in [1].

THEOREM. *Given a Young's function Φ. For $0 < s < \infty$, let*

$$g_0(s) = \sup\{\Phi^{-1}(t)/\Phi^{-1}(st)\colon 0 < t < \infty\}$$
$$g_1(s) = \limsup\{\Phi^{-1}(t)/\Phi^{-1}(st)\colon t \to \infty\}$$
$$g_2(s) = \limsup\{\Phi^{-1}(t)/\Phi^{-1}(st)\colon t \to 0+\} .$$

Then

$$\alpha_i(\Phi) = \lim_{s\to 0+} -\log g_i(s)/\log s , \qquad i = 0, 1, 2$$
$$\beta_i(\Phi) = \lim_{s\to\infty} -\log g_i(s)/\log s , \qquad i = 0, 1, 2 .$$

PRELIMINARY LEMMAS.

LEMMA 1. (a). *Suppose that h is a positive function which satisfies $h(st) \leqq h(s)h(t)$ for all $s > 0$ and $t > 0$. Let*

$$\theta(s) = -\log h(s)/\log s ,$$

and let

$$\alpha(h) = \inf\{\theta(s)\colon 0 < s < 1\},\ \beta(h) = \sup\{\theta(s)\colon s > 1\} .$$

Then

$$\alpha(h) = \lim_{s\to 0+} \theta(s) \quad and \quad \beta(h) = \lim_{s\to\infty} \theta(s) .$$

(b). *Suppose h is positive and monotone and satisfies*

$$h(mn) \leqq h(m)h(n)$$

for m, n in $\boldsymbol{Z}^+$. Let $\theta(n) = -\log h(n)/\log n$, and

$$\beta(h) = \inf\{\theta(n): n \in \boldsymbol{Z}^+\} .$$

Then $\beta(h) = \lim_{n\to\infty} \theta(n)$.

Proof. The proof of (a) is in [1], and that of (b) is in [3]. Note that the extra condition in (b), that h be monotone, is needed; otherwise, one could define $h(p)$ to be an arbitrary positive number for each prime p, and then if $n = p_1^{a_1} \cdots p_k^{a_k}$, let $h(n) = h(p_1)^{a_1} \cdots h(p_k)^{a_k}$. If $h(1) = 1$, h would satisfy (b) except that it would not be monotone, and certainly the conclusion need not be true, e.g., if $h(p) = p^p$.

LEMMA 2. (a). *Let f be a positive function on $[1, \infty[$ and for $s > 1$ let $h(s) = \sup_{t\geqq 1} f(st)/f(t)$ and $h_1(s) = \limsup_{t\to\infty} f(st)/f(t)$. If β is as in Lemma 1 (a), then*

$$\beta(h) = \beta(h_1) .$$

(b) *Let f be positive and monotone on $\boldsymbol{Z}^+$, and for $m \in \boldsymbol{Z}^+$, let*

$$h(m) = \sup_{n\in \boldsymbol{Z}^+} f(mn)/f(n) ,$$

and $h_1(m) = \limsup_{n\to\infty} f(mn)/f(n)$. If β is as in Lemma 1(b), then

$$\beta(h) = \beta_1(h) .$$

Proof. (a) Clearly h satisfies the condition of Lemma 1 (a) for $s \geqq 1$, $t \geqq 1$. Let $\beta = \beta(h)$. By definition of β and the conclusion of Lemma 1, there is a function $\varepsilon(s)$ such that $\varepsilon(s) \geqq 0$ and $\varepsilon(s) \to 0$ as $s \to \infty$ such that

$$(1) \qquad s^{-\beta} \leqq h(s) \leqq s^{-\beta+\varepsilon(s)} \text{ for } s \geqq 1 .$$

Define $\Delta(t, s) = f(st)/f(t)$. Let m be a positive integer, and s be fixed >1. Then there is a t_m so that

$$h(s^m) \leqq \Delta(t_m, s^m)s .$$

Using (1),

$$(2) \qquad \Delta(t_m, s^m) \geqq s^{-\beta m-1} .$$

Note that, for any t,

$$\Delta(ts^n, s) = f(s^{n+1}t)/f(s^n t)$$

so that

$$\prod_{n=m}^{2m-1} \Delta(t_{2m}s^n, s) = \Delta(t_{2m}, s^{2m})/\Delta(t_{2m}, s^m)$$
$$\geqq s^{-2\beta m-1}/s^{-\beta m+\varepsilon}$$
$$= s^{-\beta m-1-\varepsilon} ,$$

where $\varepsilon = \varepsilon(s^m)$.

