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INJECTIVE MODULES OVER DUO RINGS

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Let R be a ring with unit whose right and left ideals are two-sided ideals. It is shown that every Noetherian injective R-module has finite length (i.e., has a finite composition series). If I is a maximal ideal of R, then R has a universal localization, R_I at I. The condition that the injective hull of R/I is finite is characterized in terms of R_I.

1. Introduction. In this note all rings have unit and modules are unital right modules. A. Rosenberg and D. Zelinsky have shown that if R is a commutative ring and I is a maximal ideal of R, then the injective hull of R/I is finite (i.e., has finite length) if and only if the localization of R at I is Artinian (see Theorem 5 of [6, p. 379]). In this note we shall prove an extended version (Theorem 4) of their result for a class of rings which is somewhat interesting in itself. Let us call a ring R a *duo ring* if $xR = Rx$ for all $x \in R$ (equivalently all ideals are bilateral). Such rings were investigated by E. Feller [2] and G. Thierrin [7]. Trivial examples of duo rings are, of course, commutative rings and division rings. Nontrivial duo rings are not difficult to come by (e.g., any noncommutative special primary ring is duo, since the only right or left ideals are powers of the unique maximal ideal). In fact some interesting examples of duo rings have already occurred in the literature: M. Auslander and O. Goldman have shown in [1, p. 13] that there exist noncommutative maximal orders which are both duo rings and Noetherian domains. Further investigations of such rings have been carried out by G. Maury in [4].

One of the basic difficulties in extending Rosenberg and Zelinsky's result to duo rings is the existence of suitable localizations. This problem is considered in §2. Next we show in §3 that Noetherian duo rings are classical in the sense that the familiar primary decomposition theory of commutative Noetherian rings extends to duo rings. We use this fact to show that Noetherian injective modules over duo rings are finite. Finally we prove our main result in §4.

The injective hull of the module M will be denoted by $E(M)$. If A and B are subsets of M or R, then $A \mathbin{.\!\cdot} B = \{x \in R \mid xB \subseteqq A\}$ and $A \mathbin{\cdot\!.} B = \{x \in R \mid Ax \subseteqq B\}$. Also $A \backslash B$ is the set of elements in A but not B.

2. Localizations. First of all we need a suitable definition of the term "localization." The ideal P of R is prime if $AB \subseteqq P$ implies $A \subseteqq P$ or $B \subseteqq P$ for all ideals A and B of R.

DEFINITION 1. Let R be a ring and P a prime ideal of R. By a *(right) localization of R at P* we mean a nonzero ring Q together with a ring homomorphism $\phi: R \rightarrow Q$ such that images under ϕ of elements of $R \backslash P$ (elements in R but not P) are units in Q and $Q = \{\phi(a)\phi(b)^{-1} \mid a \in R \text{ and } b \in R \backslash P\}$.

REMARK 1. A localization Q of R at P is a local ring whose maximal ideal is $\phi(P)Q$. For if $\phi(a)\phi(b)^{-1}$ is a unit of Q, then so is $\phi(a)$, say $\phi(a)\phi(c)\phi(d)^{-1} = 1$. But then $ac - d \in \ker \phi \subseteqq P$, so that $a \notin P$. Hence the units of Q are precisely the elements of the set $\phi(R \backslash P)\phi(R \backslash P)^{-1}$ and the non-units form a maximal right ideal $\phi(P)Q$. Therefore Q has a unique maximal right ideal and is by definition a local ring.

DEFINITION 2. The localization Q of R at P is *universal* if every localization $\phi^*: R \rightarrow Q^*$ of R at P can be factored through $\phi: R \rightarrow Q$, i.e., there is a ring homomorphism $\lambda: Q \rightarrow Q^*$ such that $\lambda \cdot \phi = \phi^*$.

Clearly universal localizations are unique up to a ring isomorphism, if they exist. If R has a universal localization at P, we shall denote this ring by R_P. In the case of a commutative ring R, R_P exists and is just the usual ring of quotients of R by elements of $R \backslash P$. With some slight modifications, we can make the same trick work for duo rings:

THEOREM 1. *If P is a prime ideal of the duo ring R, then R has a universal localization at P.*

