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ORTHODOX SEMIGROUPS

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An orthodox semigroup is a regular semigroup in which the idempotents form a subsemigroup. The purpose of this paper is to give structure theorems for orthodox semigroups in terms of inverse semigroups and bands.

A different structure theorem for orthodox semigroups in terms of bands and inverse semigroups has already been given by Yamada in [12]; two questions posed in [12] will be answered in the negative. The present paper is the "further paper" mentioned by the author in the final paragraph of §1 [5] and in the Acknowledgement of [5].

2. Preliminaries. We use wherever possible, and usually without comment, the notations of Clifford and Preston [2]; further, for each element a in any semigroup S we define $V(a) = \{x \in S\colon axa = a$ and $xax = x\}$, the set of inverses of a in S.

RESULT 1 (from Theorem 4.6 [2]). *On any band B Green's relation $\mathscr{J}$ is the finest semilattice congruence and each $\mathscr{J}$-class is a rectangular band.*

Let $\phi\colon B \rightarrow Y$ be any homomorphism of B onto a semilattice Y snch that $\phi \circ \phi^{-1} = \mathscr{J}$. By denoting (for all $e \in B$) J_e by E_α where $e\phi = \alpha \in Y$ we obtain B as a semilattice Y of the rectangular bands $\{E_\alpha\colon \alpha \in Y\}$, i.e., $B = \bigcup_{\alpha \in Y} E_\alpha$ and for all $\alpha, \beta \in Y$ $E_\alpha \cap E_\beta = \square$ if $\alpha \neq \beta$, aud $E_\alpha E_\beta \subseteq E_{\alpha\beta}$. It is clear that $\{(e, \alpha) \in B \times Y\colon e\phi = \alpha\}$ is a subband of $B \times Y$ isomorphic to B.

RESULT 2 [9, Lemma 2.2]. *Let ρ be a congruence on a regular semigroup S. Then each ρ-class which is an idempotent of S/ρ contains an idempotent of S.*

RESULT 3 (from Theorem 13 [7]). *Let ρ be any congruence contained in $\mathscr{L}$ on any semigroup S. Then any elements a and b of S are $\mathscr{L}$-related in S if and only if $a\rho$ and $b\rho$ are $\mathscr{L}$-related in S/ρ.*

Henceforth we shall let S denote an arbitrary orthodox semigroup.

The following result is part of [3, Theorem 3]; as noted in [4] it had previously been obtained by Schein [10].

RESULT 4. *The relation* $\mathscr{Y} = \{(x, y) \in S \times S: V(x) = V(y)\}$ *is the finest inverse semigroup congruence on the orthodox semigroup* S.

From [3, Remark 1] we see that the partition of S induced by $\mathscr{Y}$ is $\{V(x): x \in S\}$. Denote the band of S by B. Then we also have from [3, Remark 1] that for any $e \in B$, $e\mathscr{Y} = J_e$ (where J_e is the $\mathscr{J}$-class of B containing e) whence, from Result 2, the semilattice of $S/\mathscr{Y}$ is $B/\mathscr{J}$ ($\mathscr{J}$ being Green's relation $\mathscr{J}$ on B).

For the remainder of this section $\mathscr{L}$ and $\mathscr{R}$ shall denote Green's relations $\mathscr{L}$ and $\mathscr{R}$ on B; as usual then L_x and R_x shall denote the $\mathscr{L}$-class and $\mathscr{R}$-class respectively of B containg an element x from B.

RESULT 5 [5, Lemma 1] or [12, Footnote 5]. *For any element* $a \in S$ *and any element* $a' \in V(a)$,

$$aV(a) = R_{aa'} \text{ and } V(a)a = L_{a'a} .$$

RESULT 6 [5, Lemma 2] or [12, Lemma 5]. *Take any elements* a *and* b *in* S.

Then

$$aV(a)(a\mathscr{Y})V(a)a = \{a\}$$

whence $a = b$ *if and only if the triple*

$$(aV(a), a\mathscr{Y}, V(a)a) = (bV(b), b\mathscr{Y}, V(b)b) .$$

Henceforth, we shall identify any one element set $\{x\}$ say, with that element x, as is usual.

