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BAXTER'S THEOREM AND VARBERG'S CONJECTURE

BENNETT EISENBERG

It is shown that Baxter's strong law for Gaussian processes provides necessary and sufficient conditions for equivalence for a wide class of Gaussian processes.

Let X be a Gaussian process with zero mean and covariance $R(s, t) = E(X(s)X(t))$. Baxter [1] proved that if R is continuous for $0 \leqq s, t \leqq 1$ with uniformly bounded second derivatives for $s \neq t$, then

$$D^+(t) = \lim_{s \downarrow t} \frac{R(s, t) - R(t, t)}{s - t}$$

and

$$D^-(t) = \lim_{s \uparrow t} \frac{R(t, t) - R(s, t)}{t - s}$$

exist and are continuous and with probability one

$$\lim_{n \to \infty} \Sigma \left[x\left(\frac{k}{2^n}\right) - x\left(\frac{k-1}{2^n}\right) \right]^2 = \int_0^1 D^-(t) - D^+(t)\, dt .$$

It follows that if Y is another Gaussian process with mean zero and covariance S continuous with bounded second derivatives for $s \neq t$ and if there exists a t with

$$D_R^-(t) - D_R^+(t) \neq D_S^-(t) - D_S^+(t)$$

then the measures μ_x and μ_y for the processes X are singular.

In the case where R and S are triangular covariances Varberg [8] has obtained a converse to this result. Varberg's Theorem:

$$\text{Let } R_i(s, t) = \begin{cases} u_i(s)v_i(t) & s \leqq t \\ v_i(s)u_i(t) & s \geqq t \end{cases},$$

where $i = 1$ or 2.
Assume furthermore

(A) $u_i(0) = 0$
(B) $v_i(t) > 0$ on [0, T]
(C) u_i'' and v_i'' exist and are continuous on $[0, T]$
(D) $v_i(t)u_i'(t) - u_i(t)v_i'(t) > 0$ on $[0, T]$

then μ_x and μ_y are mutually absolutely continuous if and only if

$$v_1(t)u_1'(t) - u_1(t)v_1'(t) = v_2(t)u_2'(t) - u_2(t)v_2'(t)$$

for $t \in [0, T]$

Otherwise $\mu_x \perp \mu_y$.

Condition A says that both processes start at 0. This is no loss in generality since Baxter's theorem involves only the increments of the processes. For a triangular covariance R,

$$D_R^-(t) - D_R^+(t) = v(t)u'(t) - u(t)v'(t)$$

so that Condition D means $D_R^-(t) - D_R^+(t) > 0$. Condition C is slightly stronger than Baxter's regularity condition on R.

Varberg conjectured that for two arbitrary Gaussian processes with covariances R and S satisfying regularity and boundary conditions of the type $A - D$, a necessary and sufficient condition for equivalence is that $D_R^-(t) - D_R^+(t) = D_S^-(t) - D_S^+(t)$.

In this paper we answer Varberg's conjecture. We also extend his theorem by requiring only one continuous derivative for each of the functions u and v.

1. Varberg's conjecture. Throughout this paper X and Y are Gaussian processes with mean zero and covariances R and S, respectively. μ_x and μ_y denote the corresponding measures on the sample paths.

Write $X \sim Y$ if μ_x and μ_y are mutually absolutely continuous over a given sigma field. Write $X \perp Y$ if μ_x and μ_y are singular.

Gaussian processes are known to be either equivalent or singular over the sigma-fields generated by x_t, for t in some parameter set. The Segal-Feldman [4, 6] equivalence conditions take the form:

Let H_x be the Hilbert space spanned by X_t in $L^2(d\mu_x)$ for t in A. Let H_y be the Hilbert space spanned by Y_t in $L^2(d\mu_y)$, for t in A. Then $X \sim Y$ if and only if the map $T: X_t \rightarrow Y_t$ extends to a linear homeomorphism between H_x and H_y (In this case we write $H_x \approx H_y$) and $I - T^*T$ is Hilbert Schmidt from H_x to H_x.

Otherwise $X \perp Y$.