Thus, if n_m is such that

$$\Delta(t_{2m}s^{n_m}, s) = \max\{\Delta(t_{2m}s^n, s): m \leqq n \leqq 2m-1\} ,$$

one has

$$\Delta(t_{2m}s^{n_m}, s)^m \geqq \prod_{n=m}^{2m-1} \Delta(t_{2m}s^n, s)$$

and thus

$$\Delta(t_{2m}s^{n_m}, s) \geqq s^{-\beta-(1/m)-(\varepsilon/m)} .$$

But

$$t_{2m}s^{n_m} \geqq s^m \longrightarrow \infty \quad \text{as} \quad m \longrightarrow \infty .$$

Hence

$$h_1(s) \geqq \limsup_{m\to\infty} \Delta(t_{2m}s^{n_m}, s) \geqq s^{-\beta}$$

for any $s > 1$, which shows that $\beta(h_1) = \beta(h)$, since obviously $h_1(s) \leqq h(s)$. (b) The proof is entirely the same, using Lemma 1 (b) rather than 1 (a).

LEMMA 3. *Suppose that f is positive and monotone on $[1, \infty[$ that*

$$h_1(s) = \limsup_{t\to\infty} \frac{f(st)}{f(t)} \quad \textit{and} \quad h_2(s) = \limsup_{n\to\infty} \frac{f(sn)}{f(n)} \quad \textit{for} \quad s > 0 .$$

Then

$$h_2(s) \leqq h_1(s) \leqq h_2(s\pm) ,$$

where the upper (lower) sign is used if f is increasing (decreasing), and

$$\beta(h_2) = \beta(h_1) .$$

Proof. Clearly $h_2(s) \leqq h_1(s)$. We treat the case that f is decreasing.

Given $\varepsilon > 0$, $s > 1$ and an integer $m \geqq 1$, there is an $N \geqq m$ such that $n \geqq m$ implies

$$\frac{f((s - sm^{-1})n)}{f(n)} \leqq h_2(s - sm^{-1}) + \varepsilon \ .$$

Since f is decreasing, and $(s - sm^{-1})n \leqq (s - sn^{-1})n = s(n - 1)$, we have

$$\frac{f(s(n-1))}{f(n)} \leqq h_2(s - sm^{-1}) + \varepsilon \ , \qquad \text{for } n \geqq N \ .$$

Now if $t \geqq N + 1$, then there is an integer $n \geqq N$ such that

$$n - 1 \leqq t < n$$

so that $f(n-1) \geqq f(t) \geqq f(n)$, and $f(s(n-1)) \geqq f(st) \geqq f(sn)$. Hence

$$\frac{f(st)}{f(t)} \leqq \frac{f(s(n-1))}{f(n)} \leqq h_2(s - sm^{-1}) + \varepsilon \quad \text{for } t \geqq N + 1 \ .$$

Thus

$$h_1(s) \leqq h_2(s - sm^{-1}) + \varepsilon \qquad \text{for all } m \text{ and } \varepsilon$$

so that

$$h_1(s) \leqq h_2(s-) \ .$$

Clearly $\beta(h_2) = \beta(h_1)$ follows from these inequalities.

Proof of the Theorem. $i = 0$. This was proved in [1], Theorem 5.5. In fact, in this case, $h(s) = g_0(s)$ for all s.

$i = 1$. We prove first that $\beta_1(\Phi)$ is as in the statement of the theorem. Then we use $\alpha_1(\Phi) = 1 - \beta_1(\Psi)$, and the well-known inequality:

$$t/\Phi^{-1}(t) \leqq \Psi^{-1}(t) \leqq 2t/\Phi^{-1}(t) \qquad \text{for all } t > 0 \tag{3}$$

to show that the expression for $\alpha_1(\Phi)$ is as stated. (For the inequality see [4], p. 13).

Let us thus take $s > 1$. We recall that if E is a set of measure u and χ_E is its characteristic function, then $\chi_E^* = \chi[0, u] = \chi_u$, and $\sigma_\Phi(\chi_u) = 1/\Phi^{-1}(1/u)$, so that

$$\begin{aligned} h(s) &= \sup\{\sigma_\Phi(E_s f^*)/\sigma_\Phi(f^*) : f \neq 0, f \in L^\Phi(I)\} \\ &\geqq \sup\{\sigma_\Phi(E_s\chi_u)/\sigma_\Phi(\chi_u) : 0 < u \leqq 1\} \\ &= \sup\{\Phi^{-1}(u^{-1})/\Phi^{-1}(su^{-1}) : 0 < u \leqq 1\} \\ &= \sup\{\Phi^{-1}(t)/\Phi^{-1}(st) : t \geqq 1\} \ . \end{aligned} \tag{4}$$

We define the latter quantity to be $k(s)$. Then (4) implies

$$\beta(h) \leqq \beta(k) \ .$$

Note that by Lemma 2,

$$\beta(k) = \beta(g_1) .$$

Now we proceed to show that $h(s) \leqq k(s/2)$ for $s \geqq 2$, which clearly implies $\beta(h) \geqq \beta(k)$ so that

$$\beta(h) = \beta(k) = \beta(g_1) .$$

Assume now that Φ is strictly increasing. We leave the other cases to the reader.