We shall now present two constructions appearing in [5]; one is of a representation of S by transformations of sets and the other is of a "maximal" fundamental orthodox semigroup containing B as the band of all idempotents (a semigroup T is called *fundamental* if the only congruence contained in $\mathscr{H}$ on T is the trivial congruence). This work has been generalized to regular semigroups in [6], where in fact the proofs and presentation are simpler than in [5]. For each result that we present we shall therefore refer to results in both [5] and [6].

For each element a in S define a transformation $\rho_a \in \mathscr{T}_{B/\mathscr{L}}$, the semigroup of all transformations of the set $B/\mathscr{L}$, by

$$V(x)x\rho_a = V(xa)xa \text{ for all } x \in B$$

and define also a transformation λ_a in $\mathscr{T}_{B/\mathscr{R}}$ by $xV(x)\lambda_a = axV(ax)$ for all $x \in B$.

That ρ_a and λ_a are transformations is shown in [5, Section 3] and also follows from [6, Remark 4]. Let (ρ, λ) be the mapping of S into $\mathcal{T}_{B/\mathcal{L}} \times \mathcal{T}^*_{B/\mathcal{R}}$ (where $\mathcal{T}^*_{B/\mathcal{R}}$ is the semigroup dual to $\mathcal{T}_{B/\mathcal{R}}$) which takes each a in S to (ρ_a, λ_a).

We define now an equivalence relation $\mathcal{U}$ on B by $\mathcal{U} = \{(e, f) \in B \times B: eBe \cong fBf\}$ and for each pair $(e, f) \in \mathcal{U}$ we let $T_{e,f}$ be the set of all isomorphisms from eBe onto fBf; for each $\alpha \in T_{e,f}$ we define further transformations $\bar{\alpha} \in \mathcal{T}_{B/\mathcal{L}}$ and $\bar{\bar{\alpha}} \in \mathcal{T}_{B/\mathcal{R}}$ [6, Section 5] by

$$L_x\bar{\alpha} = L_{x\alpha} \text{ and } R_x\bar{\bar{\alpha}} = R_{x\alpha} \text{ for all } x \in eBe\ .$$

Further, let us consider the transformations $\rho_e\bar{\alpha}$ and $\lambda_f\overline{\overline{\alpha^{-1}}}$ (products being taken in $\mathcal{P}\mathcal{T}_{B/\mathcal{L}}$ and $\mathcal{P}\mathcal{T}_{B/\mathcal{R}}$ respectively) and let us put $(\rho_e\bar{\alpha}, \lambda_f\overline{\overline{\alpha^{-1}}}) = \phi(\alpha)$ say. Define now

$$W(B) = \bigcup_{(e,f)\in\mathcal{U}} \{(\rho_e\bar{\alpha}, \lambda_f\overline{\overline{\alpha^{-1}}}) : \alpha \in T_{e,f}\}\ .$$

RESULT 7.

(i) *The set $W(B)$ is a subsemigroup of $\mathcal{T}_{B/\mathcal{L}} \times \mathcal{T}^*_{B/\mathcal{R}}$.*

(ii) *Further, $W(B)$ is a fundamental orthodox semigroup whose band of idempotents is isomorphic to B.*

(iii) *The mapping (ρ, λ) is a homomorphism of S into $W(B)$ which maps B isomorphically onto the band of idempotents of $W(B)$.*

(iv) *The congruence $(\rho, \lambda)\circ(\rho, \lambda)^{-1}$ is the maximum congruence contained in $\mathcal{H}$ on S.*

Result 7 can be obtained by the specialization to orthodox semigroups of the following results on regular semigroups from [6]: Lemma 4, Theorem 7 and Theorem 18 (vii). Alternatively, except for part (iii), Result 7 is contained in Theorems 1 and 5 of [5].

RESULT 8 [5, Theorem 2]. *Take any elements $a, b \in S$. Then $a = b$ if and only if the triple*

$$(\lambda_a, a\mathcal{Y}, \rho_a) = (\lambda_b, b\mathcal{Y}, \rho_b)\ .$$

3. The structure theorems.

LEMMA 1. *The mapping from S into $W(B) \times (S/\mathcal{Y})$ which maps each element a in S to $((\rho_a, \lambda_a), a\mathcal{Y})$ is an isomorphism.*

Proof. From Results 4 and 7 (iii) we see that the mapping is a homomorphism and from Result 8 we see that it is one-to-one.

