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MODULES OVER UNIVERSAL REGULAR RINGS

ROGER WIEGAND*

To each commutive ring R there is associated a certain commutative regular ring $\hat{R}$. The ring $\hat{R}$ is in fact an R-algebra. It is shown that ${}_R\hat{R}$ is never flat, unless R is itself regular. The functor taking R to $\hat{R}$ preserves direct limits, and, in certain cases, tensor products. It is shown that if R is weakly noetherian then the global dimension of $\hat{R}$ is less than or equal to the Krull dimension of R. Necessary and sufficient conditions that $\hat{R}$ be a quotient ring of R are determined.

In this paper we study a certain commutative (von Neumann) regular ring $\hat{R}$ associated with each commutative ring R. There is a natural homomorphism $\phi: R \rightarrow \hat{R}$, characterized by the following universal property: every homomorphism from R into a regular ring factors uniquely through ϕ. The ring $\hat{R}$ has been studied briefly in [7] and [5]. In §1 we construct $\hat{R}$ and derive its basic properties, including the universal property mentioned above. The construction uses a little sheaf theory, although once a few lemmas have been proved it will rarely be necessary to recall the sheaf-theoretic construction. In fact, in §5 we give a simple description of $\hat{R}$ that is completely nontopological. In §2 we study relationships between an R-module A and the $\hat{R}$-module $A \otimes_R \hat{R}$, and in §3 we restrict our attention to weakly noetherian rings, that is, rings with maximum condition on radical ideals. It is shown that R is weakly noetherian if and only if $A \otimes_R \hat{R}$ is $\hat{R}$-projective for every finitely generated A_R. Homological considerations are taken up in §4, and it is shown that if R is weakly noetherian then the global dimension of $\hat{R}$ is less than or equal to the Krull dimension of R. In §6 we examine how the functor taking R to $\hat{R}$ behaves with respect to tensor products and direct limits. The last section is devoted to semiprime rings, and we find necessary and sufficient conditions that $\hat{R}$ be a quotient ring of R.

We make the standing assumption that all rings are commutative with unit, and all ring homomorphisms and modules are unitary. We now establish some notation to be preserved throughout the paper. Recall that Spec(R) is the set of prime ideals of R, with the Zariski topology. If S is a subset of R, we let $V(S)$ denote the (closed) subset of Spec(R) consisting of those prime ideal that contain S, and we let $D(S) = \text{Spec}(R) - V(S)$. If $x \in \text{Spec}(R)$ let k_x denote the quotient field of the domain R/x, and for each $a \in R$ let $a(x)$ be the image of a under the natural map $R \to k_x$. Finally, for each pair of elements $a, b \in R$, let $[a, b]$ be the element of $\prod\{k_x: x \in X\}$ whose x^{th} coordinate is $a(x)\backslash b(x)$ when $x \in D(b)$ and 0 otherwise.

1. Definition and basic properties of R. A topological space is *spectral* if it is compact and T_0, if the compact open sets form an open base and are closed under finite intersections, and if every closed irreducible set is the closure of a point. Spec(R) is always spectral, and M. Hochster [5] has recently proved that every spectral space is homeomorphic to Spec(R) for some ring R. The first step of his proof is to retopologize the spectral space X by taking all compact open sets and their complement as an open subbase. This stronger topology is called the *patch* topology on X. The set X with the patch topology will be denoted by $\hat{X}$. It can be shown [5] that $\hat{X}$ is a Boolean space, that is, compact, Hausdorff, and totally disconnected. In case $X = \text{Spec}(R)$, one readily verifies that the sets $D(a) \cap V(b_1) \cap \cdots \cap V(b_n)$, $a, b_i \in R$, are clopen in the patch topology and form an open base.

Now let R be any ring and let $X = \text{Spec}(R)$. Let $\mathscr{R}$ be the disjoint union of the fields k_x. Then we may regard the elements $[a, b]$ as maps from $\hat{X}$ to $\mathscr{R}$ and we give the set $\mathscr{R}$ the strongest topology making all these maps continuous. With this topology, $\mathscr{R}$ is easily seen to be a sheaf of fields over $\hat{X}$, and we let $R = \Gamma(\hat{X}, \mathscr{R})$, the ring of global sections of $\mathscr{R}$. For each prime ideal P of R, let $\hat{P}$ be the prime ideal of $\hat{R}$ consisting of those sections that vanish at P. Let $\phi: R \to \hat{R}$ be the map $a \mapsto [a, 1]$. The following theorem is a direct consequence of the representation theory of regular rings [8]:

THEOREM 1. *$\hat{R}$ is a regular ring. The correspondence $P \to \hat{P}$ is a homeomorphism from $\hat{X}$ onto Spec($\hat{R}$). The natural map $\phi: R \to \hat{R}$ induces an isomorphism from k_x onto $\hat{R}/\hat{x}$ for each $x \in X$. If R is regular then ϕ is an isomorphism.*

We remark that $\hat{R}$ can be identified with the subring of $\prod\{k_x: x \in \hat{X}\}$ consisting of those elements that are locally (in the patch topology!) of the form $[a, b]$. (This is the definition of $\hat{R}$ given in [5].)