Proof. First note that if $x, y \in R$ and $xy \in P$, then $xRyR = xyR \subseteqq P$, so $x \in P$ or $y \in P$. Hence $R \backslash P$ is multiplicatively closed. Let K be the set of $r \in R$ for which there are elements $a, b \in R \backslash P$ such that $arb = 0$. Then K is an ideal of R. For if $arb = 0$ and $a'r'b' = 0$, then $aa'(r - r')bb' = 0$. To see this apply the fact that $xR = Rx$ for all $x \in R$ to obtain that $aa'rbb' = a^*arbb' = 0$ for some $a^* \in R$. Similarly $aa'r'bb' = 0$ and $as \in K$ if $s \in R$. If $r \in R$ and $a, b \in R \backslash P$ are such that $arb \in K$, then $a'arbb' = 0$ for suitable $a', b' \in R \backslash I$. Hence $r \in K$. Hence $r \in K$. In other words images of elements of $R \backslash I$ in the ring R/K are regular elements (i.e., nonzero divisors). Let $S = (R/K) \backslash (P/K)$. Since R/K is a duo ring, it follows that for any $u \in R/K$ and $v \in S$ there is an element $w \in R/K$ such that $uv = vw$. Thus elements of S satisfy a right Ore condition, so we may form the ring of right quotients Q of R/K by elements of S in the usual way (i.e., Q is the set of equivalence classes of ordered pairs (a, b) such that $b \notin P$ and (a, b) is equivalent to (c, d) if $da = eb$, where $dc = ed$). If we identify R/K with its image in Q, then it is clear that Q, along with the natural map $\phi: R \rightarrow R/K$, is a right localization of R at P.

To see that Q is universal, simply note that if $\phi^*: R \to Q^*$ is any other localization of R at P, then Q^* is determined up to a ring isomorphism by Ker ϕ^*. For Q^* is just the ring of right quotients of $R/\ker\phi^*$ by elements of $S^* = (R/\ker \phi^*)\backslash(P/\ker \phi^*)$. However, it is clear that $K \subseteqq \ker \phi^*$ and this implies that the ring of right quotients of $R/\ker \phi^*$ by S^* is just $Q/((\ker \phi^*)Q)$, for

$$R/\ker \phi^* \cong (R/K)(\ker \phi^*/K) .$$

REMARK 2. In the case of commutative R, the ring R_P is just the usual localization and is itself a commutative ring. It is not clear that R_P need be a duo ring if R is a duo ring. We leave this question open. The ideal K of the proof of Theorem 1 will be called the *P-component of* 0. The set of $r \in R$ for which $rb = 0$ for some $b \in R\backslash P$ will be called the *right P-component of* 0. Of course if R is commutative, then the right P-component of 0 is just K.

3. Noetherian duo rings.

Let A be an ideal of the ring R. Then by the *radical of A* we mean the inverse image in R of the prime radical of the ring R/A. As usual we call the ideal A of R a *primary ideal* if the radical of A is a prime ideal and whenever C and D are ideals of R such that CD or DC is contained in A and $C \subsetneqq A$, then D is contained in the radical of A. Also A is called an *irreducible ideal* if A cannot be written as the intersection of ideals of R which properly contain A. We need to extend the well known fact that irreducible implies primary in Noetherian commutative rings.

LEMMA 1. *Let P be the prime radical of the Noetherian duo ring R and suppose that 0 is an irreducible ideal of R. Then elements of $R\backslash P$ are regular.*

Proof. The ideal P is nilpotent by Levitzki's Theorem, since R is Noetherian. Furthermore if $r^n = 0$ for some $r \in R$, then $(rR)^n = r^nR = 0$ since R is a duo ring. Therefore P consists precisely of the nilpotent elements of R. Suppose that $r \in R\backslash P$ and $0 \neq a \in R$ are such that $ar = 0$. Then $r^n \notin P$ for all integers n. Apply the fact that R is Noetherian to the ascending chain $0 .\!\cdot\, r \subseteqq 0 .\!\cdot\, r^2 \subseteqq \cdots$ to obtain that $0 .\!\cdot r^n = 0 .\!\cdot\, r^{n+1}$ for some integer n. Let $d \in Rr^n \cap Ra$, say $d = sr^n = s'a$. Then $sr^{n+1} = s'ar = 0$, so $s \in 0 .\!\cdot\, r^{n+1} = 0 .\!\cdot\, r^n$ and $sr^n = 0 = d$. Hence $0 = Rr^n \cap Ra$, which contradicts the fact that 0 is irreducible. Hence r is right regular. Repeat the preceding argument on the chain $r \cdot\!.\, 0 \subseteqq r^2 \cdot\!.\, 0 \subseteqq \cdots$ and obtain that r is left regular (and therefore regular).