Let now E be any band and define $W(E)$ as above. Let (ρ', λ') be the homomorphism of E into $W(E)$ which corresponds to the homomorphism (ρ, λ) of S into $W(B)$ above. From Result 7(iii) (ρ', λ') is an isomorphism from E onto the band of all idempotents of $W(E)$.

Let us denote the band of $W(E)$ by $\bar{E}$ and for each $e \in E$ let us denote $e(\rho', \lambda')$ simply by $\bar{e}$. Let $\mathscr{Y}_1$ denote the finest inverse semigroup congruence on $W(E)$, as given by Result 4.

Now let T be any inverse semigroup such that there is an idempotent-separating homomorphism ψ say, from T into $W(E)/\mathscr{Y}_1$ whose range contains all the idempotents of $W(E)/\mathscr{Y}_1$; if we let Y denote the semilattice of T then from Result 2 $\psi | Y$ maps Y isomorphically onto the semilattice of $W(E)/\mathscr{Y}_1$.

Let $\mathscr{Y}_1^\natural$ denote the natural homomorphism [2, Section 1.5] of $W(E)$ onto $W(E)/\mathscr{Y}_1$; then $x\mathscr{Y}_1^\natural = x\mathscr{Y}_1$ for any $x \in W(E)$.

Considering Green's relation $\mathscr{J}$ on $\bar{E}$ we have from §2 that

$$(\mathscr{Y}_1^\natural | \bar{E}) \circ (\mathscr{Y}_1^\natural | \bar{E})^{-1} = \mathscr{J} \text{ whence}$$
$$[(\mathscr{Y}_1^\natural | \bar{E})(\psi | Y)^{-1}] \circ [(\mathscr{Y}_1^\natural | \bar{E})(\psi | Y)^{-1}]^{-1} = \mathscr{J}$$

and so we may index (Result 1) the $\mathscr{J}$-classes of $\bar{E}$ with the elements of Y as follows: for all $\bar{e} \in \bar{E}$ if $\bar{e}\ \mathscr{Y}_1^\natural(\psi | Y)^{-1} = \alpha \in Y$ then denote $J_{\bar{e}}$ by $\bar{E}_\alpha$.

Similarly, considering Green's relation $\mathscr{J}$ on E and denoting $(\rho', \lambda')(\mathscr{Y}_1^\natural | \bar{E})(\psi | Y)^{-1}$ by ξ we have $\xi \circ \xi^{-1} = \mathscr{J}$ whence we may index the $\mathscr{J}$-classes of E with the elements of Y as follows: for all $e \in E$ if $e\xi = \alpha \in Y$ then denote J_e by E_α. Clearly $e \in E_\alpha$ implies $\bar{e} \in \bar{E}_\alpha$ for all $e \in E$.

Define now $S_1 = S_1(E, T, \psi)$ by

$$S_1 = \{(x, t) \in W(E) \times T \colon x\mathscr{Y}_1 = t\psi\} .$$

THEOREM 1.

(i) *The set* $S_1 = S_1(E, T, \psi)$ *is an orthodox subsemigroup of* $W(E) \times T$, *and conversely every orthodox semigroup is obtained in this way.*

(ii) *The band of* S_1 *is isomorphic to* E.

(iii) *The maximum inverse semigroup homomorphic image of* S_1 *is isomorphic to* T.

(iv) *For each element* $x \in W(E)$ *let* $(xV(x), x\mathscr{Y}_1, V(x)x)$ *denote* x. *Then*

$$S_1 = \{((R_{\bar{e}}, t\psi, L_{\bar{f}}), t) \colon t \in T, \bar{e} \in \bar{E}_{tt^{-1}}, \bar{f} \in \bar{E}_{t^{-1}t}\} ,$$

where $R_{\bar{e}}$ *and* $L_{\bar{f}}$ *are the* $\mathscr{R}$-*class and* $\mathscr{L}$-*class respectively of* $\bar{E}$ *containing* $\bar{e}$ *and* $\bar{f}$ *respectively.*

Proof.

