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RESTRICTIONS OF BANACH FUNCTION SPACES

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Let X be a compact Hausdorfs space. Let $C(X)$ be the space of continuous complex-valued functions on X and A be a function algebra on X, that is a uniformly closed separating subalgebra of $C(X)$ containing the constants. If F is a closed subset of X we say that A interpolates on F if $A \mid F = C(F)$. By a positive measure μ we shall always mean a positive regular bounded Borel measure on X. Let F be a measurable subset of X. We say a subspace S of $L^p(\mu)$ interpolates on F if $S \mid F = L^p(F) = L^p(\mu_F)$, where μ_F is the restriction of μ to F. Let $H^p(\mu)$ be the closure of A in $L^p(\mu)$ where $1 \leqq p < \infty$, and let $H^\infty(\mu) = H^2(\mu) \cap L^\infty(\mu)$. One question we are concerned with here is whether interpolation of the algebra is sufficient to imply interpolation of its associated H^p-spaces. We therefore begin by obtaining necessary and sufficient conditions for a closed subspace of $L^p(\mu)$ to have closed restriction in $L^p(F)$. These condition are analogous to some obtained by Glicksberg for function algebras. Using these results we obtain theorems about interpolation of certain invariant subspaces, and then apply them to H^p-spaces. In particular we show that when A approximates in modulus and μ is any measure which is not a point-mass, $H^p(\mu)$ interpolates only on sets of measure zero. (One sees that A interpolates only on sets of measure zero, so our original question has a trivial answer for these algebras.) For uniformly closed weak-star Dirichlet algebras again the answer to our original question is affirmative. Finally we provide an example of an algebra which interpolates such that $H^\infty(\mu)$ interpolates and the $H^p(\mu)$ do not interpolate for $1 \leqq p < \infty$. I am indebted to a paper of Glicksberg for those techniques which inspired the present effort. Below we show that these techniques apply to the L^p situation and to other "similar" situations.

Glicksberg [3] has given necessary and sufficient conditions for interpolation of a closed subspace of $C(X)$. We show here that analogous theorems hold for subspaces of $L^p(X)$. Let $A \subset B$ be Banach spaces. $A^\perp$ will denote all bounded linear functions functionals on B which annihilate A.

THEOREM 1.1. *Let A, A_1, B all be Banach spaces with $A \subset A_1$ and $R: A_1 \to B$ a nonzero bounded linear transformation. Then $R(A)$ is closed in B if and only if $\exists c \ni : \| h - R(A)^\perp \| \leqq c \| h^* - A^\perp \|$ $\forall h \in B^*$, where $h^* = R^*h$. It follows that $c \geqq 1/\| R \|$.*

Proof. The map $R_1 = R \mid A \colon A \to R(A)$ induces a map

$$T = \psi \circ R_1^* \circ \phi \colon B^*/R(A)^\perp \longrightarrow A_1^*/A^\perp$$

where $\psi \colon A^* \to A_1^*/A^\perp$ and $\phi \colon B^*/R(A)^\perp \to R(A)^*$ are the natural isometric isomorphisms. Further for $g \in B^*$, $g - R(A)^\perp$ is taken to $g^* - A^\perp$ by T, so T is $1-1$. Now the range of R_1 is closed if and only if the range of R_1^* is closed if and only if the range of T is closed [1]. The latter fact is equivalent to: $\exists c \ni : \| h - R(A)^\perp \| \leqq c \| h^* - A^\perp \|$ for all $g \in B^*$ by the open mapping theorem. Further, $\| h^* - A^\perp \| \leqq \| R \| \, \| h - R(A)^\perp \|$ so applying the above inequality gives $c \geqq 1/\| R \|$.

The statement of the above theorem is slightly more general than those of other similar theorems appearing the literature. The proof is virtually the same as that in [3] albeit in a more general setting. See also [2]. The next corollary follows as in [3].