COROLLARY. *Every irreducible ideal of a Noetherian duo ring R is a primary ideal.*

Proof. Let A be an irreducible ideal of R. Without loss of generality $A = 0$. If P is the prime radical of R, then P is a prime ideal; for if $x, y \in R \backslash P$ and $xy \in P$, then there is a least integer n such that $(xy)^n \neq 0$. Then $(xy)^{n+1} = 0$, which implies that x or y is not regular. This contradicts Lemma 1, so P is a prime ideal. Likewise apply Lemma 1 and obtain that if xy or yx equals 0 and $x \neq 0$, then $y \in P$. Hence 0 is a primary ideal.

A standard argument (see Exercises 9–13 of [3, p. 105]) shows that every ideal in a Noetherian ring can be expressed as a finite intersection of irreducible ideals. Let R be a duo ring and collect all the irreducibles in such an expression whose radical is the same prime ideal. Then we obtain the following result:

THEOREM 2. *Let A be an ideal in the Noetherian ideal ring R. Then A is the intersection of a finite number of irreducible ideals, no two of which have the same radical.*

The proof of the next result provides us with some information about injective modules over Noetherian duo rings which we shall use in Theorem 4. This proof uses Lemma 1 and some results and techniques of E. Matlis [5] for injective modules.

THEOREM 3. *Every Noetherian injective module over a duo ring has finite length.*

Proof. Let R be a duo ring and M an injective Noetherian R-module. Let K be the (right) annihilator of M in R. Then R/K is a Noetherian ring, since R/K is a subdirect sum of R-modules R/K_i, $i = 1, \cdots, m$, where K_i is the annihilator of x_i in R and $M = x_1R + \cdots + x_mR$. Also R/K is a duo ring. Furthermore M is certainly injective as an (R/K)-module, so we may as well replace R by R/K and assume that $K = 0$ and R is Noetherian. Thus M is a finite direct sum of indecomposible injective submodules by Theorem 2.5 of [5, p. 516]. We may replace M by one of these summands and assume that M is itself indecomposible.

Now apply Theorem 2.4 of [5, p. 516] and we obtain that $M = E(R/J)$, where J is an irreducible ideal of R. Furthermore if $0 \neq x \in M$, then $M \cong E(R/(x \cdot. 0))$ by the same theorem. Let P be the radical of J. Then there is a least integer n such that $P^{n+1} \subseteq J$, since R is Noetherian. Select an element $y \in P^n \backslash P^{n+1}$ and obtain from the above remarks that $M \cong E(R/(y \cdot. J))$. Clearly $P \subseteq y \cdot. J$. Also if $a \in y \cdot. J$, then $ya \in J$. Since images of elements of $R \backslash P$ are regular in R/J by Lemma 1, $a \in P$. Therefore $y \cdot. J = P$, which is a prime ideal by the Corollary to Lemma 1. Furthermore $M \cong E(R/P)$ by a preceding remark. Now any prime duo ring is an integral domain, for $xy = 0$ implies $xRyR = 0$. Furthermore a prime duo ring obviously satisfies an Ore condition, so such a ring is contained in a right quotient division ring. Let Q be the right quotient ring of R/P. The injection of R/P into M extends to a R-homomorphism of Q into M. Such a homomorphism is clearly injective, so Q is a submodule of M and therefore Noetherian. If $x \neq 0$, we obtain from $xR = Rx$ that $x^{-1}R = Rx^{-1}$. Hence any element $a'b^{-1}$ of Q may be written as $b^{-1}a$ for suitable a. In particular there are elements $a_i, b_i \in R$, $i = 1, \cdots, n$, such that

$$Q = b_1^{-1}a_1(R/P) + \cdots + b_n^{-1}a_1(R/P) .$$

But then $Q = b_1 b_2 \cdots b_n Q$ and repeated application of the identity $x^{-1}R = Rx^{-1}$ yields that $b_1 \cdots b_n b_i^{-1} a_i = b_1 \cdots b_{i-1} ca$ for some $c \in R$. Consequently $Q = R/P$ and P is actually a maximal ideal of R.

