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# ON ITERATED $w^*$-SEQUENTIAL CLOSURE OF CONES

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**In this paper it is proved that for each countable ordinal number $\alpha \geqq 2$ there exists a separable Banach space $X$ containing a cone $P$ such that, if $J_X$ is the canonical map of $X$ into its bidual $X^{**}$, then the $\alpha$th iterated $w^*$-sequential closure $K_\alpha(J_X P)$ of $J_X P$ fails to be norm-closed in $X^{**}$. From such spaces there is constructed a separable space $W$ containing a cone $P$ such that if $2 \leqq \beta \leqq \alpha$, then $K_\beta(J_W P)$ fails to be norm-closed in $W^{**}$. Further, there is constructed a (non-separable) space $Z$ containing a cone $P$ such that if $2 \leqq \beta < \Omega$, then $K_\beta(J_Z P)$ fails to be norm-closed in $Z^{**}$.**

1. If $X$ is a real Banach space and $Y$ a subset of $X^{**}$, let $K(Y)$ be the set of elements of $X^{**}$ which are $w^*$-limits of sequences in $Y$. Let $K_0(Y) = Y$ and inductively let $K_\alpha(Y) = K(\cup_{\beta<\alpha} K_\beta(Y))$ for $0 < \alpha \leqq \Omega$, where $\Omega$ is the first uncountable ordinal. A *cone* in $X$ is a subset of $X$ which is closed under addition and under multiplication by nonnegative scalars. Our main theorem extends the result of [6] that if $P$ is a cone in $X$, then $K_1(J_X P)$ must be norm-closed but $K_2(J_X P)$ can fail to be norm-closed in $X^{**}$. By contrast it is noted that if $S$ is a compact Hausdroff space and $X = C(S)$ and $\alpha < \Omega$, then $K_\alpha(J_X X)$ is norm-closed, even though for example if $S$ is compact, metric, and uncountable, then $K_\alpha(J_X X)$ is not $w^*$-sequentially closed. It is obvious that for each Banach space $X$ and each subset $Y$ of $X^{**}$, $K_\Omega(Y)$ is $w^*$-sequentially closed and hence norm-closed.

In [7] a Banach space $X$ was exhibited such that $K_2(J_X X)$ is not norm-closed. Whether $K_\alpha(J_X X)$ can fail to be norm-closed for $2 < \alpha < \Omega$ is not known to the author. However, in the present paper it will be convenient to use constructions involving spaces studied in [7].

Section 2 is devoted to a useful relationship between $w^*$-sequential convergence and pointwise convergence of bounded sequences of functions, § 3 to further study of a space constructed in [7], and §§ 4 and 5 to preparation for and proof of the main theorems.

2. Let $S$ be a compact Hausdorff space, $B(S)$ the Banach space of bounded real functions on $S$ with the supremum norm, and $C(S)$ the closed subspace of $B(S)$ consisting of the continuous real functions on $S$. If $A$ is a subset of $B(S)$, let $L(A)$ be the set of all pointwise limits of bounded sequences in $A$, and let $L_\alpha(A)$ be defined inductively by $L_0(A) = A$ and $L_\alpha(A) = L(\cup_{\beta<\alpha} L_\beta(A))$ for each ordinal $\alpha$ such that $0 < \alpha \leqq \Omega$.

If $X$ is a norm-closed subspace of $C(S)$ and $z \in L_\Omega(X)$, then $z$ is bounded and Borel measurable and hence is integrable with respect to each finite regular Borel signed measure $\mu$ on $S$. For each $f \in X^*$ there exists a finite regular Borel signed measure $\mu_f$ on $S$ such that $f(x) = \int_S x\, d\mu_f$ for each $x \in X$ [3, p. 265], and by the Hahn-Banach theorem $\mu_f$ can be chosen so that $\|\mu_f\| = \|f\|$. If $\nu_f$ is another finite regular Borel signed measure on $S$ such that $f(x) = \int_S x\, d\nu_f$ for each $x \in X$ then also $\int_S z d\mu_f = \int_S z d\nu_f$ for each $z \in L_\Omega(X)$, by virtue of the bounded convergence theorem and transfinite induction. Hence a mapping $T$ is unambiguously defined from $L_\Omega(X)$ into the space of real functions on $X^*$ by

$$(Tz)(f) = \int_S z d\mu_f \quad (z \in L_\Omega(X),\ f \in X^*).$$

TEOREM 2.1. *If $S$ is a compact Hausdorff space and $X$ a norm-closed subspace of $C(S)$, then $T$ is an isometric isomorphism from $L_\Omega(X)$ onto $K_\Omega(J_X X)$, and $T$ maps $L_\alpha(A)$ onto $K_\alpha(J_X A)$ for each subset $A$ of $X$ and each $\alpha \leqq \Omega$.*

*Proof.* For each $z \in L_\Omega(X)$ it is trivial that $Tz$ is linear on $X^*$ and that $|(Tz)(f)| \leqq \|z\|\,\|f\|$ for every $f \in X^*$, so that $Tz \in X^{**}$ and $\|Tz\| \leqq \|z\|$. For each $t \in S$ let $f_t(x) = x(t)$ for all $x \in X$; then clearly $f_t \in X^*$ with $\|f_t\| \leqq 1$, and it is easily seen that $(Tz)(f_t) = \int_S z d\mu_{f_t} = z(t)$, so that $|z(t)| \leqq \|Tz\|\,\|f_t\| \leqq \|Tz\|$ and hence $\|z\| \leqq \|Tz\|$. Since $T$ is obviously linear, it follows that $T$ is an isometric isomorphism from $L_\Omega(X)$ into $X^{**}$.

Now let $A$ be a subset of $X$. Since the restriction of $T$ to $X$ is $J_X$, it follows that $T[L_0(A)] = TA = J_X A = K_0(J_X A)$. If $0 < \alpha \leqq \Omega$ and it is assumed that $T[L_\beta(A)] = K_\beta(J_X A)$ for each $\beta < \alpha$, then for each $z \in L_\alpha(A)$ there exists a bounded sequence $\{z_n\}$ in $\bigcup_{\beta<\alpha} L_\beta(A)$ which converges pointwise to $z$. By the bounded convergence theorem $(Tz)(f) = \lim_n (Tz_n)(f)$ for each $f \in X^*$. Since by assumption $\{Tz_n\} \subset \bigcup_{\beta<\alpha} K_\beta(J_X A)$, it follows that $Tz \in K_\alpha(J_X A)$. Conversely, if $F \in K_\alpha(J_X A)$ there exists a sequence $\{F_n\} \subset \bigcup_{\beta<\alpha} K_\beta(J_X A)$ such that $F_n \xrightarrow{w^*} F$; the sequence $\{F_n\}$ must be bounded [3, p. 60], and by assumption there exists a sequence $\{z_n\} \subset \bigcup_{\beta<\alpha} L_\beta(A)$ such that $Tz_n = F_n$ for each $n$. Now $\{z_n\}$ is bounded, and if $z(t)$ is defined to be $F(f_t)$ for each $t \in S$ it follows that $\{z_n\}$ converges pointwise to $z$ so that $z \in L_\alpha(A)$. For every $f \in X^*$, $(Tz)(f) = \lim_n (Tz_n)(f)$ by the bounded convergence theorem. Thus $F = Tz \in T[L_\alpha(A)]$, completing the proof that $T[L_\alpha(A)] = K_\alpha(J_X A)$. By transfinite induction the theorem follows.

REMARK. If $S$ is a compact Hausdorff space and $X$ is the Banach space $C(S)$, then for each $\alpha \leqq \Omega$, $L_\alpha(X)$ is the space of bounded Baire functions on $S$ of order $\leqq \alpha$ and, just as in the special case of a metric space $S$ [8, p. 132], $L_\alpha(X)$ is norm-closed in $B(S)$ and hence also $K_\alpha(J_X X)$ is norm-closed in $X^{**}$. If $S$ is a compact metric space with uncountably many elements then $S$ has a nonempty dense-in-itself kernel [1, Ch. 9, p. 34]. Hence for each countable $\alpha$ there is a subset $T$ of $S$ of Borel order exactly $\alpha$ [4, p. 207], but then it follows that $L_\alpha(X) \neq L_{\alpha+1}(X)$ [5, p. 299] and hence that $K_\alpha(J_X X) \neq K_{\alpha+1}(J_X X)$ for each countable $\alpha$.

3. The reader is now referred to the proof of Theorem 1 of [7] for the construction, for each real $c \geqq 1$, of a Banach space $X \subset C([0;3])$ having the property that there exists an $x^0 \in L_2(X)$ such that $\|x^0\| = 1$ but if $\{y^h\}$ is a bounded sequence in $L_1(X)$ which converges pointwise to $x^0$, then $\liminf_h \|y^h\| \geqq c$. The remainder of the present paper depends heavily on properties of the space $X$, and the reader will occasionally need to refer to [7]. In particular, note that $X$ is generated by a set $\{x_{pq}: p, q \in \omega\}$ of piecewise linear nonnegative functions of norm $c$ on $[0;3]$ and that $x^0$ is the pointwise limit of the sequence $\{x^p\} \subset L_1(X)$, where $x^p$ is the pointwise limit of $\{x_{pq}\}_{q\in\omega}$ and $\|x^p\| = c$ for each $p$. Each $x_{pq}$ has truncated peaks centered at certain of the points $s_{ui}$, $t_{vj}$, $2 + s_{ui}$ where $s_{ui} = 2^{-u}i$ and $t_{vj} = 2 - 2^{-v}(1 + 2^{-j})$ for $u, i, v, j \in \omega$ and $i < 2^u$. Specifically, $x_{pq}(s_{ui}) = x_{pq}(2 + s_{ui}) = 1$ if $p \geqq u$, and $x_{pq}(s_{u1}) = 1$ if and only if $p \geqq u$. Further, $x_{pq}(t_{vj}) = c$ if $v \leqq p \leqq j < p + q$ and 0 otherwise. If $\chi(S)$ denotes the characteristic function of the subset $S$ of $[0;3]$, it turns out that

$$x^p = \chi(\{s_{pi}: i < 2^p\} \cup \{2 + s_{pi}: i < 2^p\}) + c\chi(\{t_{vj}: v \leqq p \leqq j\})$$

and that

$$x^0 = \chi(\{s_{pi}: p \in \omega, i < 2^p\} \cup \{2 + s_{pi}: p \in \omega, i < 2^p\}).$$

