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CONCERNING WEB-LIKE CONTINUA

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The compact metric continuum M is said to be a web if and only if there exist two monotone upper semi-continuous decompositions G_1 and G_2 of M such that M/G_1 and M/G_2 are arcs and each element of G_1 intersects each element of G_2. It is shown that there exists in Euclidean 3-space a compact continuum M that is not a web but does have two monotone upper semicontinuous decompositions G_1 and G_2 such that (1) M/G_1 and M/G_2 are simple closed curves and (2) each element of G_1 intersects each element of G_2. Such continua are called pseudo-webs.

This solves a problem suggested to the author by Professor R. L. Moore. It is also shown that there do not exist pseudo-webs in the plane.

THEOREM 1. *Suppose M is a metric chainable continuum, J is a simple metric closed curve, and g_1 and g_2 are mutually exclusive subcontinua of $M \times J$ such that if P is a point of M, then g_1 and g_2 intersect $P \times J$. Then no subcontinum of $M \times J$ separates g_1 from g_2 in $M \times J$.*

Proof. Suppose there is a subcontinum g of $M \times J$ which separates g_1 from g_2 in $M \times J$. Let ε denote a positive number less than the distances from g_1 to $g_2 + g$ and g_2 to $g_1 + g$. There exists an ε-map f from M onto $[0, 1]$. If P is a point of M and j is in J, let $T(P, j) = (f(P), j)$. T is an ε-map from $M \times J$ onto $[0, 1] \times J$. If P is a point of $[0, 1]$, $T(g)$, $T(g_1)$, and $T(g_2)$ are mutually exclusive continua intersecting $P \times J$.

By Theorem 29 of Chapter IV of [5], there exist two mutually exclusive arcs α_1 and α_2, each intersecting $0 \times J$ and $1 \times J$, such that (1) only the endpoints of α_1 and α_2 lie on $0 \times J$ and $1 \times J$, and (2) $\alpha_1 + \alpha_2$ separates $T(g)$ from $T(g_1 + g_2)$ in $[0, 1] \times J$. $([0, 1] \times J) - (\alpha_1 + \alpha_2)$ is the sum of two mutually separated connected point sets, D and D' containing $T(g)$ and $T(g_1 + g_2)$, respectively. Let β denote $\bar{D}' \cdot (0 \times J)$. β is an arc of $0 \times J$ that intersects $T(g_1)$ and $T(g_2)$ and does not intersect $T(g)$.

Let Z be a point of $T^{-1}(\beta)$. Let Z' denote the point of M such that Z is a point of $Z' \times J$. Let P_1 and P_2 denote points of $g_1' \cdot (Z' \times J)$ and $g_2' \cdot (Z' \times J)$, respectively. Since g separates g_1 from g_2 in $M \times J$, there exist two points X_1 and X_2 of g which separate P_1 from P_2 in $(Z' \times J)$. Then $T(X_1) + T(X_2)$ separates $T(P_1)$ from $T(P_2)$ in $(0 \times J)$. Then β contains either $T(X_1)$ or $T(X_2)$. This involves a contradiction. Hence g does not separate g_1 from g_2 on $Z' \times J$. Therefore, there is a connected subset of $Z' \times J$ that intersects g_1 and g_2 but not g, and hence g does not separate g_1 from g_2 in $M \times J$.

THEOREM 2. *The Cartesian product of a metric chainable indecomposable continuum with a metric simple closed curve is a pseudo-web.*

Proof. Let M denote a chainable indecomposable continuum in the xy-plane of E^3 and J denote a simple closed curve. It will first be shown that there exist two monotone upper semi-continuous decompositions, G_1 and G_2, of $M \times J$ such that each element of G_1 intersects each element of G_2 and $(M \times J)/G_1$ and $(M \times J)/G_2$ are simple closed curves. It will then be shown that there is no monotone upper semi-continuous decomposition of $M \times J$ which is an arc with respect to its elements.

Let L denote a line in the xy-plane parallel to the y-axis not intersecting M. $M \times J$ is homeomorphic to the point set M' obtained by revolving M about L. Let H_1 denote the collection to which h belongs if and only if for some half-plane A with L on its boundary, g is $M' \cdot A$. Let P denote a point of L which is on a horizontal line intersecting M, and L' denote a line in the xy-plane distinct from L such that L' contains P and does not intersect M. L' is not perpendicular to L. Let H_2 denote the collection to which h belongs if and only if for some half-plane A with L' on its boundary, h is $M' \cdot A$.

M'/H_1 and M'/H_2 are simple closed curves. There exist an arc H_1' of elements of H_1 and an arc H_2' of elements of H_2 such that each element of H_1' intersects each element of H_2'. For each $i = 1, 2$, let G_i denote the collection to which g belongs if and only if g is a separating element of H_i' or g is $\overline{(H_i - H_i')^*}$. G_1 and G_2 are two monotone upper semi-continuous decompositions of M such that each of M/G_1 and M/G_2 is a simple closed curve and each element of G_1 intersects each element of G_2.