COROLLARY 1.2. *Let X be locally compact and A a uniformly closed subspace of $C_0(X)$. Let F be a locally compact subset of X and suppose $A \mid F \subset C_0(F)$. Then*

(i) *$A \mid F$ is uniformly closed in $C_0(F)$ if and only if $\exists c \ni :$ $\| \mu - (A \mid F)^\perp \| \leqq c \| \mu - A^\perp \|$ $\forall$ regular bounded Borel measure μ on F.*

(ii) *$A \mid F = C_0(F)$ if and only if $\exists c \ni : \| \mu_F \| \leqq c \| \mu_{F'} \|$ $\forall \mu \in A^\perp$.*

We now apply 1.1 to get the analogous conclusion for subspaces of L^p-spaces.

DEFINITION. Let μ be a fixed positive measure on X and F a measurable subset of X. Set $L^p(F) = L^p(\mu_F)$, $1 \leqq p \leqq \infty$ where μ_F is the restriction of μ to F. For $f \in L^q(F)$ let $\tilde{f}$ be the function which is f on F and 0 on F'. Note that if R is the restriction map $L^p(X) \to L^p(F)$, then $\tilde{f} = f^*$. For a subspace S of $L^p(X)$, $(S \mid F)^\perp = \{ g \in L^q(F) \mid g \perp S \mid F \}$. Clearly $\{ \tilde{f} \mid f \in (S \mid F)^\perp \} \subset S^\perp$.

THEOREM 1.3. *Let S be a closed subspace of $L^p(X)$, $1 \leqq p < \infty$, and F a measurable subset of X. Then:*

(i) *$S \mid F$ is closed in $L^p(F)$ if and only if*

$$(1) \qquad \exists c \ni : \| g - (S \mid F)^\perp \| \leqq c \| \tilde{g} - S^\perp \| \quad \forall g \in L^q(F) \; ;$$

(ii) *$S \mid F = L^p(F)$ if and only if*

$$(2) \qquad \exists c \ni : \| g \mid F \|_q \leqq c \| g \mid F' \|_q \quad \forall g \in S^\perp \; .$$

If F has positive measure it follows that $c \geqq 1$. If $p = \infty$ then the *"only if" parts of* (i) *and* (ii) *hold for* $g \in L^1(E)$ *and* $L^1(X) \cap S^{\perp}$ *respectively.*

Proof. (i) follows by applying 1.1 to the restriction map R. As $\|R\| \leqq 1$, we have $c \geqq 1$. If $S \mid F = L^p(F)$, then (1) becomes $\|g\| \leqq c \|g - S^{\perp}\|$ $\forall\, g \in L^q(F)$. In particular if $g \in S^{\perp}$,

$$\|g \mid F\| \leqq c \|\widetilde{g \mid F} - g\| = c \|g \mid F'\| .$$

This shows the "only if" part of (ii). For the "if" part of (ii) we shall use a concavity property of the q-norm; namely, if $\alpha, \beta \geqq 0$, $\alpha + \beta \leqq 1$, then $\|f\|_q \geqq \alpha \|f \mid F\|_q + \beta \|f \mid F'\|_q$. Now taking $g \in (S \mid F)^{\perp}$, and applying (2) to g shows that $(S \mid F)^{\perp} = 0$, so $S \mid F$ is dense in $L^p(F)$. Thus we need only show that $S \mid F$ is closed. Here (1) reduces to $\|g\|_q \leqq c' \|\tilde{g} - S^{\perp}\|_q$ $\forall\, g \in L^q(F)$. But if $g \in L^q(F)$ and $h \in S^{\perp}$, then $\|\tilde{g} - h\|_q \geqq \alpha \|(g - h) \mid F\|_q + \beta \|h \mid F'\|_p$ if $\alpha, \beta \geqq 0$ and $\alpha + \beta \leqq 1$. Now choose n so that $c/n + c^2/n \leqq 1$ and let $\alpha = c/n$, $\beta = c^2/n$. Then

$$\|\tilde{g} - h\|_q \geqq c/n \|g \mid F\|_q - c/n \|h \mid F\|_q + c^2/n \|h \mid F'\|_q \geqq c/n \|g \mid F\|_q$$

after applying (2). Thus setting $c' = n/c$ gives $S \mid F$ is closed and thus $S \mid F = L^p(F)$. The latter part of the conclusion is clear from the above arguments.