Regular rings are characterized by the property that the local ring at each maximal ideal is the same as the corresponding residue class field. This fact and the isomorphisms $k_x = \hat{R}/\hat{x}$ provide a convenient localization technique, which will be exploited throughout the paper. The next result is a simple illustration of this technique.

COROLLARY. *The natural map $\hat{R} \otimes_R \hat{R} \to \hat{R}$ is an isomorphism.*

Proof. Localizing at each maximal ideal $\hat{x}$ of $\hat{R}$, we have natural isomorphisms $\hat{R} \otimes_R \hat{R} \otimes_R k_x = \hat{R} \otimes_R k_x = \hat{R} \otimes_R k_x \otimes_{\hat{R}} k_x = k_x \otimes_R k_x = k_x = \hat{R} \otimes_{\hat{R}} k_x$. The globalization theorem completes the proof.

We conclude this section with a proof of the universal property stated in [7].

COROLLARY. *Every ring homomorphism from R into a commutative regular ring factors uniquely through the natural map $\phi: R \to \hat{R}$.*

Proof. Let $\psi: R \to S$ be the homomorphism in question. There can be at most one factorization, since, by the corollary above and [9], ϕ is an epimorphism in the category of (not necessarily commutative) rings. To prove that a factorization exists, we identify S with $\hat{S}$ and we let $Y = \hat{Y} = \mathrm{Spec}(S)$. The map $\mathrm{Spec}(\psi): Y \to X$ defines a continuous map $\bar{\psi}: \hat{Y} \to \hat{X}$. For each $y \in Y$, ψ induces a field homomorphism $\psi_y: k_{\bar{\psi}(y)} \to S/y$. If $\rho \in R$, we define $\sigma \in S$ by $\sigma(y) = \psi_y(\rho(\bar{\psi}(y)))$, and $\theta: \rho \to \sigma$ is the desired factorization.

2. Tensoring with $\hat{R}$.

In this section we study the relationship between an R-module A and the $\hat{R}$-module $A \otimes_R \hat{R}$. The latter module has a very convenient local description. For example, if $x \in \mathrm{Spec}(R)$, then

$$A \otimes_R \hat{R}/\hat{x} = A \otimes_R k_x = (A/xA) \otimes_{R/x} k_x \,. \tag{1}$$

As an application, suppose R is a principal ideal domain. Then $A \otimes_R R = 0$ if and only if A is a divisible torsion module.

Another local description is obtained by identifying k_x with the residue class field of the local ring R_x. Then, changing notation, we have

$$A \otimes_R \hat{R}/\hat{P} = A_P/PA_P \,. \tag{2}$$

From (2) and Nakayama's lemma, we obtain the following useful observation:

PROPOSITION 1. *Let A_R be finitely generated and let P be a prime ideal of R. Then $A \otimes_R \hat{R}/\hat{P} = 0$ if and only if $A_P = 0$. In particular, if $A \otimes_R \hat{R} = 0$ then $A = 0$.*

COROLLARY. *If $\hat{R}$ is flat as an R-module then $\hat{R} = R$.*

Proof. We need only verify that R is regular. Let I be a finitely generated ideal of R. We show I is a direct summand of R. Let J be the annihilator of I. It will suffice to show that $I + J =$

R, for then $I \cap J = IJ = 0$. If $I + J \neq R$, let P be a prime ideal containg $I + J$. Then $I \otimes_R \hat{R} \subseteqq P \otimes_R \hat{R} = P\hat{R} \subseteqq \hat{P}$, and it follows that $I \otimes_R \hat{R}/\hat{P} = 0$. By Proposition 1, $I_P = 0$, that is $J \nsubseteqq P$, a contradiction.

We remark that the functor $\_\otimes_R \hat{R}$ is not faithful, even on maps between finitely generated modules. For example, the embedding $\boldsymbol{Z}/(p^n) \to \boldsymbol{Z}/(p^{n+1})$ is killed by tensoring with $\boldsymbol{Z}$.

Let A be a finitely generated R-module. For each prime ideal P of R let $r_A(P) = r(P)$ be the minimum number of generators required for A_P over R_P. By (2) and Nakayama's lemma, $r(P)$ is the dimension of $A \otimes_R \hat{R}/\hat{P}$ as a vector space over $\hat{R}/\hat{P}$.

PROPOSITION 2. *Let A_R be finitely generated. Then $A \otimes_R \hat{R}$ is a projective $\hat{R}$-module if and only if for each n the set $U_n = \{P \in \operatorname{Spec}(R) \mid r(P) = n\}$ is compact* (in the Zariski topology).

Proof. The function $r(P)$ is always upper-semicontinuous on $X = \operatorname{Spec}(R)$. In other words, if $V_n = U_0 \cup \cdots \cup U_n$, then the sets V_n are always open in X. It follows that each U_n is compact if and only if each V_n is compact. But an open subset of X is compact if and only if it is clopen in $\hat{X}$. Therefore, the sets U_n are all compact if and only if the dimension of $A \otimes_R \hat{R}/\hat{P}$ is locally constant on $\hat{X}$, that is, if and only if $A \otimes_R \hat{R}$ is $\hat{R}$-projective [8, p. 63].

