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COMPLETIONS OF BOOLEAN ALGEBRAS WITH PARTIALLY ADDITIVE OPERATORS

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To generalize a result of Jónsson and Tarski on perfect extensions of Boolean algebras with operators, L. Henkin has introduced the notion of p-additive operation for p a positive integer. Here we use this notion to extend the analogous result of D. Monk which states that each equation without occurrences of the complementation sign has its validity preserved when passing from a Boolean algebra with operators to its completion.

We first point out very briefly the basic notions and results from [1], [2], or [3] needed in the sequel. Then the theory of completions of Boolean algebras with p_i-additive operators, f_i, is developed following the pattern of [3].

1. A Boolean algebra $\mathfrak{B} = \langle B, +, \circ, -, 0, 1\rangle$ is a completion of a Boolean algebra $\mathfrak{A} = \langle A, +, \circ, -, 0, 1\rangle$ if (i) $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, (ii) for each subset X of A such that $\Sigma^{\mathfrak{A}}_{x\in X}x$ exists in A, $\Sigma^{\mathfrak{B}}_{x\in X}x$ exists in B and $\Sigma^{\mathfrak{A}}_{x\in X}x = \Sigma^{\mathfrak{B}}_{x\in X}x$, (iii) $\mathfrak{B}$ is the least complete Boolean algebra having $\mathfrak{A}$ as a subalgebra. It is well known that every Boolean algebra $\mathfrak{A}$ has such a completion $\mathfrak{B}$ and that for every element x in B, $x = \Sigma_{x\geqq y\in A}y$.

${}^nA$ denotes the set of all n-termed sequences $x = \langle x_0, \cdots, x_{n-1}\rangle$ of elements of A. We write, for $x, y \in {}^nA$, $x \leqq y$ if $x_i \leqq y_i$ for each $i < n$. Furthermore, if $j < n$ and $x, y \in {}^nA$, $x = {}_jy$ means that $x_k = y_k$ for all $k < n$ and $k \neq j$. For p a positive integer and $X \subseteq {}^nA$, σ_pX denotes $\{y \in {}^nA\colon y = x^0 + \cdots + x^{p-1}$ for some $x^0, \cdots, x^{p-1} \in X\}$.

An operation f on ${}^nA$ to A is (i) monotonic if, given any $x, y \in {}^nA$ such that $x \leqq y$, we always have $fx \leqq fy$, (ii) p-additive if, whenever $X \subseteq {}^nA$ has cardinal number $\leqq p + 1$ and there is some $j < n$ such that $x = {}_jy$ for all $x, y \in X$, we always have

$$f(\Sigma X) = \Sigma\{fz\colon z \in \sigma_pX\}\ ,$$

(iii) completely p-additive if, whenever $X \subseteq {}^nA$, ΣX exists in ${}^nA$ and there is some $j < n$ such that $x = {}_jy$ for all $x, y \in X$, then $\Sigma\{fz\colon z \in \sigma_pX\}$ exists and equals $f(\Sigma X)$. $\Phi_p(\mathfrak{A})$ (or Φ_p if no confusion occurs) denotes the set of all p-additive operations on $\mathfrak{A}$, $\Phi^c_p(\mathfrak{A})$ that of all completely p-additive operations on $\mathfrak{A}$, we write Φ_ω for $\bigcup_{p\geqq 1}\Phi_p$ and Φ^c_ω for $\bigcup_{p\geqq 1}\Phi^c_p$. It is clear from the definition that $\Phi^c_p \subseteq \Phi_p$ for each positive integer

p. The basic result that if $f \in \Phi_\omega$ then f is monotonic is proved in [1] (Theorem 2.3).

Except when stated otherwise we assume hereafter that $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras, $\mathfrak{B}$ is a completion of $\mathfrak{A}$ and f is an n-ary operation on $\mathfrak{A}$. An operation g on ${}^nB$ to B is said to be an extension of f if for all $x \in {}^nA$ $fx = gx$. $g \restriction {}^nA$ denotes the restriction of g to ${}^nA$. Given an operation f on $\mathfrak{A}$, Monk has defined in [3] an n-ary operation f^+ on $\mathfrak{B}$ by

$$f^+x = \Sigma\{fy: x \geqq y \in {}^nA\}$$

for any $x \in B$. It is obvious from this definition that f^+ is monotonic, and that f^+ is an extension of f if f is monotonic.