LEMMA 3.1. *Let $Q$ be the norm-closed cone in $X$ generated by $\{x_{pq}: p, q \in \omega\}$. Then $Q$ coincides with*

$$Q_0 = \{\Sigma_p \Sigma_q a_{pq} x_{pq}: a_{pq} \geqq 0, \Sigma_p \Sigma_q a_{pq} < \infty\},$$

*where the indicated summations are over the set $\omega$ of all positive integers.*

*Proof.* It is clear that $Q_0$ is a cone containing $\{x_{pq}: p, q \in \omega\}$ and contained in $Q$. If $\{z_n\}$ is a sequence in $Q_0$ which converges in norm to some $x \in X$, then each $z_n$ has the form $z_n = \Sigma_p \Sigma_q a_{npq} x_{pq}$ with $a_{npq} \geqq 0$ and $\Sigma_p \Sigma_q a_{npq} < \infty$. As noted in [7] the limit $\lim_n a_{npq} \equiv a_{pq}$ exists for all $p, q$; indeed, in the notation of [7],

$$a_{pq} = c^{-1}(x(t_{pp} - 2^{-2p-q-2}) - x(t_{pp} - 2^{-2p-q-1})).$$

Clearly each $a_{pq} \geqq 0$, and if $r, s \in \omega$ then

$$\Sigma_{p\leqq r}\Sigma_{q\leqq s}a_{pq} = \lim_n \Sigma_{p\leqq r}\Sigma_{q\leqq s}a_{npq} \leqq \lim_n z_n(s_{11}) = x(s_{11});$$

hence $\Sigma_p\Sigma_q a_{pq} \leqq x(s_{11})$ and $z \equiv \Sigma_p\Sigma_q a_{pq}x_{pq} \in Q_0$.

Let $\varepsilon > 0$ be given. It follows from [7, p. 1196] that each $x_{pq}$ is continuous and vanishes at 0 and at $2 - 2^{-1}$ and hence that each element of $X$ shares these properties. Since $s_{p1} \to 0$, there exists $p_1 \in \omega$ such that $z(s') < \varepsilon$ and $x(s') < \varepsilon$ for $s' = s_{p_1+1,1}$. Since $||z_n - x|| \to 0$, there exists $n'$ such that $z_n(s') < \varepsilon$ for all $n > n'$. Thus, by [7], $\Sigma_{p>p_1}\Sigma_q a_{pq} = z(s') < \varepsilon$ and $\Sigma_{p>p_1}\Sigma_q a_{npq} = z_n(s') < \varepsilon$ for $n > n'$. Further, since $t_{1j} \to 2 - 2^{-1}$, there exists by continuity $q_1 \geqq p_1$ such that $z(t_{1,q_1}) < c\varepsilon$ and $x(t_{1,q_1}) < c\varepsilon$; hence there exists $n'' \geqq n'$ such that $z_n(t_{1\,q_1}) < c\varepsilon$ for all $n > n''$. It follows from [7] that

$$\Sigma_{p\leqq p_1}\Sigma_{q>q_1}a_{pq} \leqq \Sigma_{p\leqq q_1}\Sigma_{q>q_1-p}a_{pq} = c^{-1}z(t_{1,q_1}) < \varepsilon$$

and similarly $\Sigma_{p\leqq p_1}\Sigma_{q>q_1}a_{npq} \leqq c^{-1}z_n(t_{1,q_1}) < \varepsilon$ for all $n > n''$. Moreover, since $a_{npq} \to a_{pq}$, there exists $n_1 \geqq n''$ such that $\Sigma_{p\leqq p_1}\Sigma_{q\leqq q_1}|a_{pq} - a_{npq}| < \varepsilon$ for all $n > n_1$. Hence for $n > n_1$ the triangle inequality implies that

$$\begin{aligned} ||z - z_n|| &\leqq ||\Sigma_{p>p_1}\Sigma_q a_{pq}x_{pq}|| + ||\Sigma_{p>p_1}\Sigma_q a_{npq}x_{pq}|| \\ &+ ||\Sigma_{p\leqq p_1}\Sigma_{q>q_1}a_{pq}x_{pq}|| + ||\Sigma_{p\leqq p_1}\Sigma_{q>q_1}a_{npq}x_{pq}|| \\ &+ ||\Sigma_{p\leqq p_1}\Sigma_{q\leqq q_1}(a_{pq} - a_{npq})x_{pq}|| \\ &< 5c\varepsilon, \end{aligned}$$

since $||x_{pq}|| = c$ for all $p, q$. Thus $||z - z_n|| \to 0$ and therefore $x = z \in Q_0$, proving that $Q_0$ is norm-closed.

LEMMA 3.2. *Let* $Q_1 = \{\Sigma_p b_p x^p : b_p \geqq 0, \Sigma_p b_p < \infty\}$. *Then* $L_1(Q) = Q + Q_1$.

*Proof.* Since $L_1(Q)$ is a norm-closed cone in $B([0; 3])$ by [6, Theorem 1, p. 192] and Theorem 2.1, and since $\{x^p\}_p \subset L_1(Q)$, it is clear that $Q + Q_1 \subset L_1(Q)$. If $\{z_n\}$ is a bounded sequence in $Q$ which is pointwise convergent to some $z \in L_1(Q)$, each $z_n$ has the form $z_n = \Sigma_p\Sigma_q a_{npq}x_{pq}$ with $a_{npq} \geqq 0$ and $\Sigma_p\Sigma_q a_{npq} < \infty$. As in the proof of Lemma 3.1, for all $p, q \in \omega$ the limit $a_{pq} = \lim_n a_{npq}$ exists. For all $p, q_1 \in \omega$,

$$\Sigma_{q\leqq q_1}a_{pq} = \lim_n \Sigma_{q\leqq q_1}a_{npq} \leqq \lim_n c^{-1}z_n(t_{pp}) = c^{-1}z(t_{pp});$$

hence $\Sigma_q a_{pq} \leqq c^{-1}z(t_{pp})$ for each $p \in \omega$. Let $b_p = c^{-1}z(t_{pp}) - \Sigma_q a_{pq}$ for each $p$, and note that all the numbers $a_{pq}$ and $b_p$ are nonnegative.

For $n, p \in \omega$ let $u_{np} = \Sigma_q a_{npq}x_{pq}$ and $u_p = \Sigma_q a_{pq}x_{pq} + b_p x^p$. For each $p$, if $t \in [0; 3]$ and $t$ is not of the form $s_{pi}$, $2 + s_{pi}$, or $t_{vj}$ with $v \leqq p$

$\leqq j$, in the notation of [7, p. 1196], $x_{pq}(t) = 0$ for all sufficiently large $q$ and hence $x^p(t) = 0$, so that $u_{np}(t) \xrightarrow[n]{} u_p(t)$, If $t = s_{pi}$ or $t = 2 + s_{pi}$, then

$$u_{np}(t) = \Sigma_q a_{npq} = c^{-1}z_n(t_{pp}) \longrightarrow c^{-1}z(t_{pp}) = u_p(t) .$$

Finally, if $v \leqq p \leqq j$, then

$$\begin{aligned} u_{np}(t_{vj}) &= c\Sigma_{q>j-p}a_{npq} \longrightarrow z(t_{pp}) - c\Sigma_{q\leqq j-p}a_{pq} \\ &= c[b_p + \Sigma_{q>j-p}a_{pq}] = u_p(t_{vj}), \end{aligned}$$

proving that $\{u_{np}\}$ converges pointwise to $u_p$ on [0; 3],

For each $r \in \omega$,

$$\begin{aligned} \Sigma_{p\leqq r}(\Sigma_q a_{pq} + b_p) &= c^{-1}\Sigma_{p\leqq r}z(t_{pp}) \\ &= c^{-1}\lim_n \Sigma_{p\leqq r}z_n(t_{pp}) = \lim_n \Sigma_{p\leqq r}\Sigma_q a_{npq} \\ &\leqq \lim_n z_n(s_{11}) = z(s_{11}), \end{aligned}$$

Hence $\Sigma_p u_p \in Q + Q_1$. Let $w = z - \Sigma_p u_p$; then $w$ is easily seen to be a Baire function of the first class on [0; 3] and hence by [8, p. 143] $w$ must have a point $t_1$ of continuity in [2; 3].

At each point of the form $t = 2 + s_{ri}$ with $i$ odd, $u_p(t) = u_p(s_{11})$ for each $p \geqq r$ and hence

$$\begin{aligned} w(t) &= \lim_n(\Sigma_{p<r}u_{np}(t) + \Sigma_{p\geqq r}\Sigma_q a_{npq}) - \Sigma_p u_p(t) \\ &= \lim_n(z_n(s_{11}) - \Sigma_{p<r}u_{np}(s_{11})) - \Sigma_{p\geqq r}u_p(t) \\ &= z(s_{11}) - \Sigma_p u_p(s_{11}) = w(s_{11}). \end{aligned}$$

Since the set of such points $t$ is dense in [2; 3], $w(t_1) = w(s_{11})$. On the other hand, it follows from [7] that for each point of the form $s = 2 + s_{ri} \pm 2c_{ri_1}$ with $i$ odd, $x_{pq}(s) = 0$ whenever $p \geqq r$, and hence

$$w(s) = \lim_n \Sigma_{p<r}u_{np}(s) - \Sigma_{p<r}u_p(s) = 0.$$

Since the set of such points $s$ is also dense in [2; 3], it follows that $w(t_1) = 0$ and hence that $w(s_{11}) = 0$.