(i) Take any elements (x, t), (y, u) in S_1. Then

$$(xy)\mathscr{Y}_1^\natural = (x\mathscr{Y}_1^\natural)(y\mathscr{Y}_1^\natural) = (t\psi)(u\psi) = (tu)\psi$$

whence $(x, t)(y, u) = (xy, tu) \in S_1$ and S_1 is a subsemigroup of $W(E) \times T$. Now the set of inverses of (x, t) in $W(E) \times T$ is $V(x) \times \{t^{-1}\}$ (where of course $V(x)$ denotes the set of inverses of x in $W(E)$); take any $(x', t^{-1}) \in V(x) \times \{t^{-1}\}$. Then $x'\mathscr{Y}_1$ and $t^{-1}\psi$ are both inverses of $x\mathscr{Y}_1 = t\psi$ in $W(E)/\mathscr{Y}_1$ whence $x'\mathscr{Y}_1 = t^{-1}\psi$ and $V(x) \times \{t^{-1}\} \subseteqq S_1$. In particular S_1 is regular. Since $W(E) \times T$ is orthodox we now have that S_1 is orthodox.

Conversely, consider again the orthodox semigroup S of §2. Let $\mathscr{Y}_2$ be the finest inverse semigroup congruence on $W(B)$. Then $S(\rho, \lambda)\, \mathscr{Y}_2^{\natural}$ is an inverse semigroup homomorphic image of S so

$$\mathscr{Y} \subseteqq [(\rho, \lambda)\mathscr{Y}_2^{\natural}]\circ[(\rho, \lambda)\mathscr{Y}_2^{\natural}]^{-1} .$$

Let θ be the unique homomorphism from $S/\mathscr{Y}$ onto $S(\rho, \lambda)\mathscr{Y}_2^{\natural}$ such that $\mathscr{Y}^{\natural}\theta = (\rho, \lambda)\mathscr{Y}_2^{\natural}$ [2, Theorem 1.6].
The semilattices of $S/\mathscr{Y}$ and $S(\rho, \lambda)\mathscr{Y}_2^{\natural}$ are $B\mathscr{Y}^{\natural}$ and $B(\rho, \lambda)\mathscr{Y}_2^{\natural}$ respectively (Result 2), and moreover (for $\mathscr{J}$ on B)

$$(\mathscr{Y}^{\natural} | B)\circ(\mathscr{Y}^{\natural} | B)^{-1} = \mathscr{J} = [((\rho, \lambda)\mathscr{Y}_2^{\natural}) | B]\circ[((\rho, \lambda)\mathscr{Y}_2^{\natural}) | B]^{-1}$$

so θ maps $B\mathscr{Y}^{\natural}$ one-to-one onto $B(\rho, \lambda)\mathscr{Y}_2^{\natural}$. Thus $S_1(B, S/\mathscr{Y}, \theta)$ is defined, and further, for all $a \in S$, $((\rho_a, \lambda_a), a\mathscr{Y}) \in S_1(B, S/\mathscr{Y}, \theta)$ since $(a\mathscr{Y})\theta = a(\rho, \lambda)\mathscr{Y}_2^{\natural} = (\rho_a, \lambda_a)\mathscr{Y}_2$.

Take now any element $(x, a\mathscr{Y}) \in S_1(B, S/\mathscr{Y}, \theta)$, where $a \in S$.
Then

$$x\mathscr{Y}_2 = (a\mathscr{Y})\theta = a(\rho, \lambda)\mathscr{Y}_2^{\natural} = (\rho_a, \lambda_a)\mathscr{Y}_2$$

whence $V(x) = V((\rho_a, \lambda_a))$ in $W(B)$. Take any $a' \in V(a)$ in S. Then $(\rho_{a'}, \lambda_{a'}) \in V(x)$ in $W(B)$ and from Result 7 (iii)

$$(\rho_{a'}, \lambda_{a'})x = (\rho_e, \lambda_e) \text{ and } x(\rho_{a'}, \lambda_{a'}) = (\rho_f, \lambda_f)$$

