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A NOTE ON $C\theta\theta$-GROUPS

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A $C\theta\theta$-group is a finite group of order divisible by 3 in which centralisers of 3-elements are 3-groups. Several authors have studied such groups; in particular it is known that, given the additional hypothesis that the Sylow 3-subgroups intersect trivially, a simple $C\theta\theta$-group has abelian Sylow 3-subgroups. In this note it is proved that this additional hypothesis is superfluous.

More precisely the following will be proved:

THEOREM. *Let G be a $C\theta\theta$-group in which $0^3(G) = G$ and let M be a Sylow 3-subgroup of G. Then M is a TI-set in G.*

The proof of the theorem depends on two lemmas:

LEMMA 1. *Let H be a $C\theta\theta$-group. If any element of order 3 in H is conjugate to its inverse; or, equivalently, if any 3-local subgroup of H has even order; then Sylow 3-subgroups of H are abelian and hence TI-sets in H.*

Proof. Suppose t is an element of order 3 in H conjugate to its inverse. Let T be a Sylow 3-subgroup of H such that $t \in T$. Now the extended centraliser $C_H^*(t)$ is a Frobenius group with the 3-group $C_H(t)$ as kernel. Since $|C_H^*(t): C_H(t)| = 2$, $C_H(t)$ is abelian and every element in it is conjugate to its inverse. Now $Z(T) \leqslant C_H(t)$ so we may assume that $t \in Z(T)$. In this case $C_H(t) = T$ and so T is abelian.

LEMMA 2. *Let H be a $C\theta\theta$-group in which $0_3(H) > 1$. Then H is soluble and one of the following occurs:*

(i) *a Sylow 3-subgroup of H is normal in H*

(ii) $0^3(H) < H$.

Proof. Put $L = 0_3(H)$, $\bar{H} = H/L$. Suppose first that $|H|$ is even. Every element of L is conjugate to its inverse so, by Lemma 1, Sylow 3-subgroups of H are abelian. Clearly $L = C_H(L)$ is a Sylow 3-subgroup of H, case (i) arises, and $|\bar{H}|$ is prime to 3. $\bar{H}$ can now be regarded as a group of fixed-point-free automorphisms of L so, if p is odd, the Sylow p-subgroups of $\bar{H}$ are cyclic and the Sylow 2-subgroups are either cyclic or generalised quaternion. A group all of whose Sylow subgroups are cyclic is soluble. (See [2] Theorem 7.6.2.) On the other hand it is not difficult to show that a group having generalised quaternion Sylow 2-subgroups either involves A_4, the alternating group on 4 letters, or satisfies the hypotheses of Frobenius' theorem on the existence of a normal p-complement for $p = 2$. $|\bar{H}|$ is prime to 3 so $\bar{H}$ is soluble.

If $|H|$ is odd then it is well-known that H is soluble. Suppose that a Sylow 3-subgroup of H is not normal in H i.e., $|\bar{H}|$ is divisible by 3. A Sylow 3-subgroups of $\bar{H}$ can be regarded as a group of fixed-point-free automorphisms of $0_3'(\bar{H})$. Thus $\bar{H}$ has cyclic Sylow 3-subgroups. But the only 3′-automorphism of a cyclic 3-group has order 2 and $|\bar{H}|$ is odd. Hence, by Burnside's Theorem, $\bar{H}$ has a normal 3-complement; in particular $0^3(\bar{H}) < \bar{H}$ and so $0^3(H) < H$.

Proof of Theorem. Suppose M is not a *TI*-set in G. Then M is not abelian so, by Lemmas 1 and 2, the normaliser of every non-identity 3-subgroup of G is soluble and of odd order. In the terminology of [2], this means that the normaliser of every non-identity 3-subgroup is 3-constrained and 3-stable (see [2] p. 268) and so satisfies the conditions of [2] Theorem 8.2.11. Hence G satisfies the conditions of [2] Theorem 8.4.2. and 8.4.3.

Write $N = N(Z(J(M)))$. If N is of type (ii) in Lemma 2 then $M \cap N'$ is a proper subgroup of M. By [2] Theorem 8.4.3. $M \cap G'$ is a proper subgroup of M and so, by [2] Theorem 7.3.1. $0^3(G)$ is a proper subgroup of G. This is not the case and so N is of type (i) in Lemma 2.

Let M_0 be a maximal intersection of Sylow 3-subgroups of G contained in M. By the maximality of M_0, $M_0 = 0_3(N(M_0))$; by Lemma 2, $N(M_0)$ is soluble. Hence $C(M_0) \leqslant M_0$; in particular, $Z(M) \leqslant M_0$. Let $m \in Z(M)$ and $h \in N(M_0)$. $m, m^h \in M$ so by [2] Theorem 8.4.2. there is an element $n \in N$ such that $m^h = m^n$ i.e., $n.h^{-1} \in C(m)$. Clearly then $n.h^{-1} \in M \leqslant N$. Hence $h \in N$ and so $N(M_0) \leqslant N$. But N has a unique Sylow 3-subgroup, $N(M_0)$ does not. This contradiction proves that M is a *TI*-set in G.

COROLLARY. *A simple $C\theta\theta$-group has abelian Sylow 3-subgroup.*

Proof. This follows immediately from the theorem and work of Ferguson [1] and Herzog [3].

I am indebted to Mrs. Ferguson for letting me see a preliminary draft of her Ph.D. thesis.

References

1. P. Ferguson, Ph.D. Thesis, University of Chicago, 1969.
2. D. Gorenstein, *Finite Groups*, Harper and Row, 1968.
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