For each $r \in \omega$ let $w_r = z - \Sigma_{p<r}u_p$. Then $w_r \longrightarrow w$ in the norm topology, and $w_r$ is the pointwise limit of $\{\Sigma_{p\geqq r}u_{np}\}$. Hence

$$||w_r|| \leqq \limsup_n ||\Sigma_{p\geqq r}u_{np}|| \leqq c\lim_n \Sigma_{p\geqq r}u_{np}(s_{11}) = cw_r(s_{11})$$

and consequently

$$||w|| = \lim_r ||w_r|| \leqq c\lim_r w_r(s_{11}) = cw(s_{11}) = 0.$$

Therefore $w = 0$ and $z = \Sigma_p u_p \in Q + Q_1$, completing the proof of the lemma.

*Note.* The last paragraph of the previous proof shows that if

$\{z_n\}$ is a bounded pointwise convergent sequence in $Q$, then in the notation of that proof for each $\varepsilon > 0$ there exist $p_1, n_1 \in \omega$ such that $\Sigma_{p \geqq p_1} \Sigma_q a_{npq} < \varepsilon$ for all $n \geqq n_1$. Indeed, given $\varepsilon > 0$ there exists $p_1$ such that $cw_{p_1}(s_{11}) < \varepsilon$. Since $\limsup_n \|\Sigma_{p \geqq p_1} u_{np}\| \leqq cw_{p_1}(s_{11})$, there exists $n_1$ such that for each $n \geqq n_1$

$$\Sigma_{p \geqq p_1} \Sigma_q a_{npq} = (\Sigma_{p \geqq p_1} u_{np})(s_{11}) \leqq \|\Sigma_{p \geqq p_1} u_{np}\| < \varepsilon .$$

LEMMA 3.3. *Let* $Q_2 = \{c_0 x^0 \colon c_0 \geqq 0\}$. *Then* $L_2(Q) = L_\Omega(Q) = Q + Q_1 + Q_2$.

*Proof*. Clearly $Q + Q_1 + Q_2$ is a cone containing $L_1(Q)$ and contained in $L_2(Q)$. To prove the lemma it suffices to show that $L(Q + Q_1 + Q_2) \subseteqq Q + Q_1 + Q_2$. If $\{z_n\}$ is a bounded sequence in $Q + Q_1 + Q_2$ which is pointwise convergent to a function $z$, then each $z_n$ has the form

$$z_n = y_n + \Sigma_p b_{np} x^p + c_n x^0$$

where $y_n \in Q$, $b_{np} \geqq 0$, $c_n \geqq 0$, and $\Sigma_p b_{np} < \infty$. Since $\{z_n\}$ is bounded, the diagonal process yields a subsequence $\{z_{n_i}\}$ of $z_n$ such that $c_0 \equiv \lim_i c_{n_i}$ and $b \equiv \lim_i \Sigma_p b_{n_i p}$ exist and $b_p \equiv \lim_i b_{n_i p}$ exists for each $p \in \omega$. It is easily seen from [7, p. 1196] that these limits are finite and nonnegative, that $\Sigma_p b_p \leqq b$, and that the sequence $\{\Sigma_p b_{n_i p} x^p + c_{n_i} x^0\}$ is pointwise convergent to $\Sigma_p b_p x^p + (c_0 + b - \Sigma_p b_p) x^0$. Hence also $\{y_{n_i}\}$ is pointwise convergent, and by Lemma 3.2 its pointwise limit is in $Q + Q_1$. Since $z$ is the pointwise limit of $\{z_{n_i}\}$, it follows that $z \in Q + Q_1 + Q_2$.

REMARK. It is clear from [7] that the representation of each $z \in L_\Omega(Q)$ in the form $\Sigma_p \Sigma_q a_{pq} x_{pq} + \Sigma_p b_p x^p + c_0 x^0$ is unique.

4. Given an arbitrary countable ordinal $\alpha \geqq 2$ and a number $c \geqq 1$, we now construct a separable Banach space $X_\alpha$ containing a cone $P_\alpha$ for which there exists $z_\alpha \in L_\alpha(P_\alpha)$ such that $\|z_\alpha\| = 1$ but such that if $\{w_n\}$ is a bounded sequence in $\bigcup_{\beta < \alpha} L_\beta(P_\alpha)$ converging pointwise to $z_\alpha$, then $\underline{\lim}_n \|w_n\| \geqq c$.

Let $B_\alpha$ be the countable set $\{(2, 1)\} \cup \{(\beta, \gamma) \colon \alpha \geqq \beta > \gamma \geqq 2\}$. Then there exists a one-to-one mapping $\nu_\alpha$ from $D_\alpha$ onto $B_\alpha$, where $D_\alpha = \{1, \cdots, 2^{-1}(\alpha^2 - 3\alpha + 4)\}$ if $\alpha < \omega$ and $D_\alpha = \omega$ if $\alpha \geqq \omega$, such that $\nu_\alpha(1) = (2, 1)$. Let $U = \{0\} \cup \{n^{-1} \colon n \in D_\alpha\}$ and let $S_\alpha$ be the compact subset $[0; 6] \times U$ of $E^2$. For each real function $z$ defined on $S_\alpha$ and each $u \in U$, let

$$z^{1,u}(t) = z(t, u), \qquad z^{2,u}(t) = z(t + 3, u)$$

for $t \in [0; 3]$. Further, let $\mathscr{S}_\alpha$ be the set of all type $- \alpha$ generalized sequences $s = (s_\beta : 1 \leqq \beta \leqq \alpha)$ of positive integers.

Letting $x_{pq}$ be as in § 3 and noting by [7] that $x_{pq}(0) = x_{pq}(3) = 0$ for $p, q \in \omega$, we easily verify that for each $s \in \mathscr{S}_\alpha$ the function $x_s$ defined by

$$x_s^{1,u} = \begin{cases} x_{s_\beta s_\gamma} & \text{if } u > 0,\, u^{-1} \leqq s_1,\, \nu_\alpha(u^{-1}) = (\beta, \gamma) \\ 0 & \text{if } u > 0,\, u^{-1} > s_1 \\ 0 & \text{if } u = 0 \end{cases}$$

$$x_s^{2,u} = \begin{cases} u x_{s_\beta s_\gamma} & \text{if } u > 0,\, \nu_\alpha(u^{-1}) = (\beta, \gamma) \\ 0 & \text{if } u = 0 \end{cases}$$

is an element of $C(S_\alpha)$. Let $X_\alpha$ be the norm-closed subspace and $P_\alpha$ the norm-closed cone in $C(S_\alpha)$ generated by $\{x_s : s \in \mathscr{S}_\alpha\}$. Since $S_\alpha$ is compact metric, $C(S_\alpha)$ is separable [3, p. 340] and hence also $X_\alpha$ is separable. Note that $\|x_s\| = c$ for each $s \in \mathscr{S}_\alpha$.

For $1 \leqq \delta \leqq \alpha$ and $s \in \mathscr{S}_\alpha$ let $z_{s\,\delta}$ be defined on $S_\alpha$ by

$$z_{s,\delta}^{1,u} = u^{-1} z_{s,\delta}^{2,u} = \begin{cases} x_{s_\beta s_\gamma} & \text{if } u > 0,\, \nu_\alpha(u^{-1}) = (\beta, \gamma),\, \beta > \gamma > \delta \\ x^{s_\beta} & \text{if } u > 0,\, \nu_\alpha(u^{-1}) = (\beta, \gamma),\, \beta > \delta \geqq \gamma \\ x^0 & \text{if } u > 0,\, \nu_\alpha(u^{-1}) = (\beta, \gamma),\, \delta \geqq \beta > \gamma \end{cases}$$

$$z_{s,\delta}^{1,0} = z_{s,\delta}^{2,0} = 0.$$

Thus $\|z_{s.\delta}\| = c$ if $1 \leqq \delta < \alpha$, but $\|z_{s,\alpha}\| = 1$ for each $s \in \mathscr{S}_\alpha$. In fact, $z_{s\,\alpha}$ is independent of $s \in \mathscr{S}_\alpha$ and we simply write $z_\alpha$ instead of $z_{s\,\alpha}$.

LEMMA 4.1. *For each $s \in \mathscr{S}_\alpha$ and $1 \leqq \delta \leqq \alpha$, $z_{s,\delta} \in L_\delta(P_\alpha)$.*

*Proof.* If $\delta = 1$ and $s \in \mathscr{S}_\alpha$, then for each $q \in \omega$ let $s^q \in \mathscr{S}_\alpha$ be defined by

$$s_\beta^q = \begin{cases} q & \text{if } \beta = 1 \\ s_\beta & \text{if } 1 < \beta \leqq \alpha. \end{cases}$$

It is easy to verify that $\{x_{s^q}\}_{q=1}^\infty$ is a bounded sequence in $P_\alpha$ converging pointwise to $z_{s,1}$, so that $z_{s\,1} \in L_1(P_\alpha)$.

Proceeding by transfinite induction, assume that $1 < \delta \leqq \alpha$ and that $z_{s,\varepsilon} \in L_\varepsilon(P_\alpha)$ for each $s \in \mathscr{S}_\alpha$ and $1 \leqq \varepsilon < \delta$. Let $s \in \mathscr{S}_\alpha$ be given, and let $t^q \in \mathscr{S}_\alpha$ be defined for each $q \in \omega$ by

$$t_\beta^q = \begin{cases} s_\beta & \text{if } \delta \neq \beta \leqq \alpha \\ q & \text{if } \beta = \delta. \end{cases}$$

If $\delta$ is not a limiting ordinal, then $\delta$ has an immediate predecessor $\delta - 1$, and it is straightforward to show that the bounded sequence

$\{z_{t^q,\delta-1}\}_{q=1}^{\infty}$ in $L_{\delta-1}(P_\alpha)$ converges pointwise to $z_{s,\delta}$ on $S_\alpha$. On the other hand, if the countable ordinal $\delta$ is limiting, there exists an increasing sequence $\{\varepsilon_q\}_{q=1}^{\infty}$ of ordinals whose limit is $\delta$, and it can be verified that the bounded sequence $\{z_{t^q,\varepsilon_q}\}_{q=1}^{\infty}$ in $\bigcup_{\varepsilon<\delta} L_\varepsilon(P_\alpha)$ is pointwise convergent to $z_{s,\delta}$. Thus the lemma is proved inductively. In particular, our proof has shown that $z_\alpha$, whose norm is 1, is the pointwise limit of a sequence of elements of norm $c$ in $\bigcup_{\beta<\alpha} L_\beta(P_\alpha)$.