COROLLARY 1.4. *If S is a closed subspace of $L^p(X)$, $1 \leqq p < \infty$ and $S^{\perp} \mid F \subset (S \mid F)^{\perp}$ then $S \mid F$ is closed in $L^p(F)$.*

Proof. $(\widetilde{S \mid F})^{\perp} \subset S^{\perp}$ so in fact $S^{\perp} \mid F = (S \mid F)^{\perp}$. Taking $g \in L^q(F)$, and $h \in S^{\perp}$ we have

$$\|g - S^{\perp}\|_q \geqq \|g - S^{\perp} \mid F\|_q = \|\tilde{g} - (S \mid F)^{\perp}\|_q$$

and (1) applies.

2. Restrictions of invariant subspaces. Let X be a topological space and μ a positive measure on X. Throughout this section A will be a subalgebra of $L^{\infty}(\mu)$, and S will be a closed subspace of $L^p(X)$ for some $1 \leqq p < \infty$. We assume that S is invariant under multiplication by elements of A. A *separates in modulus* (SM) if $\forall\, \varepsilon > 0$, E, F disjoint closed sets in X, $\exists\, f \in A$ such that $|f| < \varepsilon$ a.e., on E and $|1 - |f|| < \varepsilon$ a.e., on F. Call f a separating function. A *boundedly separates in modulus* (*BSM*) if $\exists\, M \ni : \forall\, \varepsilon > 0$, E, F disjoint closed sets, $\exists$ a separating function $f \in A$ with $\|f\|_{\infty} < M$. We say that A *boundedly separates in modulus by invertible func-*

tions (BSMI) if A is BSM and the bounded separating functions can be chosen to be invertible. If A is a function algebra on X and the a.e., condition can be left out of the above then we say that A is BSM or BSMI *on* X. For example, if A approximates in modulus then A is BSM on X and if A is logmodular then A is BSMI on X. If A is weak-star-Dirichlet [7] then A may not even be BSM, but H^∞ must be BSMI because $\log V = L^\infty_R$ where V is the set of invertible elements in H^∞. This includes the case where μ is a unique representing measure on X, or more generally, is "minimal" in the sense of [7, pg. 238]. Thus BSM, etc., "localize" the separation properties to the support of the measure in question.

THEOREM 2.1. *Let F be a mesurable set in X. If A is* BSM *then $S \mid F = L^p(F)$ if and only if $g \in S^\perp \Rightarrow g \mid F = 0$. In particular, this holds if A approximates in modulus.*

Proof. 1.4 implies the "if" part. Conversely, suppose $S \mid F = L^p(F)$. Then $\exists$ c such that $g \in S^\perp \Rightarrow \| g \mid F \|_q \leqq c \| g \mid F' \|_q$. Choose $\varepsilon > 0$. Find K_n compact $\subset F \subset V_n$ open such that $\mu(V_n \sim K_n) < 1/n$. We can assume that the K_n are monotone. Suppose M is the bounding constant for the separating functions in A. Find $k \in A$ such that $\| k \|_\infty \leqq M$ and $\| k \| - 1 \mid < \varepsilon$ on K_n and $| k | < \varepsilon$ on V_n' a.e. Then for fixed $g \in S^\perp$,

$$\begin{aligned}(1-\varepsilon) \| g \mid K_n \|_q &\leqq \| kg \mid K_n \|_q \leqq \| kg \mid F \|_q \leqq c \| kg \mid F' \|_q \\ &\leqq c \| kg \mid F' \cap V_n \|_q + c \| kg \mid V_n' \|_q \\ &\leqq cM \| g \mid F' \cap V_n \|_q + c\, \varepsilon \| g \|_q .\end{aligned}$$

Letting $\varepsilon \to 0$, we have $\| g \mid K_n \|_q \leqq cM \| g \mid F' \cap V_n \|_q$. Letting $n \to \infty$, we have $g \mid F = 0$.