Therefore, in order to prove that $M \times J$ is a pseudo-web, it will be sufficient to show that there is no monotone upper semi-continuous decomposition of M which is an arc with respect to its elements.

Suppose there exists a monotone upper semi-continuous decomposition G of $M \times J$ such that M/G is an arc. Suppose g is a separating element of G and there is a point P of M such that g does not intersect $P \times J$. Let M_g denote the set of all points Q of M such that g intersects $Q \times J$. Since g is closed and connected, M_g is closed and connected. Therefore, since M_g is a proper subset of M, M_g is a subset of some composant C of M. Hence, g is a subset of $C \times J$. But $(M - C) \times J$ is connected and $\overline{(M - C)} \times J$ is $M \times J$. Therefore,

$(M \times J) - g$ is connected. It then follows that the end elements of G intersect $P \times J$ for each P in J.

Let g_1 and g_2 denote two separating elements of M/G, and let g denote an element of G between g_1 and g_2. M, J, g_1, and g_2 satisfy all the conditions of Theorem 1. Therefore, g is not a continuum. This involves a contradiction. Therefore, there is no monotone upper semi-continuous decomposition G of $M \times J$ such that $(M \times J)/G$ is an arc. Hence, M is not a web and therefore, M is a pseudo-web.

REMARKS. It can also be shown that there exists an example of a pseudo-web that contains no essential continuum of condensation. Also, in the plane, a square disc D is a web. But since D is unicoherent, it follows that if G is a monotone upper semi-continuous decomposition of D, D/G is not a simple closed curve.

Furthermore, a 2-torus does not have a dendratomic subset and therefore, by Theorem 48 of chapter V, part 1, of [5], a 2-torus is a web. However, one might wonder if the Cartesian product of a circularly chainable indecomposible continuum that is not chainable with a simple closed curve is a pseudo-web.

THEOREM 3. *There is no plane pseudo-web.*

Proof. Suppose M is a pseudo-web in the plane Σ. Then there exist two monotone upper semi-continuous decompositions G_1 and G_2 of M such that (1) each of M/G_1 and M/G_2 is a simple closed curve and (2) each element of G_1 intersects each element of G_2.

For each point P of M, let g_P denote the component containing P of the common part of the continuum of G_1 that contains P and the continuum of G_2 that contains P, and G denote the collection of all continua g_P for all points P of M. Then by Theorem 7 of Chapter V, part 2, of [5], G is a continuous curve with respect to its elements.

Let G' denote the collection to which g' belongs if and only if g' is an element g of G together with all the points not in M which are separated from an element of G by g, if there are any. Let S' denote the collection of all continua P' such that P' is either a continuum of the collection G' or a point which neither belongs to a continuum of G' nor is separated by any continuuum of G' from any other continuum of G'. Let S denote the set of all points of Σ and Σ' denote S/S'. Then Σ' is topologically equivalent to Σ or to a sphere. G' in Σ' is a continuum.

For $i = 1, 2$, let G_i' denote the collection to which g' belongs if and only if for some element g of G_i, g' is the sum of all the elements of G' that intersect g. The continuum G' together with the collections

G_1' and G_2' satisfy all the conditions of Theorem 1 of [1]. Hence G' is a simple plane web or simple web that is a subset of a sphere. Hence, there exist two monotone upper semi-continuous decompositions H_1' and H_2' of G' such that each of G'/H_1' and G'/H_2' is a dendron and if h_1' and h_2' are elements of H_1' and H_2', respectively, then $h_1' \cdot h_2'$ exists and is totally disconnected. For each $i = 1, 2$, let H_i denote the collection to which h belongs if and only if for some h' in H_i', h is the set of all points of M in Σ which belong to an element of h' in Σ. H_1 and H_2 are monotone upper semi-continuous decompositions of M such that (1) M/H_1 and M/H_2 are dendrons and (2) each element of H_1 intersects each element of H_2. H_1 and H_2 satisfy the conditions of an equivalent definition of a web given on page 297 of [5]. Hence, by Theorem 41 of Chapter V of [5], M is a web.

In conclusion, the following questions may be raised. Does there exist a compact metric continuum M that is not a web but does have two monotone upper semi-continuous decompositions G_1 and G_2 of M satisfying the conditions of a pseudo-web except that M/G_1 is an arc and M/G_2 is a simple closed curve? Also, does every pseudo-web contain uncountably many mutually exclusive webs? Does every web contain an indecomposable continuum?

REFERENCES

1. R. H. Bing, *Collections filling up a simple plane web*, Bull, Amer. Math. Soc., **51** (1945), 674–679.
2. ———, *Concerning simple plane webs*, Trans. Amer. Math. Soc., **60** (1946), 133–148.
3. M. E. Hamstrom, *Concerning webs in the plane*, Trans. Math. Soc., **74** (1953), 500–513.
4. ———, *Concerning certain types of webs*, Proc. Amer. Math. Soc., **4** (1953) 974–978.
5. R. L. Moore, *Foundations of Point Set Theory*, Amer. Math. Soc. Colloq. Pub., **13** (1962) Providence.

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