Note that if $1 \leqq \delta \leqq \Omega$, $z \in L_\delta(P_\alpha)$, $i \in \{1, 2\}$, and $u \in U$, then $z^{i,u} \in L_\delta(Q) \subseteqq L_\Omega(Q) = Q + Q_1 + Q_2$ by Lemma 3.3, and trivially $z^{i,0} = 0$.

LEMMA 4.2. *Let $1 \leqq \delta \leqq \Omega$ and $z \in L_\delta(P_\alpha)$ with*

$$z^{1,1} = \Sigma_p \Sigma_q a_{pq} x_{pq} + \Sigma_p b_p x^p + c_0 x^0.$$

*Then also $y \in L_\delta(P_\alpha)$, where*

$$y^{1,1} = y^{2,1} = \Sigma_p(b_p + \Sigma_q a_{pq})x^p + c_0 x^0,$$

$y^{2,0} = y^{1,0} = 0$, *and* $uy^{1,u} = y^{2,u} = z^{2,u}$ *for each* $u \in U \backslash \{0, 1\}$.

*Proof.* The proof will be by induction on $\delta$. If $\delta = 1$, then $z^{1,1} \in L_1(Q) = Q + Q_1$ and hence $c_0 = 0$. There exists a bounded sequence $\{w_n\}$ in $P_\alpha$ which converges pointwise to $z$ on $S_\alpha$. Since the finite linear combinations with nonnegative coefficients of elements in $\{x_s: s \in \mathscr{S}_\alpha\}$ are norm-dense in $P_\alpha$, each $w_n$ can be assumed to have the form $w_n = \Sigma_{i\in\omega} r_{ni} x_{(s^{ni})}$, where each $s^{ni} \in \mathscr{S}_\alpha$, each $r_{ni} \geqq 0$, and for each $n$ there exist only finitely many $i$ such that $r_{ni} > 0$. If $t^{ni} \in \mathscr{S}_\alpha$ is defined for all $n, i \in \omega$ by $(t^{ni})_\beta = (s^{ni})_\beta$ for $2 \leqq \beta \leqq \alpha$ and $(t^{ni})_1 = n$, then the sequence $\{w'_n\}$, where $w'_n = \Sigma_{i\in\omega} r_{ni} x_{(t^{ni})}$, is clearly a bounded sequence in $P_\alpha$. It will now be shown that $\{w'_n\}$ converges pointwise to $y$.

For each $u \in U \backslash \{0, 1\}$, $\nu_\alpha(u^{-1}) = (\beta, \gamma)$ for some $\beta, \gamma$ such that $\beta > \gamma \geqq 2$, and hence for each $n \geqq u^{-1}$,

$$\begin{aligned} w_n'^{1,u} &= u^{-1} w_n'^{2,u} = \Sigma_{i\in\omega} r_{ni} x_{(t^{ni})_\beta (t^{ni})_\gamma} \\ &= \Sigma_{i\in\omega} r_{ni} x_{(s^{ni})_\beta (s^{ni})_\gamma} = u^{-1} w_n^{2,u}; \end{aligned}$$

therefore, $w_n'^{1,u}(t) \xrightarrow[n]{} u^{-1} z^{2,u}(t) = y^{1,u}(t)$ and $w_n'^{2,u}(t) \to z^{2,u}(t) = y^{2,u}(t)$ for all $t \in [0; 3]$.

Since the situation for $u = 0$ is trivial, it remains only to consider the case in which $u = 1$. Given $n, p, q \in \omega$ let

$$a_{npq} = \Sigma\{r_{ni}: (s^{ni})_2 = p, (s^{ni})_1 = q\}.$$

Thus each $a_{npq} \geqq 0$, and for each $n$ there are only finitely many pairs $(p, q)$ for which $a_{npq} > 0$. Since $w_n^{1,1} = \Sigma_p \Sigma_q a_{npq} x_{pq}$ for each $n$, it follows from the proof of Lemma 3.2 and the note following that proof that

$\lim_n a_{npq} = a_{pq}$ for each $p, q$; that

$$\lim_n \Sigma_q a_{npq} = c^{-1} z^{1,1}(t_{pp}) = \Sigma_q a_{pq} + b_p$$

for each $p$; and that $\lim\sup_n \Sigma_{p \geqq r} \Sigma_q a_{npq} \to 0$ as $r \to \infty$. Thus given $\varepsilon > 0$, there exist $r$ and $n_1$ such that $\Sigma_{p \geqq r}(\Sigma_q a_{pq} + b_p) < \varepsilon/3c$ and $\Sigma_{p \geqq r} \Sigma_q a_{npq} < \varepsilon/3c$ for all $n > n_1$. Now $w_n'^{1,1} = \Sigma_p(\Sigma_q a_{npq}) x_{pn}$, and for each $t \in [0; 3]$ there exists $n_2(t) > n_1$ such that

$$|(\Sigma_q a_{npq}) x_{pn}(t) - (\Sigma_q a_{pq} + b_p) x^p(t)| < \frac{\varepsilon}{3r}$$

for each $n > n_2(t)$ and $p < r$. It follows easily by the triangle inequality that

$$|w_n'^{1,1}(t) - \Sigma_p(b_p + \Sigma_q a_{pq}) x^p(t)| < \varepsilon$$

for each $n > n_2(t)$. Thus

$$w_n'^{1,1}(t) = w_n'^{2,1}(t) \longrightarrow y^{1,1}(t) = y^{2,1}(t)$$

for all $t$, completing the proof for $\delta = 1$.

Now let $\delta > 1$ and assume that the statement of the lemma is true for each ordinal $\varepsilon$ such that $1 \leqq \varepsilon < \delta$. If $z \in L_\delta(P_\alpha)$, there exists a bounded sequence $\{w_n\} \subset \bigcup_{\varepsilon<\delta} L_\varepsilon(P_\alpha)$ which converges pointwise to $z$. By the induction hypothesis the sequence $\{y_n\}$ is contained in $\bigcup_{\varepsilon<\delta} L_\varepsilon(P_\alpha)$, where, if

$$w_n^{1,1} = \Sigma_{p,q} a_{npq} x_{pq} + \Sigma_p b_{np} x^p + c_n x^0,$$

then

$$y_n^{1,1} = y_n^{2,1} = \Sigma_p(b_{np} + \Sigma_q a_{npq}) x^p + c_n x^0,$$

and $y_n^{1,0} = y_n^{2,0} = 0$ and $u y_n^{1,u} = y_n^{2,u} = w_n^{2,u}$ for $u \neq 0, 1$. An easy induction argument shows that $\|f^{2,u}\| \leqq ucf^{1,1}(s_{11})$ for each $u \in U$ and $f \in L_\Omega(P_\alpha)$, and from this result it follows that the sequence $\{y_n\}$ is bounded. To see that $\{y_n\}$ converges pointwise to $y$, note first that $y_n^{1,0} = y_n^{2,0} = 0 = y^{1,0} = y^{2,0}$ for each $n$. Next, if $u \neq 0, 1$ and $t \in [0; 3]$, then

$$u y_n^{1,u}(t) = y_n^{2,u}(t) = w_n^{2,u}(t) \longrightarrow z^{2,u}(t) = u y^{1,u}(t) = y^{2,u}(t).$$

For $u = 1$, since $y_n^{1,1} = y_n^{2,1}$ and $y^{1,1} = y^{2,1}$, it remains only to show that $y_n^{1,1}(t) \to y^{1,1}(t)$ for each $t \in [0; 3]$. If $t$ is not of the form $s_{pi}$, $2 + s_{pi}$, or $t_{vj}$ with $v \leqq j$, then $y_n^{1,1}(t) = 0 = y^{1,1}(t)$. If $t = s_{p_1 i_1}$ or $2 + s_{p_1 i_1}$ with $i_1$ odd, then

$$y_n^{1,1}(t) = w_n^{1,1}(t) - \Sigma_{p<p_1} \Sigma_q a_{npq} x_{pq}(t)$$

and

$$y^{1,1}(t) = z^{1,1}(t) - \Sigma_{p<p_1}\Sigma_q a_{pq}x_{pq}(t);$$

since $w_n^{1,1}(t) \to z^{1,1}(t)$ and $a_{npq} \to a_{pq}$ (as noted in the proof of Lemma 3.1), and since there exists $q_1$ such that $x_{pq}(t) = 0$ whenever $p < p_1$ $q > q_1$, it follows that $y_n^{1,1}(t) \to y^{1,1}(t)$. Finally, if $t = t_{vj}$ with $1 \leqq v \leqq j$, then

$$\begin{aligned} y_n^{1,1}(t) &= w_n^{1,1}(t) + c\Sigma_{p=v}^{j}\Sigma_{q=1}^{j-p}a_{npq} \\ &\longrightarrow z^{1,1}(t) + c\Sigma_{p=v}^{j}\Sigma_{q=1}^{j-p}a_{pq} = y^{1,1}(t). \end{aligned}$$

This completes the induction step and hence the proof of the lemma.

LEMMA 4.3. *Let* $0 \leqq \delta \leqq \Omega$ *and* $z \in L_\delta(P_\alpha)$. *Then* $z^{1,u} \leqq u^{-1}z^{2,u}$ *for each* $u \in U\backslash\{0\}$. If

$$z^{1,1} = \Sigma_p\Sigma_q a_{pq}x_{pq} + \Sigma_p b_p x^p + c_0 x^0$$

*and if* $q_1 \in \omega$, *then*

$$z^{1,u} \leqq u^{-1}z^{2,u} - c\Sigma_p\Sigma_{q<q_1}a_{pq}$$

*for each* $u \geqq q_1^{-1}$.

