PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 3, 1972

A CONVERGENCE THEOREM WITH BOUNDARY

S. SIMONS

This paper contains a bounded-convergence type theorem that depends on the fact that certain functions attain their suprema. Among the applications discussed are Rainwater's theorem and two technical results, one used in the proof of the Choquet-Bishop-deLeeuw theorem and the other in the proof of Krein's Theorem.

The contents of this paper and the two following it were suggested by some results and techniques of R. C. James and J. D. Pryce.

The main result of this paper is Lemma 2. See [1], Lemma 2 and [5], Lemma 4 for the source of the idea. We deduce from Lemma 2 a one-sided convergence theorem (Theorem 3) and a two-sided convergence theorem (Theorem 8).

Corollary 4 is a strict generalization of the following result: if $\{f_n\}_{n\geqq 1}$ is a uniformly bounded sequence of concave uppersemicontinuous functions on a compact subset X of a real Hausdorff locally convex space and $\liminf_{n\to\infty} f_n(x) \geqq 0$ for each extreme point x of X then $\liminf_{n\to\infty} f_n(x) \geqq 0$ for each $x \in X$. (See [4], Lemma 4.3, p. 28.) The latter result is used in one proof of the Choquet-Bishop-deLeeuw theorem. (For an alternative approach see [7], Theorem 43.)

Corollary 10 extends Lebesgue's bounded convergence theorem to continuous functions on a pseudocompact space (i.e., a topological space on which every real continuous function is bounded (and hence attains its bounds)).

Corollary 11 is a strict generalization of the following result of Rainwater: let $\{x_n\}_{n\geqq 1}$ be a bounded sequence in a normed linear space E, $x \in E$ and $\langle x_n, y\rangle \to \langle x, y\rangle$ for each extreme point y of the unit ball of the dual, E', of E. Then $x_n \to x$ in $w(E, E')$. (See [4], p. 33 and [6].)

Corollary 13 is a strict generalization of the following result used in one proof of Krein's Theorem: if Y is a countably compact subset of a real linear topological space, $\{f_n\}_{n\geqq 1}$ is a sequence of continuous linear functionals on E uniformly bounded on Y and $\lim_{n\to\infty}\langle y, f_n\rangle = 0$ whenever $y \in Y$ then $\lim_{n\to\infty}\langle x, f_n\rangle = 0$ whenever $x \in \mathrm{conv}^- Y$. (See [2], 17.11, p. 158 and 17 H, p. 164.)

All vector spaces considered in this paper will be real.

1. NOTATION. We suppose that $X \neq \phi$. If $f \in l_\infty(X)$ we write $S(f) = \sup f(X)$, $I(f) = \inf f(X)$ and $\|f\| = \sup|f(X)|$. We write "conv" for "convex hull of".

2. LEMMA. *We suppose that, for all* $n \geqq 1, f_n \in l_\infty(X)$ *and* $\sup_{n\geqq 1}\|f_n\| < \infty$. *We suppose further that* $Y \subset X$ *and that, whenever* $\lambda_n \geqq 0$ *and* $\sum_{n\geqq 1}\lambda_n = 1$, *there exists* $y \in Y$ *such that* $\sum_{n\geqq 1}\lambda_n f_n(y) = S(\sum_{n\geqq 1}\lambda_n f_n)$.

Then $\sup_{y\in Y}\limsup_{n\to\infty} f_n(y) \geqq \inf S(\operatorname{conv}\{f_n : n \geqq 1\})$.

Proof. We write $A = \inf S(\operatorname{conv}\{f_n : n \geqq 1\})$ and $B = \sup_{n\geqq 1} S(f_n)$. Then $-\infty < A \leqq B < \infty$. We suppose that $\delta > 0$ is arbitrary and choose $\lambda > 0$ such that $A - \delta(1+\lambda) - B\lambda \geqq (A - 2\delta)(1-\lambda)$ (which implies that $\lambda < 1$). We choose $g_1, g_2, \cdots$ inductively so that, for all $m \geqq 1$, $g_m \in \operatorname{conv}\{f_n : n \geqq m\}$ and

$$S(\textstyle\sum_{n\leqq m}\lambda^{n-1}g_n) \leqq \inf S(\sum_{n\leqq m-1}\lambda^{n-1}g_n + \lambda^{m-1}\operatorname{conv}\{f_n : n \geqq m\}) + \delta\left(\frac{\lambda}{2}\right)^m .$$