By definition of $k(s)$,

$$\Phi^{-1}(t)/k(s) \leqq \Phi^{-1}(st) \qquad \text{for } s \geqq 1, t \geqq 1 .$$

Thus,

$$\Phi(\Phi^{-1}(t)/k(s)) \leqq st , \qquad s \geqq 1, t \geqq 1$$

and so

$$\Phi(u/k(s)) \leqq s\Phi(u) , \qquad \text{for } u \geqq \Phi^{-1}(1) . \tag{5}$$

Given $0 < f \in L^{\Phi}(I)$ with f decreasing let

$$c = \sup\{t: f(t) \geqq \Phi^{-1}(1)\} .$$

Then from (5),

$$\Phi(f(t)/k(s)) \leqq s\Phi(f(t)) , \qquad 0 \leqq t \leqq c$$

and

$$\Phi(f(t)/k(s)) \leqq \Phi(\Phi^{-1}(1)/k(s)) \leqq s , \qquad \text{for } c < t \leqq 1 . \tag{6}$$

Hence, for $s \geqq 2$, (6) implies that

$$\begin{aligned} M_\Phi(E_s f/k(s/2)) &= \int_0^{1/s} \Phi(f(st)/k(s/2))dt \\ &= s^{-1}\int_0^1 \Phi(f(t)/k(s/2))dt \\ &= s^{-1}\left(\int_0^c \frac{s}{2}\Phi(f(t))dt + \int_c^1 \frac{s}{2}\,dt\right) \\ &\leqq \tfrac{1}{2}\int_0^1 \Phi(f(t))dt + \tfrac{1}{2} \leqq 1 , \qquad \text{if } M_\Phi(f) \leqq 1 . \end{aligned}$$

Thus,

$$\rho\,(E_s f) \leqq k(s/2) \qquad \text{for } \rho\,(f) \leqq 1 ,$$

which proves that

$$h(s) \leqq k(s/2) \qquad \text{for } s \geqq 2 .$$

$i = 2$. According to Lemma 4 of [2], if F_m is the operator on sequences $\{f(n)\}$ given by

$$F_m f(n) = f([(n-1)/m] + 1) ,$$

and if

$$K(m) = \sup\{\rho(F_m f) : f \in L^\rho(Z^+), \rho(f) \leqq 1\} ,$$

then

$$\alpha_2 = \lim_{n\to\infty} \log K(m)/\log m = -\beta(K) .$$

Note that the action of F_m on $\{f(1), f(2), \cdots\}$ is to repeat each term m-times, e.g., $F_2\{f(1), f(2), \cdots\} = \{f(1), f(1), f(2), f(2), \cdots\}$. Let us define

$$p(m) = \sup_{0<u\leqq 1} \Phi^{-1}(u)/\Phi^{-1}(u/m) \qquad \text{for } m \in Z^+ .$$

Then, as in (5) above we obtain

$$\Phi(t/p(m)) \leqq m^{-1}\Phi(t) \qquad \text{provided } t \leqq \Phi^{-1}(1) . \tag{7}$$

Observe that, if $0 \leqq f \in L^\Phi$, and k is fixed, then $f(k)\chi_{\{k\}}(n) \leqq f(n)$ for all $n = 1, 2, \cdots$ so that

$$\rho_\Phi (f(k)\chi_{\{k\}}) \leqq \rho_\Phi (f) .$$

That is

$$f(k) \leqq \rho_\Phi (f)/\rho_\Phi (\chi_{\{k\}}) = \rho_\Phi (f)\Phi^{-1}(1) , \qquad \text{for all } k .$$

Hence, if $\rho_\Phi (f) \leqq 1$, we have from (7) that

$$\Phi(f(k)/p(m)) \leqq m^{-1}\Phi(f(k)) \qquad \text{for all } k \text{ and } m .$$

Thus

$$\begin{aligned} M_\Phi (F_m f/p(m)) &= \sum_{n=1}^{\infty} \Phi(F_m f(n)/p(m)) \\ &= m\sum_{n=1}^{\infty} \Phi(f(n)/p(m)) , \quad \text{by an earlier remark,} \\ &\leqq m\sum_{n=1}^{\infty} m^{-1}\Phi(f(n)) = M_\Phi (f) . \end{aligned}$$

Thus,

$$K(m) \leqq p(m)$$

But, if $f = \chi_{\{1,2,\ldots,k\}}$, then

$$\rho (F_m f)/\rho_\Phi (f) = \Phi^{-1}(k^{-1})/\Phi^{-1}(k^{-1}m^{-1}) .$$

Thus

$$K(m) \geqq \sup_{k\in Z^+} \Phi^{-1}(k^{-1})/\Phi^{-1}(k^{-1}m^{-1}) = p_1(m) , \qquad \text{say} .$$