for some idempotents $e, f \in S$. Then $(\rho_e, \lambda_e)\mathscr{R}(\rho_{a'}, \lambda_{a'})\mathscr{L}(\rho_f, \lambda_f)$ in $W(B)$ whence $e\mathscr{R}a'\mathscr{L}f$ in S (from Result 7 (iv), Result 3 and the result dual to Result 3). From [2, Theorem 2.18] there is an inverse b say, of a' in S, such that $e\mathscr{L}b\mathscr{R}f$ in S. Thus $(\rho_e, \lambda_e)\mathscr{L}(\rho_b, \lambda_b)\mathscr{R}(\rho_f, \lambda_f)$ in $W(B)$; but also $(\rho_e, \lambda_e)\mathscr{L}x\mathscr{R}(\rho_f, \lambda_f)$ in $W(B)$ and both x and (ρ_b, λ_b) are inverses of $(\rho_{a'}, \lambda_{a'})$ in $W(B)$, so from [2, Theorem 2.18] $x = (\rho_b, \lambda_b)$. Note also that $b\mathscr{Y} = a\mathscr{Y}$ (since both are inverses of $a'\mathscr{Y}$ in $S/\mathscr{Y}$). Thus $(x, a\mathscr{Y}) = ((\rho_b, \lambda_b), b\mathscr{Y})$. With an observation above this gives that

$$S_1(B, S/\mathscr{Y}, \theta) = \{((\rho_a, \lambda_a), a\mathscr{Y}) \in W(B) \times (S/\mathscr{Y}) : a \in S\} .$$

From Lemma 1 we have that S is isomorphic to $S_1(B, S/\mathscr{Y}, \theta)$.

(ii) Take any idempotent (x, α) say, in $S_1 = S_1(E, T, \psi)$. Then $x^2 = x$, $\alpha^2 = \alpha$ and $x\mathscr{Y}_1 = \alpha\psi$ whence $x\mathscr{Y}_1^{\natural}(\psi | Y)^{-1} = \alpha$ and so $x \in \bar{E}_\alpha$. Conversely, for any $\alpha \in Y$ and $x \in \bar{E}_\alpha$ we have $x\mathscr{Y}_1^\natural(\psi \mid Y)^{-1} = \alpha$ whence $x\mathscr{Y}_1 = \alpha\psi$ and $(x, \alpha) \in S_1$. Thus the band of idempotents of S_1 is $\{(x, \alpha) \in \bar{E} \times Y: \alpha \in Y, x \in \bar{E}_\alpha\}$, which is clearly isomorphic to $\bar{E}$ (Section 2).

(iii) Let π_2: $S_1 \to T$ be the function satisfying $(x, t)\pi_2 = t$ for all $(x, t) \in S_1$, and let $\mathscr{Y}_3$ denote the finest inverse semigroup congruence on S_1. Then π_2 is a homomorphism onto T, an inverse semigroup, whence $\mathscr{Y}_3 \subseteq \pi_2 \circ \pi_2^{-1}$.

Since from the proof of (i) the set of inverses of any element (x, t) in S_1 is $V(x) \times \{t^{-1}\}$ we have that

$\mathscr{Y}_3 = \{((x, t), (y, t)) \in S_1 \times S_1: V(x) = V(y) \text{ in } W(E)\}$. But for any (x, t), (y, t) in S_1 we have $x\mathscr{Y}_1 = t\psi = y\mathscr{Y}_1$ whence $V(x) = V(y)$ in $W(E)$. Thus $\pi_2 \circ \pi_2^{-1} \subseteq \mathscr{Y}_3$, giving $\pi_2 \circ \pi_2^{-1} = \mathscr{Y}_3$ and $S_1/\mathscr{Y}_3$ is isomorphic to $S_1\pi_2 = T$.

(iv) We note that it is Result 6 which enables us to let $(xV(x), x\mathscr{Y}_1, V(x)x)$ denote x, for each $x \in W(E)$.