Since

$$\frac{g_m + \lambda g_{m+1}}{1+\lambda} \in \operatorname{conv}\{f_n : n \geqq m\}, \qquad \text{for all} \quad m \geqq 1$$

$$(1) \qquad S(\sum_{n\leqq m}\lambda^{n-1}g_n) \leqq S\left(\sum_{n\leqq m-1}\lambda^{n-1}g_n + \lambda^{m-1}\frac{g_m + \lambda g_{m+1}}{1+\lambda}\right) + \delta\left(\frac{\lambda}{2}\right)^m .$$

We write $h_0 = 0$, for all $m \geqq 1$, $h_m = \sum_{n\leqq m}\lambda^{n-1}g_n$ and $h = \sum_{n\geqq 1}\lambda^{n-1}g_n$. Then, multiplying (1) by $(1+\lambda)$, for all $m \geqq 1$

$$\begin{aligned}(1+\lambda)S(h_m) &\leqq S(\lambda h_{m-1} + h_{m+1}) + \delta(1+\lambda)\left(\frac{\lambda}{2}\right)^m \\ &\leqq \lambda S(h_{m-1}) + S(h_{m+1}) + \delta(1+\lambda)\left(\frac{\lambda}{2}\right)^m\end{aligned}$$

from which

$$(2) \qquad \frac{S(h_{m+1}) - S(h_m)}{\lambda^m} \geqq \frac{S(h_m) - S(h_{m-1})}{\lambda^{m-1}} - \frac{\delta(1+\lambda)}{2^m} .$$

Since $S(h_1) - S(h_0) = S(h_1) \geqq A$, it follows from (2) and induction that, for all $m \geqq 1$,

$$(3) \quad \frac{S(h_m) - S(h_{m-1})}{\lambda^{m-1}} \geqq A - \delta(1+\lambda)\left(\frac{1}{2} + \frac{1}{4} + \cdots\right) = A - \delta(1+\lambda)$$

hence $S(h) - S(h_{m-1}) = \sum_{n\geqq m}[S(h_n) - S(h_{n-1})] \geqq \sum_{n\geqq m}\lambda^{n-1}[A - \delta(1+\lambda)]$ i.e.

$$(4) \qquad S(h) - S(h_{m-1}) \geqq \frac{\lambda^{m-1}}{1-\lambda}[A - \delta(1+\lambda)] .$$

By assumption, there exists $y \in Y$ such that $h(y) = S(h)$. Then for all $m \geqq 1$

$$\lambda^{m-1}g_m(y) = h(y) - h_{m-1}(y) - \sum_{n \geqq m+1} \lambda^{n-1} g_n(y)$$
$$\geqq S(h) - S(h_{m-1}) - \sum_{n \geqq m+1} \lambda^{n-1} B$$
from (4)
$$\geqq \frac{\lambda^{m-1}}{1-\lambda}[A - \delta(1+\lambda)] - \frac{\lambda^m}{1-\lambda}B$$

hence, from the choice of λ, $g_m(y) \geqq A - 2\delta$. Since $g_m \in \operatorname{conv}\{f_n : n \geqq m\}$, for each $m \geqq 1$ there exists $k(m) \geqq m$ such that $f_{k(m)}(y) \geqq A - 2\delta$, from which $\limsup_{n\to\infty} f_n(y) \geqq A - 2\delta$. The result follows since δ is arbitrary.