In terms of covariance functions the conditions become [3] $X \sim Y$, if and only if

$R(s, t) - S(s, t) = E(HX_sX_t)$, where H is a Hilbert Schmidt operator on H_x with $I - H$ invertible.

In place of the invertibility condition we also have [3] $X \sim Y$ if and only if $H_x \approx H_y$ and

$R(s, t) - S(s, t) = E(HX_sX_t)$, where H is Hilbert Schmidt on H_x.

For a quick application let X be an additive Gaussian process.

PROPOSITION 1. *Assume X has independent increments with $E(X_t^2) = F(t)$. Then $X \sim Y$ on $[0, T]$ if and only if*

$F(\min(s, t)) - S(s, t) = \int_{0-}^{s}\int_{0-}^{t} H(u, v)\,dF(d)\,dF(v)$, *where* H *is a Hilbert Schmidt kernel on* $L^2([0, T], dF)$ *with* $I - H$ *invertible. Notation:* $\int_{0-}$ *signifies that the mass* $F(0)$ *at* 0 *is included in the integration.*

Proof. $X_t \to c_{0t}$ in $L^2(dF)$ is an isometry of H_x to $L^2([0, t]), dF)$; where C_{0_t} is the characteristic function of $[0, T]$.

Thus $X \sim Y$ if and only if

$$F(\min(s, t)) - S(s, t) = E(HX_sX_t)$$
$$= \int_{0-}^{T}\int_{0-}^{T} \tilde{H}(u, v)C_{0s}(u)C_{0s}(v)dF(u)dF(v) = \int_{0-}^{s}\int_{0-}^{t} \tilde{H}(u, v)dF(u)dF(v)$$

where H is unitarily equivalent to $\tilde{H}$.

This generalizes Shepp's theorem [7] where X is Brownian motion and dF is Lebesgue measure.

COROLLARY. *If* X *is an additive process with* $x(0) \equiv 0$, *and* $F(\min(s, t)) - S(s, t) = \int_0^s\int_0^t H(u, v)dF(u)dF(v)$ *for* $0 \leqq s, t \leqq T$, *then there exists an interval where* $X \sim Y$.

Proof. If we consider a smaller interval $T' \subset T$

$$F(\min(s, t)) - S(s, t) = \int_0^s\int_0^t H(u, v)dF(u)dF(v)$$

for $0 \leqq s, t \leqq T'$ and the same function H. Moreover, as $T' \to 0$, the Hilbert-Schmidt norm of H acting on $L^2([0, T'], dF)$ approaches zero. Hence the operator norm of H approaches zero. But $\|H\| < 1 \Rightarrow I - H$ is invertible.

After this corollary the question might be raised, "Are there weaker conditions which imply that Gaussian processes are equivalent over the sigma field $\bigcap_{n=1}^{\infty} B_{1/n}$ where $B_{1/n}$ is generated by x_t, $0 \leqq t \leqq 1/n$?" The answer is negative.

PROPOSITION 2. *Gaussian processes are singular or equivalent locally. They are equivalent locally if and only if there exists an interval of equivalence.*

Proof. Assume there exists an interval of equivalence. Then a fortiori the processes are locally equivalent.

Assume there does not exist an interval of equivalence. Then the processes are singular over $B_{1/n}$ for every n. We may thus choose A_n in $B_{1/n}$ with $\mu_x(A_n) = 1$ and $\mu_y(A_n) = 0$. $A = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n$ lies in $\bigcap B_{1/n}$ and $\mu_x(A) = 1$ while $\mu_y(A) = 0$. Hence the processes are singular locally.

This proposition implies that two processes share the same strong laws locally if and only if there exists an interval of equivalence.

Next assume both X and Y are additive processes with $E(X_t^2) = F(t)$ and $E(Y_t^2) = G(t)$. To keep the discussion focused on Baxter's theorem assume $F(0) = G(0) = 0$ and F and G are each absolutely continuous with $F' > 0$ and $G' > 0$. Baxter's theorem would say that if F'' and G'' are bounded, then unless $F' \equiv G'$ (which implies $F \equiv G$) the processes would be singular. A theorem of Feldman extends this idea to general F and G.