COROLLARY 2.2. *Let A be* BSM. *Suppose that F_i are mesurable subsets of X and $F_0 = \bigcup_{i=1}^\infty F_i$. If $S \mid F_i = L^p(F_i)$ for $i = 1, 2, \cdots$ then $S \mid F_0 = L^p(F_0)$.*

Proof. Let $g \in S^\perp$. Then $g \mid F_i = 0$ a.e. for $i = 1, 2, \cdots$ and thus $g \mid F_0 = 0$ a.e.

THEOREM 2.3. *Let F be a closed subset of X. If A is* BSMI *then $S \mid F$ is closed in $L^p(F)$ if and only if $g \in S^\perp \Rightarrow g \mid F \in (S \mid F)^\perp$.*

Proof. "If." Apply Corollary 1.4. Here it is not necessary that F be closed.

"Only if." Find V_n open $\supset F$ such that $\mu(V_n \sim F) < 1/n$. Then

$\exists\ M > 0$ and k_n invertible in A such that $\|k_n\|_\infty \leqq M$, $|1 - |k_n|| < \varepsilon$ a.e. on F and $|k_n| < \varepsilon$ a.e. on V_n'. Now $\exists\ c$ such that 1.3 (1) holds so $g \in S^\perp \Rightarrow \|g|F - (S|F)^\perp\|_q \leqq c\|g|F'\|_q$. The same holds for $k_n g$. Thus

$$\|k_n g|F - (S|F)^\perp\|_q \leqq c\|k_n g|V_n \cap F'\|_q + c\|k_n g|V_n'\|_q \leqq cM\|g|V_n \sim F\|_q + c\varepsilon\|g\|_q .$$

Now since k_n are invertible, $k_n(S|F)^\perp = (S|F)^\perp$. Thus

$$(1-\varepsilon)\|g|F - (S|F)^\perp\|_q \leqq \|k_n g|F - (S|F)^\perp\|_q \leqq cM\|g|V_n \sim F\|_q + c\varepsilon\|g\|_q .$$

Letting $\varepsilon \to 0$ and $n \to \infty$ gives $g|F \in (S|F)^\perp$.

COROLLARY 2.4. *Let A be* BSMI. *Suppose F_i are closed subsets of X and $F = \bigcup_{i=1}^\infty F_i$. If $S|F_i$ is closed in $L^p(F_i)$ for each i, then $S|F$ is closed in $L^p(F)$.*

Proof. Take $g \in S^\perp$. Then $g|F_i \in (S|F_i)^\perp$, and by the dominated convergence theorem, it follows that $g|F \in (S|F)^\perp$.

Using the above theorem we also encounter the following phenomenon which is different from that which usually occurs in the function algebra setting.

COROLLARY 2.5. *Let F be a closed subset of X, and let A be* BSMI. *Then $S|F$ is closed in $L^p(F) \Rightarrow S|F'$ is closed in $L^p(F')$. In particular this happens if A is logmodular.*

Proof. Let $g \in S^\perp$. Then $g|F \in (S|F)^\perp$. Hence

$$\widetilde{g|F'} = g - (\widetilde{g|F}) \in S^\perp$$

and thus $g|F' \in (S|F')^\perp$, so $S|F'$ is closed.

The above is explained by the following "splitting lemma" which was pointed out to me by K.B. Laursen.

LEMMA 2.6. *Let S be a closed subspace of $L^p(\mu)$, $1 \leqq p < \infty$, and let F be a measurable subset of X. Then $S = \widetilde{S|F} \oplus \widetilde{S|F'}$ if and only if $g \in S^\perp \Rightarrow \widetilde{g|F} \in S^\perp$.*

REMARKS. The following illustrates 2.5. Let X be the union of two disjoint disks, $\mu = m_1 + m_2$ where m_1 and m_2 are the Lebesgue measures on the two circles comprising the boundary of X, and let A be the algebra of functions continuous on X and analytic on the interior of X. Then $H^1(m_1) + L^1(m_2)$ splits and neither F nor F' have measure 0.