3. THEOREM. *If the notation is as in Lemma* 2 *and* μ *is a linear functional on* $l_\infty(X)$ *dominated by* S (i.e., a positive linear functional of norm 1) *then*

$$\sup_{y \in Y} \limsup_{n\to\infty} f_n(y) \geqq \limsup_{n\to\infty} \mu(f_n) .$$

In particular, for all $x \in X$,

$$\sup_{y \in Y} \limsup_{n\to\infty} f_n(y) \geqq \limsup_{n\to\infty} f_n(x) .$$

Proof. If $\sup_{y\in Y} \limsup_{n\to\infty} f_n(y) < \limsup_{n\to\infty} \mu(f_n)$ then, by replacing $\{f_n\}$ by an appropriate subsequence, we can assume that

$$\sup_{y \in Y} \limsup_{n\to\infty} f_n(y) < \inf_{n \geqq 1} \mu(f_n) .$$

But $\inf_{n\geqq 1} \mu(f_n) = \inf \mu(\operatorname{conv}\{f_n : n \geqq 1\}) \leqq \inf S(\operatorname{conv}\{f_n : n \geqq 1\})$ and this would contradict Lemma 2.

4. COROLLARY. *We suppose that* X *is a compact convex subset of a real linear topological space* E, $Y \subset X$ *and*

(5) $\begin{cases} \textit{whenever } f \textit{ is a continuous convex function on } X \\ \textit{then there exists } y \in Y \textit{ such that } f(y) = S(f) . \end{cases}$

(a) *If, for each* $n \geqq 1, f_n$ *is a continuous convex function on* X, $\sup_{n\geqq 1} \|f_n\| < \infty$ *and* $\limsup_{n\to\infty} f_n(y) \leqq 0$ *whenever* $y \in Y$ *then* $\limsup_{n\to\infty} f_n(x) \leqq 0$ *whenever* $x \in X$.

(b) *If* E *is locally convex Hausdorff and, for each* $n \geqq 1$, g_n *is a bounded convex lower semicontinuous function on* X, $\sup_{n\geqq 1} \|g_n\| < \infty$ *and* $\limsup_{n\to\infty} g_n(y) \leqq 0$ *whenever* $y \in Y$ *then* $\limsup_{n\to\infty} g_n(x) \leqq 0$ *whenever* $x \in X$. *In particular, this result is true if* $Y = exX$ (the set of extreme points of X).

Proof.

(a) is immediate from Theorem 3.

(b) We suppose $x \in X$. Then, for all $n \geqq 1$, there exists a continuous convex function f_n on X such that $I(g_n) \leqq f_n \leqq g_n$ and $f_n(x) \geqq g_n(x) - 1/n$. (See [3], p. 222 or [4], p. 19; we can take f_n of the form $\max\{I(g_n), a_n + \langle \cdot, x_n' \rangle | X\}$ where $a_n \in R$ and $x_n \in E'$, the dual of E.) The result follows from (a) applied to $\{f_n : n \geqq 1\}$. The final observation follows from Bauer's theorem on extreme points (see [3], p. 225).

5. EXAMPLE. We write E for the set of all real sequences $\{x_n\}_{n \geqq 0}$ such that $\sum_{n \geqq 0} |x_n| < \infty$ and E' for the set of all real sequences $\{z_n\}_{n \geqq 1}$ that are eventually constant. We define $\langle \cdot, \cdot \rangle : E \times E' \to R$ by

$$\langle x, z \rangle = x_0 \lim_{n \to \infty} z_n + \sum_{n \geqq 1} x_n z_n \,.$$

We write $X = \{x : x \in E, \sum_{n \geqq 0} |x_n| \leqq 1\}$ and $Y = \{\pm e^{(1)}, \pm e^{(2)}, \cdots\} \subset X$. Then X is $w(E, E')$-compact and

(6) for all $z \in E'$ there exists $y \in Y$ such that $\langle y, z \rangle = \sup \langle X, z \rangle$.

If $z_n \in E'$ is defined by $z_{n,m} = 0$ $(m < n)$ and $z_{n,m} = 1$ $(m \geqq n)$ then, for all $y \in Y$, $\lim_{n \to \infty} \langle y, z_n \rangle = 0$ but $\lim_{n \to \infty} \langle e^{(0)}, z_n \rangle = 1$. So Corollary 4(b) fails if we weaken (5) to (6) *even if all the functions* g_n *are in* $\langle \cdot, E' \rangle | X$.