PROPOSITION 3 (Feldman) [5]: *Assume X and Y are additive processes with $F(0) = G(0) = 0$. Assume F and G are each absolutely continuous with $F' > 0$ and $G' > 0$.*

Then $X \sim Y$ if and only if $F(t) \equiv G(t)$.
Feldman used general F and G in his original theorem.

Proof. For completeness we prove this result, this time using the original Segal-Feldman equivalence conditions. (R. M. Dudley suggested this.)

$x_t \to C_{0t}$ is an isometry of H_x and $L^2(dF)$

$y_t \to C_{0t}(dG/dF)^{1/2}$ is an isometry of H_y and $L^2(dF)$.

T: $x_t \to y_t$ is then unitarily equivalent to $\tilde{T}$: $C_{0t} \to C_{0t}(dG/dF)^{1/2}$ in $L^2(dF)$.

$I - \tilde{T}^*\tilde{T}$ is then multiplication by $1 - (dG/dF)$
which is Hilbert Schmidt if and only if $(dG/dF) \equiv 1$ (and in this case $I - H = I$ is invertible).

Thus Baxter's theorem determines equivalence conditions for some processes without bounded second derivatives.

Next we answer Varberg's conjecture.

PROPOSITION 4. *Let X have covariance R, where*

(A′) $R(0, 0) = 0$

(C′) $(\partial^2 R/\partial s\partial t)$ *exists and is continuous for $s \neq t$ and* $\lim_{s=t} (\partial^2 R/\partial s\partial t)$ *exists for all s.*

(D′) $D_R^-(t) - D_R^+(t) > 0$.

Then there is an additive process Y with X equivalent to Y

on some interval. Moreover, $D_R^- - D_R^+ = D_S^- - D_S^+$.

Before proving the proposition we need some real variable lemmas.

LEMMA. *Assume $R(s, t)$ defined and continuous for $0 \leqq s \leqq t$. Assume $(\partial^2 R/\partial t \partial s)$ is continuous for $s < t$ and $\lim_{s=t} (\partial^2 R/\partial t \partial s)$ exists. Then*

$$\frac{\partial}{\partial t^+} \frac{\partial R}{\partial s^-}\Big|_{s=t} = \lim_{s=t} \frac{\partial^2 R}{\partial t \partial s} .$$

Proof.

$$\int_T^{t'} \int_0^{s'} \frac{\partial^2 R}{\partial s \partial t}\, dsdt = \int_T^{t'} \frac{\partial R}{\partial t}(s', t) - \frac{\partial R}{\partial t}(0, t)dt$$
$$= R(s', t') - R(s', T) - R(0, t') + R(0, T) .$$

Hence

$$R(s, t) = \int_T^t \int_0^s (\partial^2 R/\partial s \partial t)\, dsdt + R(s, T) + R(0, t) - R(0, T) .$$

And by continuity of R and the assumption this holds for $s = t$ as well as $s < t$. Hence $(\partial^2 R/\partial t^+ \partial s^-) = \lim\, (\partial^2 R/\partial s \partial t)$.

COROLLARY. *If R is continuous for $0 \leqq S, t \leqq T$ and if $(\partial^2 R/\partial s \partial t)$ is continuous for $s \neq t$ and has limits at $s = t$, then*

$$\frac{\partial}{\partial t^+} \frac{\partial R}{\partial s^-}\Big|_{s=t} = \frac{\partial}{\partial t^-} \frac{\partial R}{\partial s^+}\Big|_{s=t} = \lim_{s=t} \frac{\partial^2 R}{\partial s \partial t} .$$

In particular, if $(\partial R/\partial s^-)|_{s=t} = (\partial R/\partial s^+)|_{s=t}$ then $(\partial^2 R/\partial s \partial t)$ exists at $s = t$ and is continuous for $0 \leqq s, t \leqq T$.

Note that if R is symmetric and $\lim_{s=t} (\partial^2 R/\partial s \partial t)$ exists from each side of the diagonal, then the limits must be the same. Proof of Proposition 4:

Let $f(t) = D_R^-(t) - D_R^+(t)$. Then f is continuous. Let $E(Y_t^2) = F(t) = \int_0^t f(u)du$. Then by (A'), and the corollary to Proposition 1, $X \sim Y$ on some interval if and only if $F(\min\, (s, t)) - R(s, t) = \int_0^s \int_0^t H(u, v) f(u) f(v)\, dudv$, where H is in L^2.