*proof.* The first assertion is immediate by induction on $\delta$. For the second assertion suppose first that $z$ has the form $z = \Sigma_{s\in\sigma}d_s x_s$ where $\sigma$ is a finite subset of $\mathscr{S}_\alpha$ and $d_s \geqq 0$ for each $s$. Then $z^{1,1} = \Sigma_p\Sigma_q a_{pq}x_{pq}$, where

$$a_{pq} = \Sigma\{d_s: s \in \sigma,\ s_2 = p,\ s_1 = q\}.$$

Thus $\Sigma_p\Sigma_{q<q_1}a_{pq} = \Sigma\{d_s: s \in \sigma,\ s_1 < q_1\}$ and hence if $u \geqq q_1^{-1}$ and $\nu_\alpha(u^{-1}) = (\beta, \gamma)$, then

$$\begin{aligned} z^{2,u} &= u\Sigma_{s\in\sigma}d_s x_{s_\beta s_\gamma} = uz^{1,u} + u\Sigma_{s_1<u^{-1}}d_s x_{s_\beta s_\gamma} \\ &\leqq u(z^{1,u} + \Sigma_{s_1<q_1}d_s x_{s_\beta s_\gamma}) \leqq u(z^{1,u} + c\Sigma_p\Sigma_{p<q_1}a_{pq}) \end{aligned}$$

as desired.

Next, suppose $z$ is the pointwise limit of a bounded sequence $\{w_n\}_{n\in\omega}$ in $L_\Omega(P_\alpha)$ such that each $w_n$ has the desired property; i.e., for each $u \geqq q_1^{-1}$,

$$w_n^{1,u} \geqq u^{-1}w_n^{2,u} - c\Sigma_p\Sigma_{q<q_1}a_{npq}$$

where

$$w_n^{1,1} = \Sigma_p\Sigma_q a_{npq}x_{pq} + \Sigma_p b_{np}x^p + c_n x^0.$$

By the proof of Lemma 3.3 there is a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $\{\Sigma_p\Sigma_q a_{n_i pq}x_{pq}\}$ is pointwise convergent, and by the note following

Lemma 3.2 for each $\zeta > 0$ there exist $p_1$ and $i_1$ such that for each $i > i_1$,

$$\Sigma_{p\geqq p_1}\Sigma_q a_{n_ipq} < c\zeta.$$

Since $a_{n_ipq} \to a_{pq}$ for each $p$ and $q$, there exists $i_2 > i_1$ such that for each $i > i_2$,

$$\Sigma_{p<p_1}\Sigma_{q<q_1} a_{n_ipq} < \Sigma_{p<p_1}\Sigma_{q<q_1} a_{pq} + \zeta.$$

Hence, for each $i > i_2$,

$$\begin{aligned}\Sigma_p\Sigma_{q<q_1} a_{n_ipq} &< \Sigma_{p<p_1}\Sigma_{q<q_1} a_{pq} + (1+c)\zeta \\ &\leqq \Sigma_p\Sigma_{q<q_1} a_{pq} + (1+c)\zeta.\end{aligned}$$

For each $t \in [0; 3]$ and $u \geqq q_1^{-1}$,

$$\begin{aligned}z^{1,u}(t) = \lim_i w_{n_i}^{1,u}(t) &\geqq \overline{\lim}_i(u^{-1}w_{n_i}^{2,u}(t) - c\Sigma_p\Sigma_{q<q_1}a_{n_ipq}) \\ &\geqq u^{-1}z^{2,u}(t) - c[\Sigma_p\Sigma_{q<q_1}a_{pq} + (1+c)\zeta].\end{aligned}$$

Since $\zeta$ can be arbitrarily small,

$$z^{1,u} \geqq u^{-1}z^{2,u} - c\Sigma_p\Sigma_{q<q_1}a_{pq}$$

for each $u \geqq q_1^{-1}$, as desired.

The preceding paragraphs provide both the base step and the inductive step for the proof of the second assertion of the lemma.

LEMMA 4.4. *Let $G$ be the set of all $z \in L_\Omega(P_\alpha)$ such that $z^{1,1} \in Q_1 + Q_2$. If $z \in G$, then $z^{1,u} = u^{-1}z^{2,u}$ for each $u \in U\backslash\{0\}$.*

*Proof.* In the notation of Lemma 4.3, $a_{pq} = 0$ for all $p$, $q$ and hence $\Sigma_p\Sigma_{q<u^{-1}}a_{pq} = 0$. The present result now follows immediately from Lemma 4.3.

LEMMA 4.5. $L_\delta(P_\alpha) \cap G = \begin{cases} L_{\delta-1}(L_1(P_\alpha) \cap G) & \text{if } 1 \leqq \delta < \omega \\ L_\delta(L_1(P_\alpha) \cap G) & \text{if } \omega \leqq \delta \leqq \Omega. \end{cases}$

*Proof.* The result is trivial for $\delta = 1$. Let $1 < \delta < \omega$ and assume the result is true for all $\varepsilon < \delta$. Then for each $z \in L_\delta(P_\alpha) \cap G$ it follows from Lemma 4.4 that $z^{1,u} = u^{-1}z^{2,u}$ for each $u \neq 0$. Since $z \in G$, it follows that $z$ is identical with the $y$ occurring in the statement of Lemma 4.2 and hence is the pointwise limit of the bounded sequence $\{y_n\} \subset G \cap \bigcup_{1\leqq\varepsilon<\delta} L_\varepsilon(P_\alpha)$ which appears in the inductive step of the proof of Lemma 4.2. By the inductive hypothesis

$$\{y_n\} \subset \bigcup_{1\leqq\varepsilon<\delta} \mathrm{L}_{\varepsilon-1}(L_1(P_\alpha) \cap G) = L_{\delta-2}(L_1(P_2) \cap G)$$

and hence $z \in L_{\delta-1}(L_1(P_\alpha) \cap G)$. Conversely, if $z \in L_{\delta-1}(L_1(P_\alpha) \cap G)$, then $z$ is the pointwise limit of a bounded sequence $\{w_n\} \subset L_{\delta-2}(L_1(P_\alpha) \cap G)$. By the inductive hypothesis $L_{\delta-2}(L_1(P_\alpha) \cap G) = L_{\delta-1}(P_\alpha) \cap G$. Hence clearly $z \in L_\delta(P_\alpha)$, and also $z \in G$ by the proof of Lemma 3.3. Thus the proof is complete for $\delta < \omega$.

Now let $\omega \leqq \delta \leqq \Omega$ and assume the result is true for all $\varepsilon < \delta$. As in the previous case each $z \in L_\delta(P_\alpha) \cap G$ is the pointwise limit of a bounded sequence $\{y_n\} \subset G \cap \bigcup_{\varepsilon<\delta} L_\varepsilon(P_\alpha)$. By the inductive hypothesis $\{y_n\} \subset \bigcup_{\varepsilon<\delta} L_\varepsilon(L_1(P_\alpha) \cap G)$, and hence $z \in L_\delta(L_1(P_\alpha) \cap G)$. Conversely, if $z \in L_\delta(L_1(P_\alpha) \cap G)$, then $z$ is the pointwise limit of a bounded sequence $\{w_n\} \subset \bigcup_{\varepsilon<\delta} L_\varepsilon(L_1(P_\alpha) \cap G)$. By the inductive hypothesis $\{w_n\} \subset G \cap \bigcup_{\varepsilon<\delta} L_\varepsilon(P_\alpha)$ and hence $z \in G \cap L_\delta(P_\alpha)$, completing the proof of the lemma.

LEMMA 4.6. *Let $\{w_n\}$ be a bounded sequence in $\bigcup_{\varepsilon<\alpha} L_\varepsilon(P_\alpha)$ which converges pointwise on $S_\alpha$ to the function $z_\alpha$ defined earlier in the present section. If*

$$w_n^{1,1} = \Sigma_p \Sigma_q a_{npq} x_{pq} + \Sigma_p b_{np} x^p + c_n x^0$$

*for each $n \in \omega$, then* $\lim_n \Sigma_p \Sigma_q a_{npq} = 0$.

*Proof.* If the conclusion is not true, then as in the proof of Lemma 3.3 a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ exists such that $\inf_i \Sigma_p \Sigma_q a_{n_i pq} > 0$ and such that the limits $c_0 = \lim_i c_{n_i}$, $b = \lim_i \Sigma_p b_{n_i p}$, $b_p = \lim_i b_{n_i p}$, and $a_p = \lim_i \Sigma_q a_{n_i pq}$ all exist ($p \in \omega$). Since $z_\alpha^{1,1} = x^0$ by definition of $z_\alpha$, the coefficient of each $x_{pq}$ in the unique expansion of $z_\alpha^{1,1}$ must vanish and it is easily verified that $\{\Sigma_p b_{n_i p} x^p + c_{n_i} x^0\}$ and $\{\Sigma_p \Sigma_q a_{n_i pq} x_{pq}\}$ converge pointwise to $\Sigma_p b_p x^p + (c_0 + b - \Sigma_p b_p) x^0$ and $\Sigma_p a_p x^p$ respectively, as in the proofs of Lemmas 3.3 and 3.2 (note that the symbol $b_p$ is used differently in those two proofs). Hence

$$z_\alpha^{1,1} = \Sigma_p (a_p + b_p) x^p + (c_0 + b - \Sigma_p b_p) x^0 .$$

Now the uniqueness of the expansion of $z_\alpha^{1,1}$ shows that $a_p + b_p = 0$ for each $p$ and $c_0 + b - \Sigma_p b_p = 1$. Since $a_p$ and $b_p$ are nonnegative, they must both vanish for each $p$ and hence $c_0 + b = 1$. Now

$$\begin{aligned} 1 = z_\alpha^{1,1}(s_{11}) &= \lim_i (\Sigma_p \Sigma_q a_{n_i pq} + \Sigma_p b_{n_i p} + c_{n_i}) \\ &= \lim_i \Sigma_p \Sigma_q a_{n_i pq} + b + c_0 \end{aligned}$$

and hence $\lim_i \Sigma_p \Sigma_q a_{n_i pq} = 0$, contradicting our assumption and thus proving the lemma.

THEOREM 4.1. *If $\{w_n\}$ is a bounded sequence in $\bigcup_{\varepsilon<\alpha} L_\varepsilon(P_\alpha)$ which converges pointwise to $z_\alpha$, then there exists a sequence*

$\{y_n\} \subset G \cap \bigcup_{\varepsilon<\alpha} L_\varepsilon(P_\alpha)$ *such that* $||y_n - w_n|| \to 0$.