6. REMARK. As is well known, (6) implies that $\bar{Y} \supset exX$. (5) implies that every K-analytic set that contains Y must also contain X. (The statement for K_σ sets follows from Urysohn's Lemma, Corollary 4, and the fact that if $f_n \in C(X)$ and $x \in exX$ then there exists a continuous affine function g_n on X such that $g_n \geqq f_n$ and $g_n(x) \leqq f_n(x) + 1/n$. The extension to K-analytic sets follows from standard arguments.)

7. EXAMPLE. We suppose that $\mathscr{A}$ is an uncountable set and we write E for $l_\infty(\mathscr{A})$ with the topology $w(l_\infty(\mathscr{A}), l_1(\mathscr{A}))$ and $X = \{x : x \in E, \sup_{\alpha \in \mathscr{A}} |x(\alpha)| \leqq 1\}$. If f is a continuous convex function on X then, from Bauer's Theorem, there exists $x \in exX$ such that $f(x) = S(f)$. By continuity, there exists $\{g_n : n \geqq 1\} \subset l_1(\mathscr{A})$ such that $y \in X$ and $\sup_{n \geqq 1} |\langle y - x, g_n \rangle| = 0$ imply that $f(y) = f(x) = S(f)$. Hence there exists a countable subset $\mathscr{B}$ of $\mathscr{A}$ such that $y \in X$ and $\sup_{\beta \in \mathscr{B}} |y(\beta) - x(\beta)| = 0$ imply that $f(y) = S(f)$. Consequently, (5) is satisfied if we write $Y = \{y : y \in X$, for all $\alpha \in \mathscr{A}$, $y(\alpha) = 0$ or ± 1, $\{\alpha : \alpha \in \mathscr{A}, y(\alpha) \neq 0$ is countable$\}\}$. But $Y \cap exX = \phi$.

8. THEOREM. *We suppose that* $\{f_n : n \geqq 1\}$ *is as in Lemma* 2, $Y \subset X$ *and, whenever* $\lambda_n \geqq 0$ *and* $\sum_{n \geqq 1} \lambda_n = 1$, *there exist* $y, z \in Y$ *such that*

$$\sum_{n\geqq 1} \lambda_n f_n(y) = S(\sum_{n\geqq 1} \lambda_n f_n)$$

and

$$\sum_{n\geqq 1} \lambda_n f_n(z) = I(\sum_{n\geqq 1} \lambda_n f_n) .$$

If $f_n \to 0$ pointwise on Y then $f_n \to 0$ in $w(l_\infty(X), l_\infty(X)')$ and, in particular, $f_n \to 0$ pointwise on X.

Proof. From Theorem 3, if μ is a positive linear functional on $l_\infty(X)$ then $\limsup_{n\to\infty} \mu(f_n) \leqq 0$. Applying the same argument with f_n replaced by $-f_n$ we see that $\liminf_{n\to\infty}\mu(f_n) \geqq 0$. Hence $\lim_{n\to\infty}\mu(f_n) = 0$. The result follows since any element of $l_\infty(X)'$ is the difference of two positive linear functionals on $l_\infty(X)$.

9. COROLLARY. *We suppose that M is a $||\cdot||$-closed subspace of $l_\infty(X)$, $Y \subset X$ and, for all $f \in M$, there exists $y \in Y$ such that $f(y) = S(f)$. If, for all $n \geqq 1$, $f_n \in M$, $\sup_{n\geqq 1} ||f_n|| < \infty$, $f \in M$ and $f_n \to f$ pointwise on Y then $f_n \to f$ in $w(l_\infty(X), l_\infty(X)')$ and, in particular, $f_n \to f$ pointwise on X.*

Proof. This is immediate from Theorem 8.

10. COROLLARY. *We suppose that X is a pseudocompact topological space, for all $n \geqq 1$ $f_n \in C(X)$, $\sup_{n\geqq 1} ||f_n|| < \infty$, $f \in C(X)$ and $f_n \to f$ pointwise on X. Then $f_n \to f$ in $w(C(X), C(X)')$.*

Proof. This follows from Corollary 9 with $M = C(X)$, $Y = X$ and the fact (from the Hahn-Banach theorem) that $w(l_\infty(X), l_\infty(X)')$ induces $w(C(X), C(X)')$ on $C(X)$. If we wish to avoid the axiom of choice we can reprove Theorem 3 and Theorem 8 with "$l_\infty(X)$" replaced everywhere by "$C(X)$" and still obtain the result.