Also it is easy to find examples of closed subspaces of $L^1(-1,1)$ which are proper and interpolate on $(-1,0]$ and $(0,1)$. For example, let S be the set of functions f in $L^1(-1,1)$ such that $f(x) = f(-x)$ a.e.

3. Interpolation of H^p-spaces and function algebras. Throughout this section unless it is otherwise stated, we assume that A is a function algebra on a compact space X, μ is a representing measure for A which is not a point-mass and I is the corresponding maximal ideal.

PROPOSITION 3.1. *If I is* SM *in $L^\infty(\mu)$ then the only open sets on which $H^p(\mu)$ interpolates for some $1 \leqq p \leqq \infty$ are those of measure* 0.

Proof. If H^p interpolates on V open and $\mu(V) > 0$ then find K compact in V of positive measure. Find a sequence in I whose moduli converge to 1 on K and 0 on V'. This contradicts 1.3 (ii).

PROPOSITION 3.2. *If I is* BSM *in $L^\infty(\mu)$ then the only measurable sets on which $H^p(\mu)$ interpolates for some $1 \leqq p \leqq \infty$ are those of measure* 0.

Proof. Suppose H^p interpolates on a set F of positive measure. We may assume that F is closed. Since μ is assumed to not be a point-mass F' has positive measure. We can therefore choose K_n compact and monotone in F' so that $\mu(K_n) \to \mu(F')$. Find f_n in I which are uniformly bounded such that $||f_n| - 1| < 1/n$ on F and $|f_n| < 1/n$ on K_n. This contradicts 1.3 (ii).

We wish to study the relation between interpolation of the algebra A and its associated H^p-spaces. As was pointed out in the introduction, if A approximates in modulus then the situation is trivial. For if F is a closed set on which A interpolates then because F is an intersect of peak sets, we must have that $\mu(F) = 0$ by the dominated convergence theorem. So interpolation of the H^p-space follows vacuously. More generally we have the following.

PROPOSITION 3.3. *Let A be* BSM *on X, and F a closed subset of X. If A interpolates on F then $H^p(\mu)$ interpolate on F for any measure μ, and any $1 \leqq p < \infty$.*

Proof. $g \perp H^p \Rightarrow g\, d\mu \perp A \Rightarrow g\, d\mu_F = 0 \Rightarrow g \,|\, F = 0$ a.e., $\mu \Rightarrow H^p$ interpolates on F.

PROPOSITION 3.4. *If μ is a representing measure for A, and A is* BSM *in $L^\infty(\mu)$, then $H^p(\mu)$ interpolates only on sets of measure* 0 *if $1 \leqq p \leqq \infty$.*

Proof. Suppose for some p, $H^p \mid F = L^p(F)$. Let A_0 be the ideal determined by μ. Then $A_0 \subset (H^p)^\perp$ so by 2.1., $g \in A_0 \Rightarrow g \mid F = 0$ a.e. But if $f \in H^p$, then $f - \int f d\mu$ is a pointwise a.e. limit of a sequence of elements of A_0 and thus $f = \int f d\mu$ a.e. on F, so that all H^p functions are constant a.e. on F. Thus $L^p(F) =$ constants and thus μ_F is a point-mass at some point x. But μ must be continuous, for $\exists\ g \in I$ such that $g(x) \neq 0$ and applying 2.1 gives $\mu\{x\} = 0$.

PROPOSITION 3.5. *Let A be* BSMI *on X, and F a closed subset of X. If $A \mid F$ is closed then $H^p(\mu)$ restricted to F is closed for any measure μ, and any $1 \leqq p < \infty$.*

Proof. $g \perp H^p \Rightarrow g\ d\mu \perp A \Rightarrow g\ d\mu_F \in (A \mid F)^\perp \Rightarrow g\ d\mu_F \in (H^p)^\perp \Rightarrow H^p$ restricted to F is closed by 2.3.