2. First of all we modify an example in 2.6 of [1] so that it will later be clear that our main theorem is indeed an extension of Theorem 1.9 of [3]. Let A be the set of all finite or cofinite subsets of ${}^2\omega$. Define f on A by $fx = x; x$ for all $x \in A$ (here $x; x$ is the relative product of the relation x with itself, so that for any $i, j \in \omega$, we have $\langle i, j\rangle \in fx$ if and only if there is some k such that $\langle i, k\rangle \in x$ and $\langle k, j\rangle \in x$). f is then an operation on A since fx is finite when x is finite and $fx = {}^2\omega$ when x is cofinite. We claim that $f \in \Phi_2^c$: Let $X \subseteqq A$ and $\bigcup X$ exist in A. Then $f(\bigcup X) \supseteqq fy$ for each $y \in \sigma_2 X$ since f is obviously monotonic, so $f(\bigcup X) \supseteqq \bigcup_{y \in \sigma_2 X} fy$. But also if $\langle i, j\rangle \in f(\bigcup X)$, then there is a $k \in \omega$ such that $\langle i, k\rangle \in \bigcup X$ and $\langle k, j\rangle \in \bigcup X$, hence $\langle i, k\rangle \in x$ for some $x \in X$ and $\langle k, j\rangle \in x'$ for some $x' \in X$, and therefore $\langle i, j\rangle \in (x \cup x'); (x \cup x')$, hence $\langle i, j\rangle \in \bigcup_{y \in \sigma_2 X} fy$, so that $f(\bigcup X) \subseteqq \bigcup_{y \in \sigma_2 X} fy$. However, f is not in Φ_1, for let $x = \{\langle 0, 1\rangle\}$ and $y = \{\langle 1, 2\rangle\}$; then $fx = fy = \phi$, but $f(x \cup y) = \{\langle 0, 2\rangle\}$.

THEOREM 1. *If $f \in \Phi_p^c(\mathfrak{A})$ then $f^+ \in \Phi_p^c(\mathfrak{B})$.*

Proof. Suppose $f \in \Phi_p^c(\mathfrak{A})$ and $X \subseteqq {}^nB$ such that for some $j < n$ we have $x = {}_j y$ for all $x, y \in X$. We must show that

$$f^+(\Sigma X) = \Sigma\{f^+z: z \in \sigma_p X\}\ .$$

Since f^+ is monotonic we have, obviously,

$$f^+(\Sigma X) \geqq \Sigma\{f^+z: z \in \sigma_p X\}\ . \tag{1}$$

Let $v \in {}^nA$ be such that $v \leqq \Sigma X$. Then $v_j \leqq (\Sigma X)_j = \Sigma_{x \in X} x_j = \Sigma_{x \in X}\Sigma_{x_j \geqq w \in A} w$. For each $x \in X$ and $w \in A$ with $w \leqq x_j$, we now define an n-sequence $v^{xw} \in {}^nA$ by $v_k^{xw} = v_k$ if $k \neq j$ and $v_j^{xw} = v_j \cdot w$, and note that $v^{xw} \leqq x$. Then we have $v = \{v^{xw}: x \in X \text{ and } w \leqq x_j\}$, hence by the complete p-additivity of f, we get

$$fv = \Sigma\{fy\colon y \in \sigma_p\{v^{xw}\colon x \in X \text{ and } w \leqq x_j\}\}\ .$$

Let now $y \in \sigma_p\{v^{xw}\colon x \in X$ and $w \leqq x_j\}$. Then we have $y \in {}^nA$ and

$$y = v^{x^0w^0} + \cdots + v^{x^{p-1}w^{p-1}}$$

for some $x^0, \cdots, x^{p-1} \in X$ and $w^0, \cdots, w^{p-1} \in A$, where for each $i < p$, $w^i \leqq x^i_j$. Therefore $y \leqq x^0 + \cdots + x^{p-1}$, and hence

$$fy = f^+y \leqq f^+(x^0 + \cdots + x^{p-1}) \leqq \Sigma\{f^+z\colon z \in \sigma_pX\}\ .$$

Since this holds for each $y \in \sigma_p\{v^{xw}\colon x \in X$ and $w \leqq x_j\}$, we have $fv \leqq \Sigma\{f^+z\colon z \in \sigma_pX\}$, and since this inclusion holds for each $v \in {}^nA$ such that $v \leqq \Sigma X$, we get

$$(2)\qquad \Sigma\{f^+z\colon z \in \sigma_pX\} \geqq \Sigma\{fv\colon \Sigma X \geqq v \in {}^nA\} = f^+(\Sigma X)\ .$$

With (1) and (2) the proof is completed.

The assumption of Theorem 1 that f is completely p-additive cannot be weakened to $f \in \Phi_p$:

THEOREM 2. *If $f \in \Phi^c_p(\mathfrak{B})$ and $f \restriction {}^nA$ is an operation on A then $f \restriction {}^nA \in \Phi^c_p(\mathfrak{A})$.*

Proof. This is immediate from the definition of complete p-additivity and the fact that the sum is preserved from $\mathfrak{A}$ to $\mathfrak{B}$.