*Proof.* Each $w_n^{1,1}$ has the form

$$w_n^{1,1} = \Sigma_p \Sigma_q a_{npq} x_{pq} + \Sigma_p b_{np} x^p + c_n x^0.$$

By Lemma 4.2 these exists a sequence $\{y_n\} \subset \bigcup_{\varepsilon<\alpha} L_\varepsilon(P_\alpha)$ such that

$$y_n^{1,1} = y_n^{2,1} = \Sigma_p (b_{np} + \Sigma_q a_{npq}) x^p + c_n x^0,$$

and $y_n^{2,0} = y_n^{1,0} = 0$ and $u y_n^{1,u} = y_n^{2,u} = w_n^{2,u}$ for each $u \neq 0, 1$. Since obviously $\{y_n\} \subset G$, if remains only to show that $\lim_n ||y_n - w_n|| = 0$.

First note that $(y_n - w_n)^{1,0} = 0$ and $(y_n - w_n)^{2,u} = 0$ for all $u \neq 1$.

For each real $r > 0$ there exists by Lemma 4.6 an $n_r \in \omega$ such that $\Sigma_p \Sigma_q a_{npq} < r$ for all $n > n_r$. For each $u \neq 0$ there exists $q_u \in \omega$ such that $u \geqq q_u^{-1}$ and hence by Lemma 4.3,

$$\begin{aligned} u^{-1} w_n^{2,u} - cr < u^{-1} w_n^{2,u} - c \Sigma_p \Sigma_{q<q_u} a_{npq} \\ \leqq w_n^{1,u} \leqq u^{-1} w_n^{2,u} \end{aligned}$$

for each $n > n_r$. Since $y_n^{2,u} = w_n^{2,u}$ for each $u \neq 1$,

$$||(y_n - w_n)^{1,u}|| = ||u^{-1} y_n^{2,u} - w_n^{1,u}|| = ||u^{-1} w_n^{2,u} - w_n^{1,u}|| \leqq cr$$

for each $n > n_r$ and $u \neq 0, 1$.

Finally, since $z^{1,1} = z^{2,1}$ for each $z \in L_\Omega(P_\alpha)$,

$$\begin{aligned} ||(y_n - w_n)^{2,1}|| = ||(y_n - w_n)^{1,1}|| = ||\Sigma_p (\Sigma_q a_{npq} x^p - \Sigma_q a_{npq} x_{pq})|| \\ < 2cr \end{aligned}$$

for each $n > n_r$.

We have now shown that $||y_n - w_n|| < 2cr$ for each $n > n_r$, completing the proof of the theorem.

LEMMA 4.7. *Let $\zeta$ be a countable ordinal, and let $y \in L_\zeta(L_1(P_\alpha) \cap G)$. Let $\zeta' = \zeta + 1$ if $\zeta < \omega$ and $\zeta' = \zeta$ if $\zeta \geqq \omega$. If $u \in U\backslash\{0\}$ and $\nu_\alpha(u^{-1}) = (\beta, \gamma)$ with $\beta > \gamma > \zeta'$, then $y^{1,u}$ is continuous and hence has the form $y^{1,u} = \Sigma_p \Sigma_q a_{pq}^u x_{pq}$. If also $v \in U\backslash\{0\}$ and $\nu_\alpha(v^{-1}) = (\gamma, \delta)$ with $\beta > \gamma > \delta > \zeta'$, then for each $r \in \omega$, $\Sigma_p a_{pr}^u = \Sigma_q a_{rq}^v$.*

*Proof.* The proof will be by induction on $\zeta$. If $y \in L_0(L_1(P_\alpha) \cap G) = L_1(P_\alpha) \cap G$, there is a bounded sequence $\{w_n\} \subset P_\alpha$ which converges pointwise to $y$. The sequence $\{w_n\}$ can be chosen so that each $w_n$ is a finite linear combination of elements of $\{x_s : s \in \mathscr{S}_\alpha\}$, and hence there exists a countable subset $\sigma$ of $\mathscr{S}_\alpha$ such that each $w_n$ has the form $w_n = \Sigma_{s \in \sigma} b_{ns} x_s$, where each $b_{ns}$ is nonnegative and for each $n$ only a finite number of the $b_{ns}$ are nonzero. If $u \neq 0$ and $\nu_\alpha(u^{-1}) = (\beta, \gamma)$, then

$$w_n^{2,u} = u\Sigma_{s\in\sigma} b_{ns} x_{s_\beta s_\gamma} = u\Sigma_p \Sigma_q a_{npq}^u x_{pq},$$

where

$$a_{npq}^u = \Sigma\{b_{ns}: s_\beta = p,\ s_\gamma = q\}.$$

Now $y^{1,u} = u^{-1}y^{2,u}$ by Lemma 4.4 since $y \in G$; hence $y^{1,u}$ is the pointwise limit of the bounded sequence $\{\Sigma_p \Sigma_q a_{npq}^u x_{pq}\}$. The function $y^{1,u}$ is in $L_1(Q)$ and hence has the form

$$y^{1,u} = \Sigma_p \Sigma_q a_{pq}^u x_{pq} + \Sigma_p b_p^u x^p;$$

by the proof of Lemma 3.2, $a_{pq}^u = \lim_n a_{npq}^u$ for all $p$, $q$ and

$$b_p^u = c^{-1} y^{1,u}(t_{pp}) - \Sigma_q a_{pq}^u = \lim_n \Sigma_q a_{npq}^u - \Sigma_q a_{pq}^u$$

for all $p$.

Now assume further that $\nu_\alpha(u^{-1}) = (\beta, \gamma)$ with $\gamma > 1$, and let $\lambda = 2$ if $\gamma > 2$ and $\lambda = 1$ if $\gamma = 2$. Then $(\gamma, \lambda) \in B_\alpha$ so there exists $v_1 \in U\backslash\{0\}$ such that $\nu_\alpha(\nu_1^{-1}) = (\gamma, \lambda)$. Since $\{\Sigma_p \Sigma_q a_{npq}^u x_{pq}\}$ and $\{\Sigma_p \Sigma_q a_{npq}^{v_1} x_{pq}\}$ are bounded pointwise convergent sequences in $Q$, it follows from the note following Lemma 3.2 that for each real $\varepsilon > 0$ there exist integers $p_1$ and $n_1$ such that $\Sigma_{p>p_1} \Sigma_q a_{npq}^u < \varepsilon$ and $\Sigma_{p>p_1} \Sigma_q a_{npq}^{v_1} < \varepsilon$ for all $n \geqq n_1$. Since

$$\Sigma_p \Sigma_{q>p_1} a_{npq}^u = \Sigma\{b_{ns}: s_\gamma > p_1\} = \Sigma_{p>p_1} \Sigma_q a_{npq}^{v_1} < \varepsilon$$

for each $n \geqq n_1$, it follows that if $f_n = \Sigma_{p\leqq p_1} \Sigma_{q\leqq p_1} a_{npq}^u x_{pq}$,

$$||u^{-1} w_n^{2,w} - f_n|| \leqq c\Sigma\{a_{npq}^u: p > \mathrm{p}_1 \text{ or } q < p_1\} > 2c\varepsilon$$

for each $n \geqq n_1$. Since $||f_n|| \leqq ||u^{-1} w_n^{2,u}|| \leqq u^{-1} \sup_n ||w_n||$ for each $n$, it follows that for each $n \geqq n_1$, $f_n$ belongs to the compact subset

$$\mathscr{C}_{u,p_1} = \{\Sigma_{p\leqq p_1} \Sigma_{q\leqq p_1} k_{pq} x_{pq}: k_{pq} \geqq 0,\ \Sigma_{p\leqq p_1} \Sigma_{q\leqq p_1} k_{pq} \leqq u^{-1} \sup_n ||w_n||\}$$

of $C[0; 3]$. By compactness some subsequence $\{f_{n_i}\}$ of $\{f_n\}$ must converge to an element $f$ of $\mathscr{C}_{u,p_1}$, and since $\{u^{-1} w_{n_i}^{2,u}\}$ converges pointwise to $y^{1,u}$, it follows that $||y^{1,u} - f|| \leqq 2c\varepsilon$. Thus, for each $\varepsilon > 0$ there exists an $f \in C[0; 3]$, depending on $\varepsilon$, such that $||y^{1,u} - f|| \leqq 2c\varepsilon$. Since $C[0; 3]$ is complete in norm, $y^{1,u} \in C[0; 3]$ and must therefore be equal to $\Sigma_p \Sigma_q a_{pq}^u x_{pq}$.

Now if $0 \neq v \in U$ and $\nu_\alpha(v^{-1}) = (\gamma, \delta)$ with $\gamma > \delta > 1$, then for all $n$ and $r$,

$$\Sigma_p a_{npr}^u = \Sigma\{b_{ns}: s_\gamma = r\} = \Sigma_q a_{nrq}^v.$$

Since $y^{1,v} = \Sigma_p \Sigma_q a_{pq}^v x_{pq}$, it follows that

$$\Sigma_q a^v_{rq} = c^{-1}y^{1,v}(t_{rr}) = \lim_n c^{-1}v^{-1}w^{2,v}_n(t_{rr})$$
$$= \lim_n \Sigma_q a^v_{nrq} = \lim_n \Sigma_p a^v_{npr}.$$

On the other hand the bounded sequence $\{\Sigma_p \Sigma_q a^u_{npq} x_{pq}\}$ converges pointwise to $y^{1,u} = \Sigma_p \Sigma_q a^u_{pq} x_{pq}$. By the note following Lemma 3.2, for each $\varepsilon > 0$ there exist $p_1$ and $n_1$ such that $\Sigma_{p>p_1} \Sigma_q a^u_{npq} < \varepsilon$ for all $n \geqq n_1$ and also $\Sigma_{p>p_1} \Sigma_q a^u_{pq} < \varepsilon$. Hence

$$|\Sigma_p a^u_{pr} - \lim_n \Sigma_p a^u_{npr}| < 2\varepsilon + |\Sigma_{p\leqq p_1} a^u_{pr} - \lim_n \Sigma_{p \leqq p_1} a^u_{npr}|$$
$$= 2\varepsilon.$$

Since $\varepsilon$ is an arbitrary positive number,

$$\Sigma_p a^u_{pr} = \lim_n \Sigma_p a^u_{npr} = \Sigma_q a^v_{rq}.$$

This completes the proof of the lemma for $\zeta = 0$.