REMARKS. Both 3.3 and 3.5 hold because F is an intersect of peak sets. By the above it is easy to construct examples in which the H^p spaces interpolate on sets of positive measure (where μ is not a representing measure). For another example, let A be the disk algebra on the unit disk, and let $\mu = 1/2\ d\theta + 1/2\ \delta_0$ where δ_0 is the poin-mass at 0. As yet we have boon unable to construct examples which are not of this discrete type when μ is a represeting measure.

We now construct examples in which the algebra and H^∞ interpolate but in which none of the H^p-spaces, $1 \leqq p < \infty$, interpolate. Let $\{r_n\}$ be a nonnegative interpolating sequence in the open unit disk converging to 1. Then $F = \{r_n\} \cup \{1\}$ is an interpolating sequence for the disk algebra on the unit disk [6]. Let μ_n be the Poisson measures for r_n on the unit circle. Choose a sequence $\alpha_n \geqq 0$ such that $\sum_{n=1}^\infty \alpha_n \mu_n < 1/2\, d\theta$ (*). Consider the positive measure $\mu = \sum_{n=1}^\infty \alpha_n (\delta_n - \mu_n) + d\theta$ where δ_n is the point-mass at r_n. Then μ represents 0 for the disk algebra and we claim that $H^\infty(\mu)$ interpolates on F while $H^p(\mu)$ $1 \leqq p < \infty$ do not interpolate on F. To see this we need the following.

LEMMA 3.6. *$H^p(\mu) = H^p \mid F \cup T$ where H^p is the usual H^p-space for the disk algebra* $(1 \leqq p \leqq \infty)$ *on the closed unit disk.*

Proof. If $f \in H^p(d\theta)$ then $\exists\ f_n \in A \ni : f_n \to f$ in $L^p(d\theta)$. If $\hat{f}$ denotes the harmonic extension of f to H^p, then

$$\int |\hat{f}_n - \hat{f}|^p \, d\mu \leqq (1 + \sum 2\alpha_j(1 + r_j)/(1 - r_j)) \int |f_n - f| \, d\theta \longrightarrow 0 .$$

So $H^p | F \cup T \subset H^p(d\mu)$. Conversely, if $f_n \in A$ and $f_n \to f$ in $L^p(\mu)$, then $f_n \to f$ in $L^p(d\theta)$, so $f | T \in H^p(d\theta)$ and therefore extends to $g = \widehat{f | T}$ in H^p. So $g | F \cup T \in H^p(\mu)$ and $g | T = f | T$. But since the functions in $H^p(\mu)$ are determined by their values on T, we have $f = g \in H^p | F \cup T$, and we are done for $1 \leqq p < \infty$. Now

$$\begin{aligned} H^\infty(d\mu) &= H^2(d\mu) \cap L^\infty(d\mu) = [H^2 | F \cup T] \cap L^\infty(\mu) \\ &= [H^2(d\theta) \cap L^\infty(d\theta)\hat{]} \, | F \cup T = H^\infty | F \cup T , \end{aligned}$$

and this completes the proof.

Now observe that if $f \in H^p(d\mu)$, then

$$| f(r_n) |^p \leqq [(1 + r_n)/(1 - r_n)] \int | f |^p \, d\theta$$

so that $\exists \ c \ \ni$: the growth condition $|f(r_n)|^p \leqq c(1 + r_n)/(1 - r_n)$ is satisfied. Thus if we choose a (nonnegative) sequence $\{x_n\}$ such that $x_n^p(1 - r_n)/(1 + r_n) \to \infty$ and such that $\Sigma x_n^p(1 + r_n)\alpha_n/(1 - r_n) < \infty$, we obtain an element of $L^p(\mu_F)$ which is not the restriction of a function from $H^p(d\mu)$. Such a sequence can be found for example by finding $\beta_n \geqq 0$ to satisfy (*) and setting $\alpha_n = \beta_n^2$ and $x_n = (\beta_n)^{-1/p}$.

Since H^∞ interpolates on F, we see that $H^\infty(d\mu)$ interpolates on F by 3.6.