LEMMA 3. *If p is any positive integer and $x \in {}^nB$, then*

$$\sigma_p\{y \in {}^nA\colon y \leqq x\} = \{y \in {}^nA\colon y \leqq x\}\ .$$

Proof. Obvious.

THEOREM 4. *If $f \in \Phi^c_p(\mathfrak{B})$ and $f \restriction {}^nA$ is an operation on A, then $f = (f \restriction {}^nA)^+$.*

Proof. For any $x \in {}^nB$, we have

$$fx = f(x_0, \cdots, x_{n-1}) = f(\Sigma_{x_0 \geqq y_0 \in A}y_0, \cdots, \Sigma_{x_{n-1} \geqq y_{n-1} \in A}y_{n-1})\ .$$

Using repeatedly the fact that f is completely p-additive, we get

$$fx = \Sigma_{y_0 \in \sigma_p\{y_0 \in A: y_0 \leqq x_0\}}, \cdots, \Sigma_{y_{n-1} \in \sigma_p\{y_{n-1} \in A: y_{n-1} \leqq x_{n=1}\}}fy$$

and then, by Lemma 3,

$$\begin{aligned} fx &= \Sigma_{x_0 \geqq y_0 \in A}, \cdots, \Sigma_{x_{n-1} \geqq y_{n-1} \in A}fy = \Sigma_{x \geqq y \in {}^nA}fy \\ &= \Sigma_{x \geqq y \in {}^nA}(f \restriction {}^nA)y = (f \restriction {}^nA)^+x \end{aligned}$$

as desired.

As in [3] it follows now that each completely p-additive operation on $\mathfrak{A}$ has exactly one extension which is a completely p-additive operation on $\mathfrak{B}$, and so there is a one-one correspondence between the set of completely p-additive operations on $\mathfrak{A}$ and the set of the ones on $\mathfrak{B}$ which extend those on $\mathfrak{A}$.

Also established as in [3] is:

THEOREM 5.
(i) $+^+ = +$.
(ii) $\cdot^+ = \cdot$.
(iii) *If* $f = A \times \{a\}$, *then* $f^+ = B \times \{a\}$.
(iv) *If* $fx = x_i$ *for each* $x \in {}^nA$ (where $i < n$), *then* $f^+x = x_i$ *for each* $x \in {}^nB$.

If f is any m-ary operation and $g_0, \cdots, g_{m-1}$ are n-ary operations on A, one composes them to obtain the operation $f[g_0, \cdots, g_{m-1}]$, i.e., the n-ary operation h such that $hx = f(g_0x, \cdots, g_{m-1}x)$ for every $x \in {}^nA$.

THEOREM 6. *If f is m-ary, $f \in \Phi_p^c(\mathfrak{A})$ and $g_0, \cdots, g_{m-1}$ are n-ary monotonic operations on A, then*

$$(f[g_0, \cdots, g_{m-1}])^+ = f^+[g_0^+, \cdots, g_{m-1}^+] .$$

Proof. Assume that the conditions of the theorem hold. If $x \in {}^nB$, we then have, as in the proof of Theorem 1.8 of [3],

$$f^+[g_0^+, \cdots, g_{m-1}^+]x \geqq (f[g_0, \cdots, g_{m-1}])^+x .$$

Also

$$f^+[g_0^+, \cdots, g_{m-1}^+]x = f^+(\Sigma_{x \geqq y^0 \in {}^nA}\ g_0y^0, \cdots, \Sigma_{x \geqq y^{m-1} \in A}\ g_{m-1}y^{m-1}) .$$

By Theorem 1 we have $f^+ \in \Phi_p^c(\mathfrak{B})$ and using repeatedly this fact, we get

$$f^+[g_0^+, \cdots, g_{m-1}^+]x = \Sigma_{u_0 \in \sigma_p\{g_0y^0 : x \geqq y^0 \in {}^nA\}}, \cdots, \Sigma_{u_{m-1} \in \sigma_p\{g_{m-1}y^{m-1} : x \geqq y^{m-1} \in {}^nA\}} f^+u .$$

Now if $u \in {}^mA$ is such that for each $k < m$, $u_k \in \sigma_p\{g_ky^k : x \geqq y^k \in {}^nA\}$, then $u_k = \Sigma_{i<p} g_k y^{k,i}$ where for each $i < p$, $y^{k,i} \in {}^nA$ and $y^{k,i} \leqq x$. If $z = \Sigma\{y^{k,i} : k < m \text{ and } i < p\}$ then $z \in {}^nA$ and $z \leqq x$. For $k < m$ we have $g_kz \geqq g_ky^{k,i}$ for all $i < p$, hence $g_kz \geqq \Sigma_{i<p} g_ky^{k,i} = u_k$ by monotonicity of g_k. Thus

$$\begin{aligned} f^+u &\leqq f^+(g_0z, \cdots, g_{m-1}z) \leqq \Sigma\{f(g_0z, \cdots, g_{m-1}z) : x \geqq z \in {}^nA\} \\ &= (f[g_0, \cdots, g_{m-1}])^+x . \end{aligned}$$

Since this inclusion holds for each u with $u_k \in \sigma_p\{g_k y^k : x \geqq y^k \in {}^nA\}$ for each $k < m$, we have $f^+[g_0^+, \cdots, g_{m-1}^+]x \leqq (f[g_0, \cdots, g_{m-1}]^+x$, and this completes our proof.