Take any element $(x, t) \in S_1$. Considering Green's relations $\mathscr{R}$ and $\mathscr{L}$ on $\bar{E}$ we have

$$(x, t) = ((xV(x), x\mathscr{Y}_1, V(x)x), t) = ((R_{xx'}, t\psi, L_{x'x}), t)$$

for any $x' \in V(x)$, from Result 5. Now $t^{-1}\psi = (t\psi)^{-1}$ and $x'\mathscr{Y}_1 = (x\mathscr{Y}_1)^{-1} = (t\psi)^{-1}$ so

$$(xx')\mathscr{Y}_1^\natural = (x\mathscr{Y}_1^\natural)(x'\mathscr{Y}_1^\natural) = (t\psi)(t\psi)^{-1} = (t\psi)(t^{-1}\psi) = (tt^{-1})\psi$$

giving that $(xx')\mathscr{Y}_1^\natural(\psi \mid Y)^{-1} = tt^{-1}$ and $xx' \in \bar{E}_{tt^{-1}}$. Similary $x'x \in \bar{E}_{t^{-1}t}$ and so

$$S_1 \subseteq \{((R_{\bar{e}}, t\psi, L_{\bar{f}}), t): t \in T, \bar{e} \in \bar{E}_{tt^{-1}}, \bar{f} \in \bar{E}_{t^{-1}t}\} .$$

Conversely take any $t \in T$ and any $\bar{e} \in \bar{E}_{tt^{-1}}$ and $\bar{f} \in \bar{E}_{t^{-1}t}$; then $\bar{e}\mathscr{Y}_1 = (tt^{-1})\psi$ and $\bar{f}\mathscr{Y}_1 = (t^{-1}t)\psi$. Consider $((R_{\bar{e}}, t\psi, L_{\bar{f}}), t)$. Take any element $x \in W(E)$ such that $x\mathscr{Y}_1 = t\psi$. Then $(\bar{e}x\bar{f})\mathscr{Y}_1^\natural = (\bar{e}\mathscr{Y}_1^\natural)(x\mathscr{Y}_1^\natural)(\bar{f}\mathscr{Y}_1^\natural) = [(tt^{-1})\psi](t\psi)[(t^{-1}t)\psi] = t\psi$. Take any $x' \in V(x)$ and put $\bar{e}x\bar{f} = y$ and $\bar{f}x'\bar{e} = y'$. Then $y' \in V(y)$[10, Theorem 1.10], whence $y'\mathscr{Y} = (t\psi)^{-1} = t^{-1}\psi$. Thus $(yy')\mathscr{Y}_1 = (tt^{-1})\psi$ giving $yy' \in \bar{E}_{tt^{-1}}$ and similarly $y'y \in \bar{E}_{t^{-1}t}$. Now $\bar{e}$, $yy' \in \bar{E}_{tt^{-1}}$, a rectangular band, so

$$yy' = (\bar{e}x\bar{f})(\bar{f}x'\bar{e}) = \bar{e}yy'\bar{e} = \bar{e}$$

and similarly $y'y = \bar{f}$. Thus

$$((R_{\bar{e}}, t\psi, L_{\bar{f}}), t) = ((yV(y), y\mathscr{Y}_1, V(y)y), t) = (y, t) \in S_1 .$$

Therefore

$$S_1 = \{((R_{\bar{e}}, t\psi, L_{\bar{f}}), t): t \in T, \bar{e} \in \bar{E}_{tt^{-1}}, \bar{f} \in \bar{E}_{t_{-1}t}\} .$$

REMARK 1. Let Z denote the semilattice of $S/\mathscr{Y}$ and index the $\mathscr{J}$-classes of B with the elements of Z in the natural way. For each element $a \in S$ let $(aV(a), a\mathscr{Y}, V(a)a)$ denote a and consider the $\mathscr{R}$ and $\mathscr{L}$-classes of B. Then the method used to prove (iv) also gives that

$$S = \{(R_e, v, L_f) \in (B/\mathscr{R}) \times (S/\mathscr{Y}) \times (B/\mathscr{L}) \colon v \in S/\mathscr{Y}, e \in E_{vv^{-1}} \text{ and } f \in E_{v^{-1}v}\} .$$