EXAMPLE 1. Suppose $\operatorname{Spec}(R)$ has a noncompact open set U. Write $U = D(I)$, I an ideal of R, and let $C = R/I$. Then, for $P \in \operatorname{Spec}(R)$, we have $C_P = 0$ if and only if C is annihilated by an element not in P, that is, if and only, if $P \in U$. Thus $U_0 = U$, and by Proposition 2 $C \otimes_R \hat{R}$ is not projective.

For each A_R let $d(A) = \sup \{r_A(P) \mid P \in \operatorname{Spec}(R)\}$. (Since $r(P) \leqq r(Q)$ whenever $P \subseteqq Q$, it suffices to take the supremum over all maximal ideals.) If R is regular, it is known [8, p. 57] that $d(A)$ is equal to the minimum number of generators required for A_R. Combining this fact with the remark preceding Proposition 2, we have the following result:

PROPOSITION 3. *Let A_R be finitely generated. Then $(A \otimes_R \hat{R})_{\hat{R}}$ can be generated by $d(A)$ elements, and no fewer.*

3. Noetherian spectral spaces. Recall that a topological space is *noetherian* if it has ascending chain condition on open sets, or, equivalently, if every open subset is compact. A ring R will be called *weakly noetherian* if $\operatorname{Spec}(R)$ is noetherian, that is, if R has maximum condition on intersections of prime ideals.

THEOREM 2. *The following conditions on the ring* R *are equivalent:* (a) R *is weakly noetherian,* (b) $(A \otimes_R \hat{R})_{\hat{R}}$ *is projective for each finitely generated* A_R, *and* (c) $(C \otimes_R \hat{R})_{\hat{R}}$ *is projective for each cyclic* C_R.

Proof. (a) $\Rightarrow$ (b) by Proposition 2, (b) $\Rightarrow$ (c) trivially, and (c) $\Rightarrow$ (a) by Example 1.

Every spectral space (in fact every T_0 space) has a natural partial ordering: $x \leqq y$ if and only if $y \in \{x\}^-$. The *dimension* $\dim(X)$ of the spectral space X is the greatest integer n such that there is a sequence $x_0 < \cdots < x_n$ in X. If no such integer exists we say $\dim(X) = \infty$. The dimension of $\mathrm{Spec}(R)$ is just the Krull dimension of R, that is, the supremum of lengths of chains of prime ideals in R.

Recall that the *derived space* Y' of a space Y is the complement in Y of the set of isolated points of Y. A transfinite sequence $\{Y^{(\alpha)}\}$ of closed subsets of Y is defined as follows: $Y^{(0)} = Y$, $Y^{(\alpha+1)} = (Y^{(\alpha)})'$, and $Y^{(\beta)} = \cap \{Y^{(\alpha)}: \alpha < \beta\}$ if β is a limit ordinal. Suppose now that Y is a Boolean space. We call Y *superatomic* if the associated Boolean algebra is superatomic, that is, $Y^{(\xi)} = \varnothing$ for some ordinal ξ. If ξ is the smallest ordinal such that $Y^{(\xi)} = \varnothing$, then by compactness ξ cannot be a limit ordinal, and we define $\lambda(Y) = \xi - 1$.

If S is a subset of the spectral space X we shall write S^p for the set S with the topology inherited from $\hat{X}$. In case S is closed in $\hat{X}$, is is easily shown that S is a spectral space and $\hat{S} = S^p$.

THEOREM 3. *If* X *is a noetherian spectral space then* $\hat{X}$ *is superatomic. If, in addition,* $\dim(X) < \infty$, *then* $\lambda(X) \leqq \dim(X)$.

Proof. Let S be an arbitrary nonempty subset of X, and let x be a maximal member of S. I claim that x is an isolated point of S^p. To see this, notice that $X - (x)^-$, being compact open, is clopen in $\hat{X}$. Since $\{x\} = S \cap \{x\}^-$, the claim follows. Setting $S = \hat{X}^{(\alpha)}$, we see that $\hat{X}^{(\alpha+1)}$ is properly contained in $\hat{X}^{(\alpha)}$ whenever $\hat{X}^{(\alpha)} \neq \varnothing$. It follows that $\hat{X}$ is superatomic.

The second assertion is proved by induction on $\dim(X)$. If $\dim(X) = 0$, every element of X is maximal, and therefore an isolated point of $\hat{X}$. Hence $\lambda(\hat{X}) = 0$. Now assume $\dim(X) \geqq 1$ and let X' denote the set $\hat{X}'$ with the relative topology as a subset of X. Then X' is a noetherian spectral space. Moreover, X' contains none of the maximal members of X, so $\dim(X') < \dim(X)$. Then $\lambda(\hat{X}) = \lambda(\hat{X}') + 1 = \lambda((X')^\wedge) + 1 \leqq \dim(X') + 1 \leqq \dim(X)$.