By Lemma 2, with $f(t) = 1/\Phi^{-1}(1/t)$, if

$$\hat{p}(m) = \limsup_{t\to\infty} \Phi^{-1}(1/t)/\Phi^{-1}(1/tm)$$

and

$$\hat{p}_1(m) = \limsup_{n\to\infty} \Phi^{-1}(1/n)/\Phi^{-1}(1/nm) ,$$

then $\beta(p) = \beta(\hat{p})$ and $\beta(p_1) = \beta(\hat{p}_1)$. And, by Lemma 3, $\beta(\hat{p}) = \beta(\hat{p}_1)$. Hence $\beta(K) = \beta(p) = \beta(p_1)$ and we thus obtain

$$\begin{aligned}\alpha_2(\Phi) &= \lim_{m\to\infty} -\log g_2(m^{-1})/\log(m^{-1}) \\ &= \lim_{s\to 0+} -\log g_2(s)/\log s .\end{aligned}$$

As in the case $i = 1$, we can obtain the result for $\beta_2(\Phi)$ from this and the inequality (3).

REMARK. One can relate the functions $g_i(s)$ to the functions $K_i(s)$ which are given by

$$\begin{aligned}K_0(s) &= \sup\{\Phi(st)/\Phi(t) : 0 < t < \infty\} \\ K_1(s) &= \limsup\{\Phi(st)/\Phi(t) : t\to\infty\} \\ K_2(s) &= \limsup\{\Phi(st)/\Phi(t) : t\to 0+\}\end{aligned}$$

whenever these make sense (e.g., if Φ is strictly increasing).

In fact K_i is left-continuous and increasing and its right continuous inverse is $1/g_i$. Formally this is clear, since if we let

$$\theta_t(s) = \Phi(st)/\Phi(t)$$

for any $t > 0$, then θ_t is convex and increasing and

$$\theta_t^{-1}(u) = \Phi^{-1}(u\Phi(t))/t$$

so that

$$\theta^{-1}_{\Phi^{-1}(t)}(s) = \Phi^{-1}(st)/\Phi^{-1}(t) .$$

Thus we would expect

$$\inf_{t>0} \theta^{-1}_{\Phi^{-1}(t)}(s) = 1/g_0(s) = \left(\sup_{t>0} \theta_t\right)^{-1}(s) = K_0^{-1}(s) ,$$

and similarly for lim inf as $t \to 0+$ or $t\to\infty$. This intuitive argument can be justified as is done in [1] for the special case $i = 0$. This shows that if σ_Φ, s_Φ are the exponents defined in [6] and [7] for the case that $\Omega = [0, 1]$, then $\alpha_1 = 1/s_\Phi$, $\beta_1 = 1/\sigma_\Phi$.

EXAMPLE. Let

$$\begin{aligned}\Phi(t) &= e^t - t - 1 , & t \geqq 0 \\ \Psi(t) &= (1 + t)\log(1 + t) , & t \geqq 0 .\end{aligned}$$

Then, for K_0 as above

$$K_0(s) = \begin{cases} s^2, \ 0 \leqq s \leqq 1 \\ \infty, \ s > 1 \ . \end{cases}$$

For, $\Phi(st) - s^2\Phi(t) = \sum_{n=2}^{\infty}((s^n - s^2)/n!)t^n \leqq 0$, for $0 \leqq s \leqq 1$, showing that $K_0(s) \leqq s^2$ for $0 \leqq s \leqq 1$ and $\Phi(st)/\Phi(t) \to s^2$ as $t \to 0$. Also

$$\lim_{t\to\infty} \Phi(st)/(t) = \infty \text{ for } s > 1, \text{ showing } K_0(s) = \infty \ , \qquad s > 1 \ .$$

Thus,

$$g_0(s) = \max (s^{-1/2}, 1) \ .$$

For large t, $\Phi(t) \sim e^t$ so $\Phi^{-1}(t) \sim \log t$, and hence

$$g_1(s) = 1 \ .$$

For small t, $\Phi(t) \sim \frac{1}{2}t^2$ so $\Phi^{-1}(t) \sim \sqrt{2t}$, and thus

$$g_2(s) = s^{-1/2} \ .$$

Hence

$$(\alpha_0, \beta_0) = (\tfrac{1}{2}, 0), (\alpha_1, \beta_1) = (0, 0), (\alpha_2, \beta_2) = (\tfrac{1}{2}, \tfrac{1}{2}) \ .$$

REMARK. The result given in Lemma 2 seems to be new. There is a corresponding result for subadditive function on $[0, \infty[$, which may be obtained by setting $x = \log t$, $y = \log s$ and

$$F(x) = \exp(f(\log x))$$

in that lemma.

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