For the induction step let $0 < \zeta < \Omega$, assume the desired result holds for each $\eta < \zeta$, and let $y$, $\zeta'$, $u$, $\beta$, and $\gamma$ be as in the statement of the lemma. Then there exists a bounded sequence $\{y_n\}$ in $\bigcup_{\eta<\zeta} L_\eta(L_1(P_\alpha) \cap G)$ which converges pointwise to $y$. Since $1 < \zeta' < \gamma \leqq \alpha$, there exists $v_1 \in U\backslash\{0\}$ such that $\nu_\alpha(v_1^{-1}) = (\gamma, \zeta')$. For each $n$ there exists $\eta_n < \zeta$ such that $y_n \in L_{\eta_n}(L_1(P_\alpha) \cap G)$, and it follows that $\beta > \gamma > \zeta' > \eta'_n$ for each $n$, where $\eta'_n$ is de fined in terms of $\eta_n$ as $\zeta'$ was defined in terms of $\zeta$. By the induction assumption $y^{1,u}_n$ and $y^{1,v_1}_n$ are continuous and have the form $y^{1,u}_n = \Sigma_p \Sigma_q a^u_{npq} x_{pq}$ and $y^{1,v_1}_n = \Sigma_p \Sigma_q a^{v_1}_{npq} x_{pq}$, and $\Sigma_p a^u_{npr} = \Sigma_q a^{v_1}_{nrq}$ for all $n$ and $r$.

As in the proof for $\zeta = 0$, for each $\varepsilon > 0$ there exist $n_1$ and $p_1$ such that $\Sigma_{p>p_1} a^u_{npq} < \varepsilon$ and $\Sigma_{p>p_1} \Sigma_q a^{v_1}_{npq} < \varepsilon$ for all $n \geqq n_1$. Hence, since $\Sigma_p a^u_{npr} = \Sigma_q a^{v_1}_{nrq}$ for all $n$ and $r$, it follows that for $n \geqq n_1$, the distance between $y^{1,u}_n$ and the compact subset

$$\mathscr{D}_{p_1} = \{\Sigma_{p\leqq p_1} \Sigma_{q \leqq p_1} k_{pq} x_{pq} \colon k_{pq} \geqq 0,\ \Sigma_{p \leqq p_1} \Sigma_{q \leqq p_1} k_{pq} \leqq \sup_n \|y^{1,u}_n\|\}$$

of $C[0; 3]$ is less than $2\varepsilon c$. Since $\{y^{1,u}_n\}$ converges pointwise to $y^{1,u}$, the compactness of $\mathscr{D}_{p_1}$ implies that $\|y^{1,u} - w\| \leqq 2\varepsilon c$ for some continuous $w$ depending on $\varepsilon$. Then the completeness of $C[0; 3]$ implies that $y^{1,u} \in C[0; 3]$ and therefore, since also $y^{1,u} \in L_1(Q)$, that $y^{1,u}$ has the form $\Sigma_p \Sigma_q a^u_{pq} x_{pq}$.

If also $0 \neq v \in U$ and $\nu_\alpha(v^{-1}) = (\gamma, \delta)$ with $\beta > \gamma > \delta > \zeta'$, then $y^{1,v}$ and each $y^{1,v}_n$ are continuous and have form corresponding to $y^{1,u}$ and $y^{1,u}_n$ respectively. Further, by the induction assumption, $\Sigma_p a^u_{npr} = \Sigma_q a^v_{nrq}$ for all $n$ and $r$. Hence

$$\Sigma_q a^v_{rq} = c^{-1} y^{1,v}(t_{rr}) = \lim_n c^{-1} y^{1,v}_n(t_{rr}) = \lim_n \Sigma_q r^v_{nrq}$$
$$= \lim_n \Sigma_p a^u_{npr}.$$

Exactly as in the last part of the proof for $\zeta = 0$ it is seen that

$\Sigma_p a^u_{pr} = \lim_n \Sigma_p a^u_{npr}$. This completes the proof of the induction step and hence of the lemma.

LEMMA 4.8. *If* $y \in L_\zeta(L_1(P_\alpha) \cap G)$ *for some countable* $\zeta$ *and if* $u, v \in U\backslash\{0\}$ *with* $\nu_\alpha(u^{-1}) = (\beta, \gamma)$ *and* $\nu_\alpha(v^{-1}) = (\beta, \delta)$ *for certain ordinals* $\beta, \gamma, \delta$ *then in the expression*

$$y^{1,u} = \Sigma_p \Sigma_q a^u_{pq} x_{pq} + \Sigma_p b^u_p x^p + c^u x^0$$

*and the corresponding expression for* $y^{1,v}$ *it must be true that* $y^{1,u}(2^{-1}) = y^{1,v}(2^{-1})$, $c^u = c^v$, *and* $b^u_p + \Sigma_q a^u_{pq} = b^v_p + \Sigma_q a^v_{pq}$ *for each* $p$.

*Proof.* By Lemma 4.5, $y \in G$. Hence, by Lemma 4.4, $y^{1,u} = u^{-1}y^{2,u}$ and $y^{1,v} = v^{-1}y^{2,v}$.

If $\zeta = 0$, then $y$ is the pointwise limit of a bounded sequence $\{y_n\}$ of functions of the form $y_n = \Sigma_{s \in \sigma_n} b_{ns} x_s$, where $\sigma_n$ is a finite subset of $\mathscr{S}_\alpha$ and each $b_{ns}$ is nonnegative. For each $p$ and $n$,

$$u^{-1}y^{2,u}_n(t_{pp}) = c\Sigma\{b_{ns}: s_\beta = p\} = v^{-1}y^{2,v}_n(t_{pp}).$$

Since $\{y^{2,u}_n\}$ converges pointwise to $y^{2,u}$,

$$y^{1,u}(t_{pp}) = u^{-1}y^{2,u}(t_{pp}) = v^{-1}y^{2,v}(t_{pp}) = y^{1,v}(t_{pp})$$

for each $p$, and hence it follows immediately that

$$\begin{aligned} b^u_p + \Sigma_q a^u_{pq} &= c^{-1}y^{1,u}(t_{pp}) = c^{-1}y^{1,v}(t_{pp}) \\ &= b^v_p + \Sigma_q a^v_{pq} \end{aligned}$$

for each $p$. Since $y^{1,u}$ and $y^{1,v}$ are Baire functions of the first class, $c^u = 0 = c^v$. Hence

$$y^{1,u}(2^{-1}) = \Sigma_p(b^u_p + \Sigma_q a^u_{pq}) = y^{1,v}(2^{-1}).$$

For the induction step let $\zeta > 0$ and assume the statement of the lemma holds for each $\eta < \zeta$. By hypothesis there exists a bounded sequence $\{y_n\}$ in $\bigcup_{\eta<\zeta} L_\eta(L_1(P_\alpha) \cap G)$ which converges pointwise to $y$. Under the usual notation the relations

$$b^u_{np} + \Sigma_q a^u_{npq} = b^v_{np} + \Sigma_q a^v_{npq},$$

$c^u_n = c^v_n$, and $y^{1,u}_n(2^{-1}) = y^{1,v}_n(2^{-1})$ must hold for all $n$ and $p$. It is seen immediately that $y^{1,u}(2^{-1}) = y^{1,v}(2^{-1})$ and $y^{1,u}(t_{pp}) = y^{1,v}(t_{pp})$ for all $p$, from which the remaing desired relations for $y^{1,u}$ and $y^{1,v}$ follow. The proof is thus complete.

THEOREM 4.2. *Let* $\zeta$ *be a countable ordinal, and let* $\zeta'$ *be defined as in Lemma 4.7. If* $y \in L_\zeta(L_1(P_\alpha) \cap G)$ *and* $0 \neq u \in U$ *with* $\nu_\alpha(u^{-1}) = (\beta, \gamma)$

*and* $\beta > \zeta'$, *then* $y^{1,u} \in Q + Q_1$.

*Proof.* If $\zeta = 0$, then $y \in L_1(P_\alpha)$ and hence trivially $y^{1,u} \in L_1(Q)$, which is equal to $Q + Q_1$ by Lemma 3.2.

If $\zeta > 0$ and the desired result is true for each $\eta < \zeta$, then $2 \leqq \zeta' < \beta \leqq \alpha$ and hence there exists $v \in U\backslash\{0\}$ such that $\nu_\alpha(v^{-1}) = (\beta, \zeta')$. There exists a bounded sequence $\{y_n\}$ in $\bigcup_{\eta<\zeta} L_\eta(L_1(P_\alpha) \cap G)$ which converges pointwise to $y$. Since $\beta > \zeta' > \eta'$ for each $\eta < \zeta$ it follows from Lemma 4.7 that each $y_n^{1,v}$ is continuous and hence belongs to $Q$. Hence $y^{1,v} \in L_1(Q) = Q + Q_1$. Thus in the usual notation for $y^{1,u}$ and $y^{1,v}$ it follows that $c^v = 0$, but then also $c^u = 0$ by Lemma 4.8, hence $y^{1,u} \in Q + Q_1$, and the proof is complete.

The following theorem justifies the claim made at the beginning of the present section.

THEOREM 4.3. *The element* $z_\alpha \in L_\alpha(P_\alpha)$ *has the property that* $\|z_\alpha\| = 1$ *but that if* $\{w_n\}$ *is a bounded sequence in* $\bigcup_{\beta<\alpha} L_\beta(P_\alpha)$ *converging pointwise to* $z_\alpha$, *then* $\underline{\lim}_n \|w_n\| \geqq c$.