EXAMPLE 2. A noetherian spectral space X_n such that $\dim(X_n) =$

n and $\lambda(X_n) = n$. Let $R_1, \cdots, R_n$ be principal ideal domains, each with infinitely many maximal ideals. Let $Y_i = \mathrm{Spec}(R_i)$ and let $X_n = Y_1 \times \cdots \times Y_n$ with the usual product topology. By [5, Th. 7], X_n is a spectral space, and it is easily seen that X_n is noetherian. Since the partial ordering on X_n is the product ordering, it follows that $\dim(X_n) = n$. Now $\hat{Y}_i$ is the one-point compactification of the discrete set of maximal ideals of R_i, and the patch functor, being a right adjoint [5], preserves products. Therefore $\hat{X}_n = \hat{Y}_1 \times \cdots \times \hat{Y}_n$, and clearly $\lambda(\hat{X}_n) = n$.

EXAMPLE 3. A noetherian spectral space X such that $\lambda(\hat{X}) = \omega$. Let $X = \{0\} \cup \{(i, j): 1 \leqq i \leqq j < \omega\}$. Topologize X by taking as a closed subbase the sets $S_{ij} = \{(k, j): i \leqq k \leqq j\}$. Then every proper closed set is finite, so X is certainly noetherian. Since $\hat{X}^{(n)} = \{0\} \cup \{(i, j): 1 \leqq i \leqq j - n < \omega\}$, we see that $\lambda(\hat{X}) = \omega$.

4. **Homological properties.** Let R be a (commutative) regular ring and let $X = \mathrm{Spec}(R)$. We say R is superatomic if its Boolean algebra of idempotents is superatomic, or, in the terminology of the last section, if X is superatomic. In this case we let $\lambda(R) = \lambda(X)$.

PROPOSITION 4. *Let R be a superatomic regular ring, and suppose $\lambda(R)$ is finite. Then gl. dim$(R) \leqq \lambda(R)$.*

Proof. We argue by induction on $\lambda(R)$. If $\lambda(R) = 0$ then $\mathrm{Spec}(R)$ is discrete and therefore finite. It follows that R is a finite direct product of fields, and gl. $\dim(R) \leqq 0$. Assume $\lambda(R) = n \geqq 1$, and let J be the socle of R. Then $\mathrm{Spec}(R/J) = V(J) = X'$, the derived space of X. Therefore $\lambda(R/J) = n - 1$, and by induction gl. $\dim(R/J) \leqq n - 1$. By [4, Cor. 4], gl. $\dim(R) \leqq n$.

THEOREM 4. *If R is weakly noetherian, the global dimension of $\hat{R}$ is less than or equal to the Krull dimension of R.*

Proof. Immediate from Theorem 3 and Proposition 4.

PROPOSITION 5. *Let R be a superatomic regular ring. Then* gl. $\dim(R) = \sup \{h.\ \dim_R(S) \mid {}_R S$ *is simple*$\}$.

Proof. We first show every nonzero R-module contains a simple module. It suffices to show that every proper ideal I is contained in an ideal J such that J/I is simple. Let $X = \mathrm{Spec}(R)$ and let $C = V(I)$. Clearly $C' \subseteqq X'$, and by induction $C^{(\xi)} = \varnothing$ for some ordinal ξ. In particular, $C' \neq C$, so there is a point $x \in C$ such that

$C - \{x\}$ is closed in X, say $C - \{x\} = V(J)$. Clearly J/I is simple. Now if A is an arbitrary R-module, define a transfinite sequence of submodules A_α as follows: $A_0 = 0$; if A_α is defined and $A_\alpha \neq A$, let $A_{\alpha+1}/A_\alpha$ be a simple submodule of A/A_α; if β is a limit ordinal, let $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$. By Auslander's lemma, h. $\dim_R(A) \leqq \sup\{\text{h. } \dim(A_{\alpha+1}/A_\alpha)\}$, [1, Proposition 3].

In particular, if $\hat{R}$ is superatomic, then gl. $\dim(\hat{R}) = \sup \{\text{h. } \dim_{\hat{R}}(\hat{R}/\hat{P}) : P \in \operatorname{Spec}(R)\}$. If we look at a particular prime, we can considerably sharpen the estimate of Theorem 4. Recall that the dimension $\dim(P)$ of P is the Krull dimension of R/P, that is, the supremum of lengths of chains of primes $P = P_0 \subset \cdots \subset P_n$ in R.

THEOREM 5. *Let P be a prime ideal of R. Assume $D(P)$ is compact and R/P is weakly noetherian. Then* h. $\dim_{\hat{R}}(\hat{R}/\hat{P}) \leqq \dim(P)$.

Proof. Let $X = \operatorname{Spec}(R)$, $V = V(P)$, $D = D(P)$. Then $\hat{X}$ is the disjoint union $D^p \cup V^p$, and each of these sets is clopen. Let $d = \dim(P)$. If $d = 0$ then $\{P\} = V$. It follows that $\hat{P}$ is a direct summand of $\hat{R}$; therefore $\hat{R}/\hat{P}$ is projective. Now assume $d > 0$, and let A be an arbitrary $\hat{R}$-module. Let $S = R/P$ and consider the $\hat{S}$-module $B = (A/PA) \otimes_S \hat{S}$. Let $Z \in \operatorname{Spec}(S)$ be the 0-ideal. We shall show that

$$\operatorname{Ext}^d_{\hat{R}}(\hat{P}, A) = \operatorname{Ext}^d_{\hat{S}}(\hat{Z}, B) . \tag{3}$$

The right-hand side of (3) is 0 by Theorem 4. Since A was arbitrary, it will follow that h. $\dim_{\hat{R}}(\hat{P}) \leqq d - 1$, and therefore that h. $\dim_{\hat{R}}(\hat{R}/\hat{P}) \leqq d$.