COROLLARY 1 (to the proof). *Consider the arbitrary band E and any inverse semigroup U. Then there exists an orthodox semigroup whose band is E and whose maximum inverse semigroup image is isomorphic to U if and only if there is a homomorphism from U into $W(E)/\mathscr{Y}_1$ which maps the idempotents of U one-to-one onto the idempotents of $W(E)/\mathscr{Y}_1$.*

Let us now define a subset $S_2 = S_2(E, T, \psi)$ of $(E/\mathscr{R}) \times T \times (E/\mathscr{L})$ by

$$S_2 = \{(R_e, t, L_f) \colon t \in T, e \in E_{tt^{-1}} \text{ and } f \in E_{t^{-1}t}\} .$$

Take any element (R_e, t, L_f) in S_2. Then $\bar{e} \in \bar{E}_{tt^{-1}}$ and $\bar{f} \in \bar{E}_{t^{-1}t}$ whence $((R_{\bar{e}}, t\psi, L_{\bar{f}}), t) \in S_1$, where $R_{\bar{e}}$ and $L_{\bar{f}}$ are the $\mathscr{R}$-class and $\mathscr{L}$-class respectively of $\bar{E}$ containing $\bar{e}$ and $\bar{f}$ respectively. Clearly now we may define a mapping Ψ of S_2 into S_1 by

$$(R_e, t, L_f)\Psi = ((R_{\bar{e}}, t\psi, L_{\bar{f}}), t)$$

for any element $(R_e, t, L_f) \in S_2$. It is also clear that Ψ is one-to-one and it is routine to show that Ψ is onto S_1. Thus Ψ is a one-to-one correspondence between S_2 and S_1.

Let us denote by juxtaposition the unique multiplication on S_2 which makes Ψ an isomorphism from S_2 onto S_1; then for any elements (R_e, t, L_f) and (R_g, u, L_h) in S_2

$$(R_e, t, L_f)(R_g, u, L_h) = [(R_e, t, L_f)\Psi(R_g, u, L_h)\Psi]\Psi^{-1} .$$

From Result 6 and Theorem 1 (iv) $((R_{\bar{e}}, t\psi, L_{\bar{f}}), t)$ denotes the element $(R_{\bar{e}}(t\psi)L_{\bar{f}}, t)$ of S_1; thus

$$(R_e, t, L_f)\Psi = (R_{\bar{e}}(t\psi)L_{\bar{f}}, t)$$

for any element (R_e, t, L_f) in S_2.

For each idempotent $x \in W(E)$ let $\tilde{x}$ denote $x(\rho', \lambda')^{-1}$; then $\bar{\tilde{x}} = x$ for all $x \in \bar{E}$ and $\tilde{\bar{e}} = e$ for all $e \in E$. Then for any elements (R_e, t, L_f) and (R_g, u, L_h) in S_2

$$(R_e, t, L_f)(R_g, u, L_h) = (R_{\widetilde{zz'}}, tu, L_{\widetilde{z'z}})$$

where (in $W(E)$)$R_{\bar{e}}(u\psi)L_{\bar{f}} = x$, $R_{\bar{g}}(u\psi)L_{\bar{h}} = y$, $xy = z$ and $z' \in V(z)$; this is because $(tu)\psi = (xy)\mathscr{Y}_1 = z\mathscr{Y}_1$ and

$$(R_{\widetilde{zz'}}, tu, L_{\widetilde{z'z}})\Psi = ((R_{zz'}, (tu)\psi, L_{z'z}), tu) = (z, tu) = (xy, tu) .$$

We restate these facts in the next theorem.

THEOREM 2. *Let $S_2 = S_2(E, T, \psi)$ be the subset of $(E/\mathscr{R}) \times T \times (E/\mathscr{L})$ given by*

$S_2 = \{(R_e, t, L_f)\colon t \in T, e \in E_{tt^{-1}}$ *and* $f \in E_{t^{-1}t}\}$ *and let a multiplication on S_2 be given by (for any elements (R_e, t, L_f) and (R_g, u, L_h) in S_2)*

$$(R_e, t, L_f)(R_g, u, L_h) = (R_{\widetilde{zz'}}, tu, L_{\widetilde{z'z}})$$

where (for the $\mathscr{R}$ and $\mathscr{L}$-classes of $\bar{E}$ we have) $R_{\bar{e}}(t\psi)L_{\bar{f}} = x$, $R_{\bar{g}}(u\psi)L_{\bar{h}} = y$, $xy = z$ and $z' \in V(x)$(all in $W(E)$). Then $S_2(E, T, \psi)$ is a semigroup isomorphic to $S_1(E, T, \psi)$.