For $s \neq t$ we have $(\partial^2(F(\min(s, t)) - R(s, t))/\partial s \partial t)$ exists and is continuous.

On the diagonal $s = t$,

$$\frac{\partial}{\partial s^+} (F(\min(s, t) - R(s, t))\Big|_{s=t} = -\frac{\partial R}{\partial s}(s, t)\Big|_{s=t}$$

and

$$\frac{\partial}{\partial s^-}(F(\min(s, t)) - R(s, t))\Big|_{s=t} = f(s) - \frac{\partial R}{\partial s^-}\Big|_{s=t} .$$

By definition $f(t) = (\partial R/\partial s^-)(t, t) - (\partial R/\partial s^+)(t, t)$.

Thus $(\partial(F(\min(s, t)) - R(s, t))/\partial s)$ is continuous everywhere. For $s \neq t$, $(\partial^2(F(\min(s, t)) - R(s, t))/\partial t\partial s) = -(\partial^2 R/\partial t\partial s)$. Using hypothesis C' and applying the corollary of the lemmas $(\partial^2(F(\min(s, t)) - R(s, t))/\partial^t\partial^s)$ exists at $s = t$ and equals $\lim_{s=t}(\partial^2 R/\partial t\partial s)$. Hence, it is a continuous function everywhere. Since $f(u) \neq 0$ by (D') we may write

$$H(u, v) = \frac{\partial^2(F(\min(u, v)) - R(u, v))}{\partial u\partial v}\ (f(u)f(v)).$$

By the corollary to Proposition 1 there is an interval of equivalence of X and Y.

COROLLARY. *If R and S satisfy A', C' and D' then there exists an interval where $X \sim Y$ if and only if there is an interval with $D_R^- - D_R^+ = D_S^- - D_S^+$.*

Proof. They are both equivalent to the same additive process on some interval.

The strict positivity of $D_R^- - D_R^+$ is essential since these conditions are not sufficient for the equivalence of differentiable processes, for example. It would be nice to remove the phrase "there exists an interval", but this cannot be done without complicated positive definiteness conditions on R and S. Consider the case of the Brownian motion with covariance min (s, t) and the Brownian bridge with $S(s, t) = \begin{cases} s(1-t) & s \leqq t \\ t(1-s) & s > t \end{cases}$. Both covariances start at zero, are smooth and satisfy $D^-(t) - D^+(t) \equiv 1$. Still the processes are singular over $[0, T]$, $T \geqq 1$. Varberg handles this situation by Condition B, that $v_i(t) > 0$. Unfortunately there is no corresponding condition for general covariance functions.

2. Equivalent processes with triangular covariance functions.

Next we extend Varberg's theorem. We assume u and v each have but one bounded continuous derivative.

PROPOSITION 5. *Let X have covariance function*

$$R(s, t) = \begin{cases} u(s)v(t) & s \leqq t \\ v(s)u(t) & t < s \end{cases}$$

where $v > 0$, (u/v) is right continuous and increasing. Then $X \sim Y$ over $[0, T]$ if and only if

$$R(s, t) - S(s, t) = v(s)v(t) \int_{0-}^{s} \int_{0-}^{t} H(u, v) d\left(\frac{u}{v}\right) d\left(\frac{u}{v}\right),$$

where H is a Hilbert Schmidt kernel on $L^2([0, T], d(u/v))$ with $I - H$ invertible.

Proof. Since $v > 0$, $X \sim Y$ if and only if $X/v \sim Y/v$. But X/v is an additive process with covariance (u/v) (min (s, t)).
The result then follows from Proposition 1.

COROLLARY. *Assume* $R_i(s, t) = \begin{cases} u_i(s)\, v_i(t) & s \leqq t \\ v_i(s)\, u_i(t) & t < s \end{cases}$, $i = 1, 2$.

Assume A, B, and D, but in place of C assume that u_i' and v_i' are continuous.