We identify V with $\operatorname{Spec}(S)$ and let $\tilde{A}$ (resp. $\hat{B}$) be the usual sheaf of modules over $\hat{X}$ (resp. V^p) corresponding to A (resp. B). If $Q \in V$, a straightforward computation shows that $B/(Q/P)^\wedge B \cong A/\hat{Q}A$, that is, under the identification $V = \operatorname{Spec}(S)$, $\tilde{A}$ and $\tilde{B}$ have the same stalks over V^p. Since all the isomorphisms are natural, we conclude that $\tilde{B} = \tilde{A} \mid V^p$. Now by [10] we have $\operatorname{Ext}^d_{\hat{S}}(\hat{Z}, B) = H^d(V^p - \{P\}; \tilde{B}) = H^d(V^p - \{P\}; \tilde{A})$, and $\operatorname{Ext}^d_{\hat{R}}(\hat{P}, A) = H^d(\hat{X} - \{P\}; \tilde{A}) = H^d(V^p - \{P\}; \tilde{A}) \oplus H^d(D^p; \tilde{A})$. But $\tilde{A} \mid D^p$ is acyclic since D^p is a Boolean space [10]. Therefore $H^d(D^p; \tilde{A}) = 0$, and (3) is verified.

The estimates given in Theorems 4 and 5 are in general very rough. For example, if R is any countable ring then $\hat{R}$ is countable (proof in the next section) and therefore hereditary. On the other hand, if R is a ring such that $\operatorname{Spec}(R)$ is the space X_2 of Example 2, with $|Y_1| = \aleph_0$ and $|Y_2| = \aleph_1$, then $\hat{R}$ is *not* hereditary. (In fact, it can be shown that gl. $\dim(\hat{R}) = 2$.) I know of no example of a weakly noetherian ring R such that gl. $\dim(\hat{R}) = n > 2$, but I conjecture that the space X_n would provide one, if we were to take $|Y_i| <$

$|Y_{i+1}|$, $1 \leqq i < n$.

5. **The ring $\bar{R}$.** For each $a \in R$ let $\bar{a}$ denote the saturated multiplicative set generated by a, that is, the set of elements of R that divide some power of a, Let $\mathscr{M} = \{\bar{a}\colon a \in R\}$. If $S, T \in \mathscr{M}$ let ST denote the smallest saturated multiplicative set containing $S \cup T$. Then $ST \in \mathscr{M}$; in fact, if $S = \bar{a}$ and $T = \bar{b}$ then $ST = \overline{ab}$. For each $S, T \in \mathscr{M}$ there is a map $R_S \otimes_R R_T \rightarrow R_{ST}$ defined by $[u/s] \otimes [v/t] \rightarrow [uv/st]$. These maps make the R-module $\bar{R} = \bigoplus \Sigma R_S$ into an R-algebra.

We will define a natural homomorphism $\Phi\colon \bar{R} \rightarrow \hat{R}$. First, we make two trivial observations: (1) Every element of R_S ($S \in \mathscr{M}$) can be written in the form $[u/s]$, with $\bar{s} = S$. (2) If $s, t \in R$, then $\bar{s} = \bar{t}$ if and only if $D(s) = D(t)$. Now if $\sigma \in R_S$, write $\sigma = [u/s]$, with $\bar{s} = S$, and define $\Phi_S(\sigma) = [u, s] \in \hat{R}$, (in the notation of § 1). Then the maps Φ_S are well-defined homomorphisms of R-modules and induce an algebra homomorphism $\Phi\colon \bar{R} \rightarrow \hat{R}$.

THEOREM 6. *Φ is surjective.*

Proof. We first show every idempotent is in $\operatorname{Im}(\Phi)$. For each set U clopen in $\hat{X}$ ($X = \operatorname{Spec}(R)$), let $e_U \in \hat{R}$ be the corresponding idempotent: $e_U(x) = 1$ if $x \in U$ and 0 if $x \notin U$. Let G be the set of clopen sets U for which $e_U \in \operatorname{Im}(\Phi)$. Clearly G is closed under finite Boolean combinations. Since G contains every set of the form $D(s)$, $S \in R$, it follows that every clopen set is in G. Now let $\rho \in \hat{R}$. By the remark following Theorem 1, there are clopen sets U_i covering $\hat{X}$ and element $[a_i, b_i] \in \hat{R}$ such that $\rho(x) = [a_i, b_i]\ (x)$ for each $x \in U_i$. Refining, we may assume the U_i form a finite disjoint cover. Then $\rho = \Sigma_i e_{U_i}[a_i, b_i] \in \operatorname{Im}(\Phi)$.