We complete the proof by showing that $P^n = 0$ for some integer n. For it then follows that R is a local Noetherian ring with nilpotent maximal ideal P. Since P^i/P^{i+1} must be a finite dimensional (R/P)-vector space for all integers i, one obtains that R is (right) Artinian. Consequently M must be Artinian, which was to be shown. Now if P_1 is any other prime ideal of R and $E(R/P_1) \cong E(R/P)$, then we can think of R/P_1 and R/P as embedded in $E(R/P)$ and obtain that $R/P_1 \cap R/P \neq 0$. But nonzero elements of R/P_1 have right annihilator P_1 in R and likewise for nonzero elements of R/P. Hence $P = P_1$. It follows from the remarks in the previous paragraph that if $0 \neq x \in M$, then $P^m \subseteqq x \cdot 0$ for some integer m. Form the ascending chain of submodules of M given by $A_i = \{x \in M \mid xP^i = 0\}$, $i = 1, 2, \cdots$. Then there is an integer n such that $A_n = A_{n+1}$, since M is Noetherian. But we have shown that M is the union of the A_i, $i = 1, 2, \cdots$. Hence $M = A_n$ and $MP^n = 0$. But M is a faithful R-module, so $P^n = 0$ and the proof is complete.

3. The main theorem.

Note that if I is a maximal right ideal of the duo ring R, then I is an ideal of R and R/I is a division ring, so R_I exists. We now generalize Rosenberg and Zelinskys' result. First we need the following lemma:

LEMMA 2. *Let R be a duo ring with a maximal ideal I and let M be a nonzero R-module with the property that right multiplication of N by an element of $R \backslash I$ is a one-to-one map of N onto N for all submodules N of M. Then M can be made into an R_I-module in such a way that R- and R_I-submodule lattices of M coincide.*

Proof. Let $K = M \cdot 0$. Then M is a faithful (R/K)-module. Also

$(R/K)_I = R_I/KR_I$, since $K \subseteq I$. So it is sufficient to prove the lemma in the case that $K = 0$ and we may assume that M is a faithful R-module. Right multiplication by an element of $R\backslash I$ is a one-to-one map of N onto N, where N is any submodule of M. In particular if $ar = 0$ with $a \in R$ and $r \in R\ I$, then $(Ma)r = 0$. But $MaR = MRa = Ma$, since R is a duo ring. Hence Ma is a submodule of M and therefore $Ma = 0$. Since M is a faithful R-module, $a = 0$. Similarly if $ra = 0$, then $Mra = Ma = 0$ and $a = 0$. Hence elements of $R\backslash I$ are regular elements of R. Therefore R is actually a subring of R_I. So if $r \in R\backslash I$, we obtain from $rR = Rr$ that $r^{-1}R = Rr^{-1}$.

Since right multiplication by an element of $R\backslash I$ is a one-to-one map of M onto M, we can define, for all $m \in M$ and $r \in R\ I$, $mr^{-1} = m'$, where $m'r = m$. For any $a \in R$ define $m(ab^{-1}) = (ma)b^{-1}$. If $ab^{-1} = cd^{-1}$, then we can select an element $e \in R$ such that $d^{-1}b = ed^{-1}$, since $d^{-1}R = Rd^{-1}$. Thus we obtain equations

$$(m(ab^{-1}))(bd) = mad, \text{ and } m(cd^{-1})(de) = mce .$$

But $ad = ce$, so we obtain from these equations that $m(ab^{-1}) = m(cd^{-1})(ded^{-1}b^{-1})$. Since $ded^{-1}b^{-1}=1$, we conclude that $m(ab^{-1}) = m(cd^{-1})$ and multiplication by elements of R_I is well-defined. Similarly it follows readily that the above definition makes M into an R_I-module. If N is any R-submodule of M, then $Nr = N$ for each $r \in R\backslash I$. Hence $Nr^{-1} = N$ and in general $NR_I = N$. So N is an R_I-submodule of M. Obviously R_I-submodules are R-submodules, and the lemma follows.

THEOREM 4. *Let I be a maximal ideal of the duo ring R. The following are equivalent*:

(1) *R_I is Artinian and the I-component of* 0 *is the right I-component of* 0.