Then $X_1 \sim X_2$ on $[0, T]$ if and only if

$$(*) \qquad u_1'(t)v_1(t) - u_1(t)v_1'(t) \equiv u_2'(t)v_2(t) - u_2(t)v_2'(t).$$

Proof. Assume (*) holds. Then for $s \neq t$, $(\partial(R_1 - R_2)/\partial s)$ exists and is continuous.

At $s = t$ we have $(\partial/\partial s^+)\,(R_1 - R_2)|_{s=t} = u_1(t)v_1'(t) - u_2(t)v_2'(t)$

$$\left.\frac{\partial(R_1 - R_2)}{\partial s^-}\right|_{s=t} = u_1'(t)v_1(t) - u_2'(t)v_2(t)\,.$$

By assumption these are equal.

This time the factorability makes checking the continuity of $(\partial^2(R_1 - R_2)/\partial t\partial s)$ easy.

Thus $R_1(s, t) - R_2(s, t) = E(H_1X_1(S)X_1(t))$ and $R_1(s, t) - R_2(s, t) = E(H_2X_2(S)X_2(t))$.

Both relations together imply $H_x \approx H_y$ (see the lemma which follows) so that $X \sim Y$.

Unlike the case in Varberg's proof the more difficult half is to prove that if the processes are equivalent the Baxter condition holds. (Baxter's theorem does not apply here.)

However, we now have

$$\frac{R_1(s, t) - R_2(s, t)}{v(s)v(t)} = \int_0^t \int_0^s H(x, y) d\left(\frac{u}{v}\right) d\left(\frac{u}{v}\right) = \int_0^t \int_0^s \hat{H}\, dxdy\,.$$

By conditions on R_1 and R_2

$(\partial^2(R_1 - R_2)/\partial s\partial t)$ is continuous and bounded for $s \neq t$.

Hence the same holds for $R_1(s, t) - R_2(s, t)/v(s)v(t)$. Since

$$\hat{H} = \frac{\partial^2}{\partial s\partial t}\left(\frac{R_1(s, t) - R_2(s, t)}{v(s)v(t)}\right) \text{ a.e.,}$$

we may assume $\hat{H}$ is continuous and bounded for $s \neq t$. It is then an easy argument to show that $\int_0^t \hat{H}(x, y)dx$ is a continuous function of y for each t.

Hence $(\partial/\partial^s) \int_0^s \int_0^t H(x, y)dxdy$ is a continuous function of s. But

$$R_1(s, t) - R_2(s, t) = v(s)v(t) \int_0^s \int_0^t H(x, y)\, dxdy$$

so $\partial R_1(s, t) - R_2(s, t)/\partial s\,|_{s=t}$ must exist. Hence (*) holds.

LEMMA. *If* $R(s, t) - S(s, t) = E(H_1 X_s X_t)$ *and* $S(s, t) - R(s, t) = E(H_2 Y_s Y_t)$, *where* H_1 *and* H_2 *are Hilbert-Schmidt then* $H_x \approx H_y$.

Proof.

$$S(s, t) = E(Y_s Y_t) = E((I - H)X_s X_t)$$

$$||\Sigma C_k Y_{t_k}||^2 = E((I - H)(\Sigma C_k X_{t_k})\, \Sigma C_k X_{t_k} \leqq k_1 ||\Sigma C_k X_{t_k}||^2 .$$

Similarly,

$$||\Sigma C_k X_{t_k}||^2 \leqq k_2 ||\Sigma C_k Y_{t_k}||^2 .$$

Recently, Yeh [9] has found singularity conditions of the Baxter type using different assumptions on u and v.

In the case where both processes are stationary Belayev has proved a generalized Baxter result: If $E((x_i(h) - x_i(0)^2) = \phi_i(h)$ and if $\lim_{h=0} h^2/\phi_i(h) = 0$, then $X_1 \perp X_2$ unless $\lim_{h=0} (\phi_1(h)/\phi_2(h)) = 1$.

The idea of relating a process to an equivalent additive or otherwise simple process could be useful in understanding these results as well as in extending many strong laws more easily, proved for the simpler processes.

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