COROLLARY. *Let R be a ring and let $X = \operatorname{Spec}(R)$. Then $\hat{R}$* is *isomorphic to the subring of $\prod\{k_x\colon x \in X\}$ consisting of all finite sums of elements of the form $[a, b]$.*

COROLLARY. *$|\hat{R}| = |R/N|$, where N is the set of nilpotent elements of R.*

Proof. If R has no nilpotents, clearly $\phi\colon R \rightarrow \hat{R}$ is one-to-one. Also, it is easily verified that $\hat{R} \cong (R/N)\hat{}$. It follows from the corollary that $|\hat{R}| = |R/N|$, at least in case R is infinite. But if R is finite then R/N is semisimple, and $R/N = (R/N)\hat{} = \hat{R}$.

One might guess that the map $\Phi\colon \bar{R} \rightarrow \hat{R}$ is the start of a convenient flat resolution of $\hat{R}$. Unfortunately, this does not seem to be the case; there is no simple criterion for a "nonhomogeneous" element of $\bar{R}$ to be in $\mathrm{Ker}(\Phi)$. Even though the computation of $\mathrm{Tor}_n^R(\hat{R}, A)$ appears formidable, we can simplify the problem to a certain extent:

PROPOSITION 6. *For each $\hat{R}$-module A and each $n \geqq 0$ there is a natural isomorphism* $\mathrm{Tor}_n^R(\hat{R}, A) = A \otimes_{\hat{R}} \mathrm{Tor}_n^R(\hat{R}, \hat{R})$.

Proof. If $n = 0$ this is the first corollary to Theorem 1. Assume the statement holds for n. Then, since R is regular; $\mathrm{Tor}_n^R(\hat{R}, \_)$ is an exact functor from $\hat{R}$-modules to R-modules. The long exact sequence then shows that $\mathrm{Tor}_{n+1}^R(\hat{R}, \_)$ is a right exact functor of $\hat{R}$-modules. Since this functor preserves coproducts, $\mathrm{Tor}_{n+1}^R(\hat{R}, A) = A \otimes_{\hat{R}} \mathrm{Tor}_{n+1}^R(\hat{R}, \hat{R})$.

COROLLARY. *If* $\mathrm{Tor}_p^R(\hat{R}, \hat{R}) = 0$ *for each* $p > 0$, *then* gl. $\dim(\hat{R}) \leqq$ gl. $\dim(R)$.

Proof. Straightforward induction shows that if A is an $\hat{R}$-module and $\mathrm{Tor}_p^R(\hat{R}, A) = 0$ for each $p > 0$, then h. $\dim_{\hat{R}}(A) \leqq$ h. $\dim_R(A)$. Now apply Proposition 6.

Unfortunately, the hypotheses of the corollary are not likely to hold under very general conditions. In fact, $\mathrm{Tor}_1^Z(\hat{Z}, \hat{Z}) \neq 0$, since the torsion subgroup of $\hat{Z}$ is easily seen to be $\bigoplus \Sigma_p Z/(p)$.

6. The functor $R \to \hat{R}$. Let k be a fixed commutative ring, and let $\mathscr{C}_k$ denote the category of commutative unitary k-algebras. Let $\mathscr{V}_k$ be the full subcategory of $\mathscr{C}_k$ whose objects are the regular k-algebras. If $R \in \mathscr{C}_k$ clearly $\hat{R} \in \mathscr{V}_k$. Suppose $\theta\colon R \to S$ is a morphism in $\mathscr{C}_k$. Let $\hat{\theta}\colon \hat{R} \to \hat{S}$ be the unique homomorphism such that $(R \xrightarrow{\theta} \hat{R} \xrightarrow{\hat{\theta}} \hat{S}) = (R \xrightarrow{\theta} S \xrightarrow{\phi} \hat{S})$. From the construction of $\hat{\theta}$ (§1) it is clear that $\hat{\theta}$ is a k-algebra homomorphism. Thus $V_k\colon \mathscr{C}_k \to \mathscr{V}_k$ taking R to $\hat{R}$ is a functor. In fact, using [6, p. 128], we can say much more.

PROPOSITION 7. *$\mathscr{V}_k$ is a full, coreflective subcategory of $\mathscr{C}_k$. In fact $V_k\colon R \mapsto \hat{R}$ is the left adjoint of the inclusion $\mathscr{V}_k \to \mathscr{C}_k$.*

COROLLARY. *If R is the direct limit of the k-algebras R_i then $\hat{R} = \varinjlim_i(\hat{R}_i)$. If R and S are k-algebras, then $(R \otimes_k S)\hat{} = (\hat{R} \otimes_k \hat{S})\hat{}$. If either of the natural maps $R \otimes_k R \to R$ or $S \otimes_k S \to S$ is an isomorphism then $(R \otimes_k S)\hat{} = \hat{R} \otimes_k \hat{S}$.*