11. COROLLARY. *We suppose that F is a normed linear space with dual F' and completion $\tilde{F}$, X is the unit ball of F', $Y \subset X$ and*

(7) *for all $x \in \tilde{F}$ there exists $y \in Y$ such that $\langle x, y\rangle = ||x||$.*

If, for all $n \geqq 1$, $x_n \in F$, $\sup_{n\geqq 1} ||x_n|| < \infty$, $x \in F$ and $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in Y$ then $x_n \to x$ in $w(F, F'')$. In particular, this result is true if $Y = exX$.

Proof. The result is immediate from Corollary 9 with $M = \{\langle x, \cdot\rangle | X : x \in \tilde{F}\}$. (We observe that if $x \in \tilde{F}$ then $\langle x, \cdot\rangle | X$ is continuous with respect to the topology induced on X by $w(F', F)$ although $\langle x, \cdot\rangle$

is not necessarily continuous with respect to $w(F', F)$. So the final comment follows from Bauer's theorem and not the Krein-Milman theorem.)

12. REMARK. We can use Example 5 to show that Corollary 11 fails if we weaken (7) to

$$\text{for all } x \in F \text{ there exists } y \in Y \text{ such that } \langle x, y\rangle = \|x\| .$$

We can use Example 7 to show that, even though (7) is satisfied, it may happen that $Y \cap exX = \phi$.

(In the first case we take F to be the E' of Example 5 with the supremum norm. Then $F' = E$ and $\tilde{F} = c$. In the second case we take F to be $l_1(\mathscr{A})$ with the l_1 norm. Then $F' = l_\infty(\mathscr{A})$.)

13. COROLLARY. *We suppose that* $\phi \neq Y \subset E$ *and* $\{f_n\}_{n \geqq 1}$ *is a sequence of real functions on* E, *uniformly bounded on* Y. *We write*

$$X = \{x\colon x \in E, \inf(\sum_{n \geqq 1} \lambda_n f_n)(Y) \leqq \sum_{n \geqq 1} \lambda_n f_n(x) \leqq \sup(\sum_{n \geqq 1} \lambda_n f_n)(Y) \text{ whenever } \lambda_n \geqq 0 \quad (n \geqq 1) \text{ and } \sum_{n \geqq 1} \lambda_n = 1\} .$$

If all the functions $\sum_{n \geqq 1} \lambda_n f_n$ *attain their infima and suprema on* Y *and* $\lim_{n \to \infty} f_n(y) = 0$ *whenever* $y \in Y$, *then* $\lim_{n \to \infty} f_n(x) = 0$ *whenever* $x \in X$. *If* E *is a linear topological space and each* f_n *is continuous and affine on* E *then* $X \supset \text{conv}^- Y$ *and it suffices that* Y *be pseudo-compact.*

Proof. This follows immediately from Theorem 8.

REFERENCES

1. R. C. James, *Weakly compact sets*, Trans. Amer. Math. Soc., **13** (1964), 129-140.
2. J. L. Kelley and I. Namioka, *Linear Topological Spaces*, Van Nostrand, 1963.
3. P. A. Meyer, *Theory of Probability*, Blasisdell, New York, 1965.
4. R. R. Phelps, *Lectures on Choquet's Theorem*, Van Nostrand Mathematical Studies #7, 1966.
5. J. D. Pryce, *Weak compactness in locally convex spaces*, Proc. Amer. Math. Soc., **17** (1966), 148-155.
6. J. Rainwater, *Weak convergence of bounded sequences*, Proc. Amer. Math. Soc., **14** (1963), 999.
7. S. Simons, *Minimal sublinear functionals*, Studia Math., **37** (1970), 37-56.

Received January 28, 1971. This research was supported in part by NSF grant number 20632.

UNIVERSITY OF CALIFORNIA, SANTA BARBARA