4. Some counter-examples.

4.1. Let T denote the bicyclic semigroup [2, Section 1.12]. We shall construct a band B which is an ω-chain of rectangular bands and such that there is no orthodox semigroup S with band B and with T as a homomorphic image.

Let Y be the semilattice of T; then Y is an ω-chain. For each $\alpha \in Y$ let E_α be a rectangular band such that, for all $\alpha, \beta \in Y$, if $\alpha \neq \beta$ then $E_\alpha \cap E_\beta = \square$ and $|E_\alpha| \neq |E_\beta|$. Put $B = \bigcup_{\alpha \in Y} E_\alpha$ and, following Clifford [1] extend the multiplications of the bands $\{E_\alpha\colon \alpha \in Y\}$ to a multiplication for B as follows: for any $e, f \in B$, where $e \in E_\alpha$ and $f \in E_\beta$ say, define

$$ef = \begin{cases} e \text{ if } \alpha < \beta \\ ef \text{ as in } E_\alpha \text{ if } \alpha = \beta \\ f \text{ if } \alpha > \beta. \end{cases}$$

Note that if $\alpha > \beta$ then $ef = fe = f$. It is routine to show that this multiplication is associative (alternatively see [8]) and that then the band B is an ω-chain Y of the rectangular bands $\{E_\alpha\colon \alpha \in Y\}$. Also, if $e \in E_\alpha$ and $f \in E_\beta$ $(\alpha, \beta \in Y)$ then $eBe = \{e\} \cup (\bigcup_{\gamma<\alpha} E_\gamma)$ whence eBe is isomorphic to fBf if and only if $\alpha = \beta$. From [5, Main Theorem] any orthodox semigroup, S say, with band B is a union of groups. But any homomorphic image of a semigroup which is a union of groups is also a union of groups; thus T is not the maximum inverse semigroup homomorphic image of S.

REMARK 2. The band B just defined is one of a class of bands called, by the author, almost commutative bands; a band E is called *almost commutative* if, for any $e, f \in E$, $J_e \neq J_f$ implies $ef = fe$. It is easily shown (See [8]) that a band E is almost commutative if and only if, for $e, f \in E$, $J_e > J_f$ implies $e > f$ (where $J_e > J_f$ means that $E^1eE^1 \supset E^1fE^1$ [2, Section 2.1] and $e > f$ means that $ef = fe = f \neq e$[2, Section 1.8]). A determination of the structure of almost commutative bands in terms of semilattices is given in [8].

REMARK 3. The band B and inverse semigroup T above answer in the negative the first question posed on page 269 [12]. We now briefly give alternative examples of a different nature. Let E consist of the matrices

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&1\\0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

and let T_1 consist of the matrices

$$\begin{pmatrix}0&0\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

Under matrix multiplication E is a band, T_1 is an inverse semigroup with semilattice isomorphic to $E/\mathscr{J}$, and there is no orthodox semigroup S say, with band E and such that $S/\mathscr{Y}$ is isomorphic to T_1.

4.2. We now give two non-isomorphic orthodox semigroups S_1 and S_2 whose bands are isomorphic and whose maximum inverse semigroup homomorphic images are isomorphic. This answers the second question on page 269 [12] in the negative. The referee has pointed out that this question has also been essentially answered in the last remark of Yamada [13].

Let S_1 consist of the matrices

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\1&0\end{pmatrix}, \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\1&-1\end{pmatrix}$$

and let S_2 consist of the matrices

$$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\1&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}.$$

Under matrix multiplication S_1 and S_2 are orthodox semigroups.

The bands of S_1 and S_2 are both two-element left zero semigroups with an identity adjoined and the maximum inverse semigroup homomorphic images are both two-element groups with a zero adjoined. But $\mathscr{H}$ is a congruence on S_2 and not on S_1, so S_1 and S_2 are not isomorphic.

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