Proof. Since V_k has a right adjoint, V_k is right-continuous. In particular, V_k preserves coproducts (when they exist) and direct limits. Since the ordinary direct limit of regular rings is obviously regular, the direct limit in $\mathscr{V}_k$ is the ordinary direct limit, and the first assertion follows. To prove the second statement, notice that $R \otimes_k S$ is the coproduct of R and S in $\mathscr{C}_k$. Therefore $(R \otimes_k S)^\wedge = \hat{R} \amalg \hat{S}$, where $\amalg$ denotes the coproduct in $\mathscr{V}_k$. But if A and B are regular, then $(A \otimes_k B)^\wedge = \hat{A} \amalg \hat{B} = A \amalg B$. In particular, $\hat{R} \amalg \hat{S} = (\hat{R} \otimes_k \hat{S})^\wedge$. To prove the last statement of the corollary, assume $R \otimes_k R \to R$ is an isomorphism. Then by [9] $k \to R$ is an epimorphism in $\mathscr{S}$, the category of not necessarily commutative rings. Since $R \to \hat{R}$ is also an epimorphism in $\mathscr{S}$, the map $k \to \hat{R}$ is an epimorphism in $\mathscr{S}$. By [9] again, $\hat{R} \otimes_k \hat{R} \to \hat{R}$ is an isomorphism. (A direct proof using associativity formulas is also easy.) The desired conclusion now follows from the next lemma.

LEMMA. *Let R and S be k-algebras. Assume $R \otimes_k R \to R$ is an isomorphism and S is regular. Then $R \otimes_k S$ is regular.*

Proof. Let A and B be R-S-bimodules. Then $A \otimes_{(R\otimes S)} B = A \otimes_{(R\otimes S)} (R \otimes_k B) = (A \otimes_R R) \otimes_S B = A \otimes_S B$. Therefore $(A \otimes_{(R\otimes S)} -)$ is an exact functor. Since A was arbitrary, $R \otimes_k S$ is regular.

COROLLARY: *If S is an arbitrary R-algebra then $(\hat{R} \otimes_R S)^\wedge = \hat{R} \otimes_R \hat{S}$.*

7. **Semiprime rings.** Let R be a semiprime ring, that is, a ring with no nonzero nilpotent elements. Then the natural map ϕ: $R \to \hat{R}$ is an embedding, and we identify R with its image in $\hat{R}$. Since R is nonsingular (that is, not an essential extension of the annihilator of any nonzero element), the maximal quotient ring of R is the injective hull of $_RR$. (See pp. 58, 64, and Theorem 1. + 2. on p. 69 of [3].) Therefore $\hat{R}$ is a quotient ring of R if and only if $_R\hat{R}$ is an essential extension of $_RR$.

THEOREM 7. *The following condition on a semiprime ring R are equivalent:*

(a) *$\hat{R}$ is a quotient ring of R.*

(b) *Every nonempty subset of Spec(R) that is open in the patch topology contains a nonempty set of the form $D(s)$.*

(c) *Distinct compact open subsets of Spec(R) have distinct closures.*

(d) *If I is a finitely generated ideal of R and $r \notin \sqrt{I}$ then there is an $s \in R$ such that $sI = 0$ but $sr \neq 0$.*

Proof. (a) $\Rightarrow$ (b): Let $X = \text{Spec}(R)$ and let U be a nonempty open subset of $\hat{X}$. We may assume U is clopen in $\hat{X}$. Let $e \in \hat{R}$ be the idempotent with support U, and choose $r \in R$ such that $0 \neq re \in R$. Then $\varnothing \neq D(re) \subseteqq U$.

(b) $\Rightarrow$ (a): Let σ be a nonzero element of $\hat{R}$. Suppose first that there exist $a, b \in R$ and a clopen set $V \subseteqq \hat{X}$ such that $V \subseteqq D(b)$, and $\sigma(x) = a(x)/b(x)$ on V and 0 outside V. Then $V \cap D(a) \neq \varnothing$, so let $\varnothing \neq D(s) \subseteqq V \cap D(a)$. Then $sb\sigma = sa \in R$. Also, $sa \neq 0$, since $0 \neq s \in \sqrt{a}$ and R is semiprime. In general, we can write $\sigma = \sigma_1 + \cdots + \sigma_n$, where each of the σ_i is of the above form, and $\hat{R}\sigma_k \cap \sum_{i \neq k} \hat{R}\sigma_i = 0$ for each k. We may assume $\sigma_1 \neq 0$. I claim that for each k there is an $r_k \in R$ such that $0 \neq r_k(\sigma_1 + \cdots + \sigma_k) \in R$. For $k = 1$ this has already been verified. Assume, inductively, that r_k has been chosen. If $r_k\sigma_{k+1} = 0$, let $r_{k+1} = r_k$. If not, let $0 \neq sr_k\sigma_{k+1} \in R$, and let $r_{k+1} = sr_k$.

(b) $\Rightarrow$ (c): Let U and V be compact open subsets of X, and suppose $U \not\subseteqq V$. Then $U - V$ is a nonempty clopen subset of $\hat{X}$, so let $\varnothing \neq D(s) \subseteqq U - V$. Then $D(s) \subseteqq \bar{U}$ but $D(s) \cap \bar{V} = \varnothing$, so $\bar{U} \not\subseteqq \bar{V}$.