In Theorem 6 the condition that $f \in \Phi_p^c$ cannot be replaced by $f \in \Phi_p$, as the example following Theorem 1.7 of [3] shows.

THEOREM 7. *Let* $f_0, \cdots, f_{k-1} \in \Phi_\omega^c(\mathfrak{A})$ *and let* $\tau(f_0, \cdots, f_{k-1}) = \rho(f_0, \cdots, f_{k-1})$ *be an equation which holds for all* $x \in {}^nA$. *Then the corresponding equation* $\tau(f_0^+, \cdots, f_{k-1}^+) = \rho(f_0^+, \cdots, f_{k-1}^+)$ *holds for all* $x \in {}^nB$.

The proof of Theorem 7 is similar to that of 3.8 of [1] except we use Theorems 5 and 6 here.

We adopt terminology slightly different from that in [1] and say that a system $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, f_i \rangle_{i \in I}$ is a *Boolean algebra with partially additive operators* if $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra and $f_i \in \Phi_\omega$ for each $i \in I$, that $\mathfrak{A}$ is *completely partially additive* if $f_i \in \Phi_\omega^c$ for each $i \in I$, and that $\mathfrak{A}$ is *complete* if $\mathfrak{A}$ is completely partially additive and $BL\mathfrak{A}$ (the Boolean part of $\mathfrak{A}$) is complete. We may now extend the notion of completion to Boolean algebras with partially additive operators and call a system

$$\mathfrak{B} = \langle B, +, \cdot, -, 0, 1, g_i \rangle_{i \in I}$$

a *completion* of a Boolean algebra with partially additive operators $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, f_i \rangle_{i \in I}$ in case $BL\mathfrak{B}$ is completion of $BI\mathfrak{A}$ and for each $i \in I$, $g_i = f_i^+$. Theorem 2 then yields:

THEOREM 8. *If* $\mathfrak{A}$ *is a Boolean algebra with partially additive operators which is completely partially additive, then there is a completion of* $\mathfrak{A}$ *which is complete.*

If we associate an equational logic $L_\mathfrak{A}$ with a class to which a given Boolean algebra with partially additive operators $\mathfrak{A}$ belongs, and call a term σ of $L_\mathfrak{A}$ positive if the complementation sign does not occur in σ, and an equation $\tau = \rho$ positive, if both τ and ρ are positive, then we immediately obtain the following extensions of other of Monk's theorems:

THEOREM 9. *If* $\mathfrak{B}$ *is a completion of a completely partially additive Boolean algebra* $\mathfrak{A}$, *then a positive equation* $\tau = \rho$ *holds in* $\mathfrak{A}$ *if and only if it holds in* $\mathfrak{B}$.

THEOREM 10. *With* $\mathfrak{A}$ *and* $\mathfrak{B}$ *as in Theorem* 9, *if* Γ *is a conjunction or disjunction of formulas of the form* $\sigma = 0$ *or* $\sigma \neq 0$ *where*

σ is positive, and if τ and ρ are positive, then $\Gamma \to \tau = \rho$ holds in $\mathfrak{A}$ if and only if it holds in $\mathfrak{B}$.

Finally, Theorem 1.12 of [3] can also be extended to

THEOREM 11. *Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be Boolean algebras with partially additive operators, $\mathfrak{A}$ completely partially additive, $\mathfrak{B}$ a completion of $\mathfrak{A}$, $\mathfrak{A}$ a subalgebra of $\mathfrak{C}$, $\mathfrak{C}$ complete and $BL\mathfrak{A}$ a regular subalgebra of $BL\mathfrak{C}$* (i.e., a subalgebra for which the sum is preserved from $\mathfrak{A}$ to $\mathfrak{C}$). *Then there is an isomorphism f from $\mathfrak{B}$ into $\mathfrak{C}$ such that $Id \restriction A \subset f$* (where Id is the identity map).

Proof. As in the proof of Theorem 1.12 of [3], if we define $fb = \Sigma_{b \geqq a \in A}\, a$ for any $b \in B$, then f is a complete Boolean isomorphism into, and $Id \restriction A \subseteqq f$. To show that f preserves non-Boolean operations, we may then use Theorem 2.8 of [1] and our Lemma 3.

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