*Proof.* By Lemma 4.1 and the remarks preceding it we know that $z_\alpha \in L_\alpha(P_\alpha)$ and $\|z_\alpha\| = 1$. If $\{w_n\}$ is a bounded sequence in $\bigcup_{\beta<\alpha} L_\beta(P_\alpha)$ converging pointwise to $z_\alpha$, then by Theorem 4.1 there exists a sequence $\{y_n\}$ in $G \cap \bigcup_{\beta<\alpha} L_\beta(P_\alpha)$ such that $\|y_n - w_n\| \to 0$. Clearly $\underline{\lim}_n \|w_n\| = \lim_n \|y_n\|$. Now by Lemma 4.5,

$$\{y_n\} \subset \begin{cases} L_{\alpha-2}(L_1(P_\alpha) \cap G) & \text{if } 2 \leqq \alpha < \omega \\ \bigcup_{\beta<\alpha} L_\beta(L_1(P_\alpha) \cap G) & \text{if } \omega \leqq \alpha < \Omega. \end{cases}$$

Defining $\zeta'$ as in Lemma 4.7, one sees easily that each $y_n \in L_{\zeta_n}(L_1(P_\alpha) \cap G)$ for some $\zeta_n$ such that $\alpha > \zeta_n'$. Now there exists $u_1 \in U\backslash\{0\}$ such that $\nu_\alpha(u_1^{-1}) = (\alpha, \gamma)$ for some $\gamma < \alpha$; for example, take $\gamma = 1$ if $\alpha = 2$ and $\gamma = 2$ if $\alpha > 2$. Then by Theorem 4.2, $y_n^{1,u_1} \in Q + Q_1 = L_1(Q)$ for each $n$. Now $z_\alpha^{1,u_1} = x^0$ by definition, and hence $\underline{\lim}_n \|y_n^{1,u_1}\| \geqq c$ by Theorem 1 of [7]. It follows that

$$\underline{\lim}_n \|w_n\| = \underline{\lim}_n \|y_n\| \geqq \underline{\lim}_n \|y_n^{1,u_1}\| \geqq c.$$

COROLLARY 4.1. *Let* $T$ *be the mapping of Theorem* 2.1 *for the space* $X_\alpha$, *and let* $G_\alpha = Tz_\alpha$. *Then* $G_\alpha \in K_\alpha(J_{X_\alpha} P_\alpha)$ *and* $\|G_\alpha\| = 1$, *but if* $\{F_n\}$ *is a sequence in* $\bigcup_{\beta<\alpha} K_\beta(J_{X_\alpha} P_\alpha)$ *such that* $F_n \xrightarrow{w^*} G_\alpha$, *then* $\underline{\lim}_n \|F_n\| \geqq c$.

*Proof.* It is immediate from Theorem 2.1 that $G_\alpha \in K_\alpha(J_{X_\alpha} P_\alpha)$ and $\|G_\alpha\| = 1$. If $\{F_n\} \subset \bigcup_{\beta<\alpha} K_\beta(J_{X_\alpha} P_\alpha)$ and $F_n \xrightarrow{w^*} G_\alpha$, then by Theorem 2.1 the sequence $\{T^{-1}F_n\}$ is in $\bigcup_{\beta<\alpha} L_\beta(P_\alpha)$ and $\|T^{-1}F_n\| = \|F_n\|$ for each

$n$. Now $\sup_n \| T^{-1}F_n \| = \sup_n \| F_n \| < \infty$ since $\{F_n\}$ is $w^*$-convergent. For each $t \in S_\alpha$ let $f_t \in X_\alpha^*$ be defined as in the proof of Theorem 2.1. Then

$$(T^{-1}F_n)(t) = F_n(f_t) \longrightarrow G_\alpha(f_t) = z_\alpha(t)$$

for each $t$, and hence

$$\underline{\lim}_n \| F_n \| = \underline{\lim}_n \| T^{-1}F_n \| \geqq c.$$

5. Our main theorems will now be proved through consideration of product spaces, as defined in [2, p. 31], of spaces of the type $X_\alpha$. Since $X_\alpha$, $P_\alpha$, and $G_\alpha$ depend on the given number $c \geqq 1$ as well as on $\alpha$, the objects mentioned will henceforth be indicated with double subscripts as $X_{c,\alpha}$, $P_{c\,\alpha}$, and $G_{c,\alpha}$ respectively. Recall that if $I$ is a set and $X_s$ is a Banach space for each $s \in I$, then the product spaces $\Pi_{l_1(I)} X_s^*$ and $\Pi_{m(I)} X_s^{**}$ are respectively the dual and bidual of the Banach space $\Pi_{c_0((I)} X_s$ under the natural identifications.

THEOREM 5.1. *For each countable ordinal* $\alpha \geqq 2$ *let* $Y_\alpha$ *be the Banach space* $\Pi_{c_0(\omega)} X_{n^2,\alpha}$ *and let*

$$Q_\alpha = \bigcap\nolimits_{n \in \omega} \{y \in Y_\alpha \colon y(n) \in P_{n^2,\alpha}\}.$$

*Then* $Y_\alpha$ *is separable, and* $Q_\alpha$ *is a norm-closed cone in* $Y_\alpha$ *such that* $K_\alpha(J_{Y_\alpha} Q_\alpha)$ *is not norm-closed in* $Y_\alpha^{**}$.

*Proof.* It is evident that $Y_\alpha$ is separable and $Q_\alpha$ is a closed cone in $Y_\alpha$. An easy transfinite induction argument shows that for each $n$ the functional $F_n$ belongs to $K_\alpha(J_{Y_\alpha} Q_\alpha)$, where $F_n(n) = G_{n^2,\alpha}$ and $F_n(i) = 0$ for all $i \neq n$. Hence $\Sigma_{n=1}^m n^{-1} F_n \in K_\alpha(J_{Y_\alpha} Q_\alpha)$ for each positive integer $m$, and therefore $\Sigma_{n \in \omega} n^{-1} F_n \in \overline{K_\alpha(J_{Y_\alpha} Q_\alpha)}$. If $\{H_k\}$ were a sequence in $\bigcup_{\beta<\alpha} K_\beta(J_{Y_\alpha} Q_\alpha)$ such that $H_k \xrightarrow{w^*} \Sigma_n n^{-1} F_n$, then for each $i \in \omega$ it would follow that

$$\{H_k(i)\}_k \subset \bigcup\nolimits_{\beta<\alpha} K_\beta(J_{X_{i^2,\alpha}} P_{i^2,\alpha})$$

and

$$H_k(i) \xrightarrow{w^*} \Sigma_n n^{-1} F_n(i) = i^{-1} G_{i^2,\alpha}.$$

It would then result by Corollary 4.1 that

$$\underline{\lim}_k \| H_k \| \geqq \underline{\lim}_k \| H_k(i) \| \geqq i,$$

but then since $i$ is arbitrary the sequence $\{H_k\}$ would be unbounded in norm, contradicting the fact that a $w^*$-convergent sequence in $Y_\alpha^{**}$ must be bounded [3, p. 60]. Hence $\Sigma_n n^{-1} F_n \notin K_\alpha(J_{Y_\alpha} Q_\alpha)$, and the proof is complete.

THEOREM 5.2. *For each countable ordinal $\alpha \geqq 2$ there exists a separable Banach space $W_\alpha$ containing a norm-closed cone $R_\alpha$ such that if $2 \leqq \beta \leqq \alpha$, then $K_\beta(J_{W_\alpha} R_\alpha)$ is not norm-closed in $W_\alpha^{**}$.*

*Proof.* Let $A_\alpha = \{\beta: 2 \leqq \beta \leqq \alpha\}$ and for each $\beta \in A_\alpha$ let $Y_\beta$ and $Q_\beta$ be as defined in Theorem 5.1. Let $W_\alpha = \Pi_{c_0(A_\alpha)} Y_\beta$ and $R_\alpha = \bigcap_{\beta \in A_\alpha} \{w \in W_\alpha: w(\beta) \in Q_\beta\}$. Then the Banach space $W_\alpha$ is separable since $A_\alpha$ is countable, and $R_\alpha$ is clearly a norm-closed cone in $W_\alpha$. For each $\beta \in A_\alpha$ there exists by Theorem 5.1 a sequence $\{\phi_{\beta,n}\}$ in $K_\beta(J_{Y_\beta} Q_\beta)$ which coverges in norm to an element $\phi_{\beta,0} \in Y_\beta^{**}$ not in $K_\beta(J_{Y_\beta} Q_\beta)$. If $\psi_{\beta,n}$ is defined for each integer $n \geqq 0$ by $\psi_{\beta,n}(\gamma) = 0$ for $\gamma \neq \beta$ and $\psi_{\beta,n}(\beta) = \phi_{\beta,n}$, it is easily shown that $\{\psi_{\beta,n}\}_{n \in \omega} \subset K_\beta(J_{W_\alpha} R_\alpha)$ and $\{\psi_{\beta,n}\}$ converges in norm to $\psi_{\beta\, 0}$, but that $\psi_{\beta,0} \notin K_\beta(J_{W_\alpha} R_\alpha)$. Hence for each $\beta \in A_\alpha$, $K_\beta(J_{W_\alpha} R_\alpha)$ fails to be norm-closed in $W_\alpha^{**}$.

THEOREM 5.3. *There exists a Banach space $Z$ contaning a norm-closed cone $P$ such that if $\beta$ is a countable ordinal $\geqq 2$, then $K_\beta(J_Z P)$ fails to be norm-closed in $Z^{**}$.*

*Proof.* The proof is almost identical with that of Theorem 5.2. Let $A = \{\beta: 2 \leqq \beta < \Omega\}$, $Z = \Pi_{c_0(A)} Y_\beta$, and $P = \bigcap_{\beta \in A} \{z \in Z: z(\beta) \in Q_\beta\}$. Since $A$ is uncountable, the Banach space $Z$ is nonseparable. It is clear that $P$ is a closed cone in $Z$. The pooof that $K_\beta(J_Z P)$ fails to be norm-closed in $Z^{**}$ for each $\beta \in A$ is identical with the corresponding part of the proof of Theorem 5.2, in which it was shown that $K_\beta(J_{W_\alpha} R_\alpha)$ fails to be norm-closed in $W_\alpha^{**}$ for each $\beta \in A_\alpha$.

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