(c) $\Rightarrow$ (d): If $r \notin \sqrt{I}$ then $D(r) \not\subseteqq D(I)$. Choose a point $x \in \overline{D(r)} - \overline{D(I)}$. Let $D(s)$ be a neighborhood of x that misses $D(I)$. Then $sI = 0$ but $sr \neq 0$.

(d) $\Rightarrow$ (b): Recall that the sets $D(r) \cap V(I)$, I a finitely generated ideal of R, form an open base for X. If $D(r) \cap V(I) \neq \varnothing$, then $r \notin \sqrt{I}$. Choose $s \in R$ such that $sI = 0$ and $sr \neq 0$. Then $\varnothing \neq D(sr) \subseteqq D(r) \cap V(I)$.

COROLLARY: *If $\hat{R}$ is a quotient ring of R and R is either weakly noetherian or semihereditary, then $R = \hat{R}$.*

Proof. Suppose R is weakly noetherian, and let M be an arbitrary maximal ideal of R. Then $X - \{M\} = D(M)$ is a compact open subset of $X = \operatorname{Spec}(R)$. By (c) of Theorem 7, $D(M)$ must be closed in X. It follows easily that M is a direct summand of R. Therefore R is semisimple with minimum condition.

Now assume R is semihereditary. By (d) of Theorem 7, R has the following property: (*) Every finitely generated proper ideal of R has nonzero annihilator. But Bass [2, Theorem 5.4] has shown that condition (*) is equivalent to the condition that every finitely generated projective submodule of a projective module is a direct summand. Therefore every finitely generated ideal of R is a direct summand, and R is regular.

F. L. Sandomierski has pointed out that if R satisfies (*) then the weak dimension of every ${}_RA$ is strictly less than h. $\dim_R(A)$ (if A is not projective). By induction, it suffices to show ${}_RA$ is flat whenever $h.\ dim_R(A) \leqq 1$. Write $A = P/K$ with P and K projective. Let $K \oplus T = F$, a free module. Then $A = (P \oplus T)/F$. But F is the direct limit of free submodules of finite rank, and (*) implies that each of these is a direct summand of $P \oplus T$. It follows that A is a direct limit of projective modules and therefore is flat.

A module of projective dimension 2 may fail to be flat. In fact, we give an example of a non-regular ring with global dimension 2 that satisfies the conditions of Theorem 7.

EXAMPLE 4. For each positive integer k let d_k be the product of the first k primes, and let R_k be the ring of integers modulo d_k. Let $P = \bigoplus \Sigma R_k$, and let R be the subring of $\prod R_k$ generated by P and the identity element. (R consists of all "eventually constant" sequences.) Since R/P is isomorphic to the ring of integers, R is not regular. For each $x \in R$ let $x(k) \in R_k$ denote the k^{th} coordinate of x. Let $x, y_1, \cdots, y_n \in R$. It is easy to see that $x \in \sqrt{(y_1, \cdots, y_n)}$ if and only if, for each k, $x(k) \in \sqrt{(y_1(k), \cdots, y_n(k))}$. We now show that R satisfies condition (d) of Theorem 7. Assume $x \notin \sqrt{(y_1, \cdots, y_n)}$. Since each of the rings R_k is regular, there is some k and an $r \in R_k$ such that $rx(k) \neq 0$ but $ry_i(k) = 0$ for each $i \leqq n$. If s is the element of R with r in the k^{th} position and 0's elsewhere, clearly $sx \neq 0$ and $sy_i = 0$ for each i, as desired. To show that gl. dim$(R) = 2$, we know gl. dim$(R) \geqq 2$, by the corollary to Theorem 7. But since Soc$(R) =$ P and $R/P \cong Z$, it follows from [4, Cor. 4] that gl. dim$(R) \leqq 2$.

REFERENCES

1. M. Auslander, *On the dimension of modules and algebras* (III) Nagoya Math. J., **9** (1955), 67-77.

2. H. Bass, *Finitistic dimension and a homological generalization of Semi-primary rings*, Trans. Amer. Math. Soc., **95** (1960), 466-488.

3. C. Faith, *Lectures on Injective Modules and Quotient Rings*, Lecture Notes in Mathematics 49, Springer (1967).

4. K. L. Fields, *The global dimension of residue rings*, Pacific J. Math., **32** (1970), 345-349.

5. M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc., **142** (1969), 43-60.

6. B. Mitchell, *Theory of Categories*, Academic Press, New York, 1965.

7. J. P. Olivier, *Anneaux absolument plats universals et epimorphisms d'anneaux*, C. R. Acad. Sci. Paris, Ser. A et B **266** (1968), 317.

8. R. S. Pierce, *Modules over commutative regular rings*, Memoirs Amer. Math. Soc., **70** (1967).

9. L. Silver, *Non-commutative localizations and applications*, J. Algebra, **7** (1967), 44-76.

10. R. Wiegand, *Sheaf cohomology of locally compact, totally disconnected spaces*, Proc. Amer. Math. Soc., **20**, No. 2, 533-538.

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UNIVERSITY OF WISCONSIN