PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 3, 1972

ON PTAK'S COMBINATORIAL LEMMA

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A new proof of Ptak's combinatorial lemma on the existence of convex means, is given.

The purpose of this note is to show how the "getting near the inf of S" technique used in [3], Lemma 2 can be used to prove, and indeed generalize, Ptak's combinatorial lemma ([1], (1.3)). (We note, in passing, that [1], (2.1) is an easy consequence of [3], Lemma 2 and that [1], (3.3), Krein's theorem, is proved in [4], Theorem 16). We have already given a proof of Ptak's lemma in [2] using lattice theory and the Hahn-Banach theorem. The proof given here is elementary — as was Ptak's original proof.

1. NOTATION. If $X \neq \emptyset$ we write $l_u(X)$ for the set of all functions from X into $[-\infty, \infty)$. Even though $l_u(X)$ is not a vector space, any *convex* combination of elements of $l_u(X)$ is well defined. We write "conv" for "convex hull of". We write "S_X" for "supremum on X".

2. LEMMA. *We suppose $X \neq \emptyset$. If G is a nonempty convex subset of $l_u(X)$, $A, B, C \in R$ and, for all $g \in G$, $A < B \leqq S_X(g) \leqq C$ then there exists $h \in G$ such that, if $X' = \{x: x \in X, h(x) > A\}$ then* $\inf S_{X'}(G) \geqq A$.

Proof. We choose $\lambda > 0$ so that $\lambda(C - A) < B - A$ and then $h \in G$ so that $S_X(h) < \inf S_X(G) + (B-A)\lambda/(1+\lambda)$. If $g \in G$ then, since $(h+\lambda g)/(1+\lambda) \in G$, $(1+\lambda)S_X(h) < S_X(h+\lambda g) + (B-A)\lambda$ hence there exists $x \in X$ (depending on g) such that

$$h(x) + \lambda g(x) > (1+\lambda)S_X(h) - (B-A)\lambda . \tag{1}$$

We first deduce from (1) that $h(x) > (1+\lambda)S_X(h) - \lambda C - (B-A)\lambda \geqq (1+\lambda)B - \lambda C - (B-A)\lambda > A$, from the choice of λ; hence $x \in X'$. Again from (1), $\lambda g(x) > \lambda S_X(h) - (B-A)\lambda \geqq \lambda A$ from which $g(x) > A$. We have proved that $\inf S_{X'}(G) \geqq A$, as required.

3. THEOREM. *We suppose $X \neq \emptyset$, Y is infinite,*

$$f: X \times Y \to [-\infty, \infty) ,$$

$B, C \in R$, $\delta > 0$ and, for all $g \in \operatorname{conv} f(\cdot, Y)$, $B \leqq S_X(g) \leqq C$. Then there exist $x_1, x_2, \cdots \in X$ and distinct $y_1, y_2, \cdots \in Y$ such that $f(x_p, y_m) \geqq$

$B - S$ *whenever* $1 \leqq m \leqq p$.

Proof. From Lemma 2, there exists $g_1 \in \operatorname{conv} f(\cdot, Y)$ such that if $X_1 = \{x\colon x \in X,\ g_1(x) > B - \delta/2\}$ then $\inf S_{X_1}(\operatorname{conv} f(\cdot, Y)) \geqq B - \delta/2$. There exists finite $F_1 \subset Y$ such that $g_1 \in \operatorname{conv} f(\cdot, F_1)$. Clearly $\inf S_{X_1}(\operatorname{conv} f(\cdot, Y \backslash F_1)) \geqq B - \delta/2$. Proceeding inductively we find $g_n \in \operatorname{conv} f(\cdot, Y\backslash F_1 \backslash \cdots \backslash F_{n-1})$ such that, if

$$X_n = \{x\colon x \in X_{n-1},\ g_n(x) > B - \delta/2 - \cdots - \delta/2^n\}$$

then $\inf S_{X_n}(\operatorname{conv} f(\cdot, Y\backslash F_1 \backslash \cdots F_{n-1})) \geqq B - \delta/2 - \cdots - \delta/2^n$ and finite $F_n \subset Y \backslash F_1 \backslash \cdots \backslash F_{n-1}$ such that $g_n \in \operatorname{conv} f(\cdot, F_n)$. In this way we obtain a family $[F_n\colon n \geqq 1\}$ of disjoint finite subsets of Y, $g_n \in \operatorname{conv} f(\cdot, F_n)$ such that, for all $n \geqq 1$,

$$\bigcap_{m \leqq n} \{x\colon x \in X,\ g_m(x) \geqq B - \delta\}\ (\supset X_n) \neq \emptyset\ .$$

Since $g_1 \in \operatorname{conv} f(\cdot, F_1)$,

$$\{x\colon x \in X,\ g_1(x) \geqq B - \delta\} \subset \bigcup_{y \in F_1} \{x\colon x \in X,\ f(x, y) \geqq B - \delta\}$$

hence there exists $y_1 \in F_1$ such that, for arbitrarily large $n \geqq 2$,

$$\{x\colon x \in X,\ f(x, y_1) \geqq B - \delta\} \cap \bigcap_{2 \leqq m \leqq n} \{x\colon x \in X,\ g_m(x) \geqq B - \delta\} \neq \emptyset\ .$$

This relationship must clearly then hold for all $n \geqq 2$. Proceeding inductively we find $y_n \in F_n$ such that, for all $1 \leqq p < n$,

$$\bigcap_{m \leqq p} \{x\colon x \in X,\ f(x, y_m) \geqq B - \delta\} \cap \bigcap_{p+1 \leqq m \leqq n} \{x\colon x \in X,\ g_m(x) \geqq B - \delta\} \neq \emptyset$$

and so, in particular,

$$\bigcap_{m \leqq p} \{x\colon x \in X,\ f(x, y_m) \geqq B - \delta\} \neq \emptyset$$

from which the required result follows ($y_1, y_2, \cdots$ are distinct because $F_1, F_2, \cdots$ disjoint).

Ptak's Lemma. We suppose that Y is an infinite set and that X is a nonvoid family of subsets of Y. We write $P(Y)$ for the collection of all positive, real valued functions λ on Y such that $\{y\colon y \in Y,\ \lambda(y) > 0\}$ is finite and $\sum_{y \in Y} \lambda(y) = 1$; for $x \subset Y$ we write $\lambda(x) = \sum_{y \in x} \lambda(y)$. If

$$B = \inf_{\lambda \in P(Y)}\ \sup_{x \in X}\ \lambda(x) > 0$$

then there exist $x_1, x_2, \cdots \in X$ and distinct $y_1, y_2, \cdots \in Y$ such that, for each $p \geqq 1$, $\{y_1, \cdots, y_p\} \subset x_p$.

Proof. We define $f: X \times Y \to R$ by $f(x, y) = 1$ if, and only if, $y \in x$. By hypothesis, $\inf S_X(\operatorname{conv} f(\cdot, Y)) = B$. From Theorem 3 with $\delta = B/2$, there exist $x_1, x_2, \cdots \in X$ and distinct $y_1, y_2, \cdots \in Y$ such that $f(x_p, y_m) \geqq B/2$, hence $y_m \in x_p$, whenever $1 \leqq m \leqq p$.

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Received January 28, 1971. This research was supported in part by NSF grant number 20632.

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