(2) *$E(R/I)$ has finite length.*

(3) *$E(R/I)$ is Noetherian.*

Proof. Suppose that (1) is true and let J be the right component of 0. Then $J \subseteq I$ and I/J is contained in the maximal ideal of R_I. But R_I is Artinian, so $(I/J)^n = 0$ for some integer n. Hence $I^n \subseteq J$. Let $t \in I^n$ and select $r \in R\backslash I$ such that $tr = 0$. Then let $E = E(R/I)$ and obtain that $(Et)r = 0$. Right multiplication of elements of E by r is a one-to-one map of E into E. For if $0 \neq m \in E$ and $mr = 0$, then select $s \in R$ such that $0 \neq ms \in R/I$ (possible since E is an essential extension of R/I) and obtain that $0 = mr = mrR = mRr$ and $msr = 0$. This is a contradiction, since R/I is a division ring. It follows from $(Et)r = 0$ that $Et = 0$. Consequently $t \in E \cdot. 0$. Therefore $I^n \subseteq E \cdot. 0$. Hence if N is a submodule of E, then $NI^n = 0$ for some integer n. Induct on n to show that $Nr = N$ for all $r \in R\backslash I$. If $n =$

1, then N is an (R/I)-vector space and it is clear that $Nr = N$. If the assertion is true for integers less than n and $NI^n = 0$, then NI is a submodule of M such that $(NI)I^{n-1} = 0$, so $NIr = NI$ by induction. Select $s \in R$ such that $1 - sr \in I$ (possible since R/I is a division ring). Then if $n \in N$, we have $n - nsr \in NI = NIr$, say $n - nsr = n'r$, $n' \in N$. We obtain that $n = (n' + ns)r \in Nr$ and hence $N = Nr$. It follows that $N = Nr$ for all submodules N of E and $r \in R \backslash I$. Apply Lemma 2 to E and obtain that E is an R_I-module with the same submodule lattice as the R-module E. Let $L = IR_I$ be the maximal ideal of R_I and we obtain that $EL^n = 0$. Form the chain $R_I \supseteqq L \subseteqq \cdots \supseteqq L^n = 0$ and set $E_i = \{x \in E \mid xL^i = 0\}$ for $i = 1, \cdots, n$. Then $E = E_n$ and E_{i+1}/E_i is isomorphic to a submodule of $\mathrm{Hom}_{R_I}(L^i/L^{i+1}, E)$ by Lemma 1 of [6, p. 373]. Now any image of L^i/L^{i+1} is annihilated by L. Furthermore if $m \in E \backslash (R/I)$, then $mI \neq 0$ (else mR is irreducible and meets (R/I) trivially). Also $R/I \cong R_I/L$. Hence

$$\mathrm{Hom}_{R_I}(L^i/L^{i+1}, E) = \mathrm{Hom}_{(R_I/L)}(L^i/L^{i+1}, R_I/L) ,$$

which is just the dual space of L^i/L^{i+1}. Since R_I is Artinian, L^i/L^{i+1} is finite dimensional and so E_{i+1}/E_i is Artinian. It follows that E is an finite R_I-module and therefore a finite R-module. So (1) implies (2).

The equivalence of conditions (2) and (3) follows from Theorem 3.

Finally suppose that (2) is true. As above, let J be the right I-component of 0, $E = E(R/I)$ and $K = E \cdot\!\!. \, 0 \subseteqq I$. Since E is finite, it is certainly Noetherian. We showed in the last part of the proof of Theorem 3 that if $E(R/I)$ is Noetherian, then $I^n \subseteqq K$ for some positive integer n. If x is any element of R such that $x \cdot\!\!. \, 0 \subseteqq I$, then $x \notin K$ by Lemma 6 of [6, p. 377] (simply lift the composition of maps $xR \rightarrow R/(x \cdot\!\!. \, 0) \rightarrow R/I$ to a map $R \rightarrow E$). Consequently if $x \in K$, then there is an element $r \in R \backslash I$ such that $xr = 0$. Hence x belongs to J, the right I-component of 0. Certainly J is contained in the I-component of 0, which we denote by T. Thus $I^n \subseteqq K \subseteqq J \subseteqq T$. Then R/T is a local ring with nilpotent ideal I/T. Consequently elements of $(R/T) \backslash (I/T)$ are already units in R/T and $R_I = R/T$. Also $R/T \cong (R/K)/(T/K)$, which is Artinian since R/K is Artinian. Hence R_I is Artinian. We now obtain, exactly as in the proof that (1) implies (2), that $Nr = N$ for every submodule N of E and $r \in R \backslash I$. Consequently if $x \in T$, say $axb = 0$ for $a, b \in R \backslash I$, then $(Eax)b = 0$. Therefore $0 = (Ea)x = Ex$ and $x \in K = E \cdot\!\!. \, 0$. It follows that $K = J = T$, which completes the proof that (2) implies (1).

REMARK 3. We have not been able to decide whether or not condition (1) of Theorem 4 may be replaced by the apparently weaker condition that R_I is Artinian. In any case the condition that right I-component equal I-component is vacuous if R is commutative.

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