Thus one may ask for conditions that will force interpolation of H^p-spaces to follow from interpolation of the algebra. The following is one such condition.

THEOREM 3.7 *Let A be a function algebra on X, μ a representing measure for A, and A_0 the corresponding maximal ideal. Suppose that $H^p(\mu) = H^\alpha(\mu) \cap L^p(\mu)$, $\alpha \leqq p$. If A_0 is weak-star dense in $H^\alpha(\mu)^\perp$, then interpolation of A on a closed set F implies interpolation of $H^p(\mu)$ on F for all $\alpha \leqq p < \infty$ with integer conjugates q.*

Proof. The conclusion deals only with $1 \leqq \alpha \leqq p \leqq 2$. Suppose $1 < \alpha$ and $A | F = C(F)$. Then $\exists$ c $\ni$: $|| \mu_F || \leqq c || \mu_{F'} ||$ for every $\mu \in A^\perp$. Now choose $g \in A_0$. Then $g^q d\mu \in A^\perp$ so $\int_F | g |^q d\mu \leqq c \int_{F'} | g |^q d\mu$ or (*) $|| g | F ||_q \leqq c^{1/q} || g | F' ||_q$. Since A_0 is dense in $H^p(\mu)^\perp$ also, we have (*) holds for every $g \in H^p(\mu)^\perp$ and thus $H^p(\mu)$ interpolates on F. Suppose $\alpha = 1$. For $g \perp H^1(\mu)$ we have $|| g | F ||_q \leqq c^{1/q} || g | F' ||_q$ for $q = 2, 3, \cdots$, and thus letting $q \to \infty$ we have $|| g | F ||_\infty \leqq || g | F' ||_\infty$ so that $H^1(\mu)$ also interpolates on F.

COROLLARY 3.8. *If A is a function algebra which is weak-star-Dirichlet in $L^\infty(\mu)$ then A interpolates only on sets of μ measure* 0.

Proof. A satisfies the hypotheses of 3.7 [7] and thus H^1 interpolates on F. But H^1 is invariant under H^∞ which is BSMI so that F has μ measure 0 by 3.4.

It is also clear from 3.4 that when A is weak-star-Dirichlet, H^p interpolate only on sets of measure 0 for $1 \leqq p \leqq \infty$. Using the invariant subspace theorem we have the following.

THEOREM 3.9. *Let A be weak-star-Dirichlet. If F is closed and $H^p(\mu)$ restricted to F is closed for some $1 \leqq p < \infty$, then $\mu(F) = 0$, or $\mu(F') = 0$.*

Proof. Since H^p is invariant under H^∞ which is BSMI, applying 2.3 and 2.6 we have $H^p = \widetilde{H^p \mid F} \oplus \widetilde{H^p \mid F'}$. Now if F has positive measure, then $\widetilde{H^p \mid F}$ is a simply invariant subspace of L^p and by the invariant subspace theorem [7, 4.16], $\widetilde{H^p \mid F} = qH^p$ where $|q| = 1$ a.e. But $q \in \widetilde{H^p \mid F}$ so we have $\mu(F') = 0$.

The example preceding 3.7 is clearly not weak-star-Dirichlet because the measure μ is not minimal. In addition we have the following.

COROLLARY 3.10. *In the example preceding* 3.7, *A_0 is not weak-star dense in $H^1(\mu)^\perp$.*

Proof. We only need to verify that $H^p(\mu) \supset H^1(\mu) \cap L^p(\mu)$. But if $f \in H^1(\mu) \cap L^p(\mu)$ then $f \mid T = g \mid T$ where

$$g \in H^1(d\theta) \cap L^p(d\theta) = H^p(d\theta) .$$

So as $\hat{g} \mid F \cup T \in H^p(\mu)$, and $\hat{g}$ and f agree on T, we have

$$f = \hat{g} \mid F \cup T \in H^p(\mu) .$$

Finally we remark that 1.3 should hold for function spaces whose duals restrict in some sense and whose norm satisfies the concavity condition. We hope to consider such examples at a later date.

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