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ESTABLISHING ISOMORPHISM BETWEEN TAME PRIME KNOTS IN E^3

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The "formula" of a polygonal knot in E^3 is defined by appropriate labeling of the crossings in the regular projection of the knot. Admissible transformations of such formulas are defined (for example, cancellation of the consecutive symbols x and x^{-1}), and prime formulas are defined. It is shown that if two knots have formulas which are equivalent by applications of admissible transformations, and one of the formulas is prime, then the knots are equivalently embedded in E^3.

Since each tame knot type includes a finite polygon, we restrict our attention to polygonal knots in E^3. Such a knot is the image of a one-to-one continuous mapping g of $[0, 1)$ into E^3 such that (1) $g(t)$ approaches $g(0)$ as t approaches 1, and (2) the image of g is the union of a finite number of straight line intervals. We may of course restrict our attention to such knots $K = \mathrm{Im}(g)$ as lie in general position in E^3; that is, π (defined by $\pi(x, y, z) = (x, y, 0)$) is one-to-one on K except at a finite number of points, called the double points of K, where π is precisely two-to-one, and no vertex of K is a double point.

Let $x_1, x_2, \cdots, x_n$ be the points of $[0, 1)$ mapped two-to-one by $f = \pi g$, arranged in their natural order. The formula of the knot K is then

$$f(x_1)^{e(1)} f(x_2)^{e(2)} \cdots f(x_n)^{e(n)}$$

where $e(i)$ is 1 or -1 according to the following rule: If $f(x_i) = f(x_j)$ and the z-coordinate of $g(x_i)$ exceeds that of $g(x_j)$, then $e(i) = 1$ and $e(j) = -1$. In practice we suppress the positive superscripts. For example, the formula of the trefoil knot drawn in the ordinary way can be written $ab^{-1}ca^{-1}bc^{-1}$, where a, b, and c are the three crossings in the plane projection of the trefoil. If there are no double points, the knot has empty formula denoted by 1.

Let a knot formula F be given. By an admissible operation on F is meant the application to F of one of the following ten transformations.

(1) Reversal of the order of symbols of F.

(2) Coding; that is, consistent substitution of different symbols for the symbols of F, while preserving superscripts.

(3) Negation of all superscripts in F.

(4) Cyclic permutation of the symbols of F, as for example re-

placing

$$x \cdots yz \cdots w$$

by

$$z \cdots wx \cdots y .$$

(5) Replacing

$$\cdots ab \cdots ca^{-1} \cdots b^{-1}c^{-1} \cdots$$

by

$$\cdots ba \cdots a^{-1}c \cdots c^{-1}b^{-1} \cdots ,$$

with any order of occurrence of these symbols in pairs or as pairs, provided only that the following three conditions are met:

(a) No other changes are made in F;

(b) Two pairs of adjacent symbols have like superscripts; and

(c) Between each pair of pairs of symbols, there occurs at least one symbol whose inverse occurs between one of the other two pairs of pairs of symbols. We consider this condition to be satisfied also for two pairs of symbols if no other symbols occur between them, but this is allowable for only one of the three pairs of pairs.

(6) If F has the form

$$a_1a_2 \cdots a_{n-1}a_nb_1b_2 \cdots b_m ,$$

with superscripts unimportant, and is such that for each i, $a_i^{-1} = a_j$ and $b_i^{-1} = b_j$ for some j, then F may be replaced by

$$a_1a_2 \cdots a_{n-1}b_1b_2 \cdots b_ma_n .$$

(7) If a symbol is adjacent to its inverse, both may be deleted. Moreover, if x does not occur in F, then either xx^{-1} or $x^{-1}x$ may be inserted anywhere in F.

(8) In the case that between the two occurrences of a symbol all symbols have the same superscript, then all of these symbols and their inverses may be deleted from F.

(9) If F has the form $zPz^{-1}Q$, where P and Q are sequences of symbols such that x is a symbol of P if and only if x^{-1} is a symbol of P, and Q' denotes the symbols of Q in the same order but with superscripts negated, then F may be transformed into PQ'.

(10) If two symbols x and y of F are adjacent with the same superscript, their inverses x^{-1} and y^{-1} elsewhere in F are also adjacent, and transformation (9) does not apply with either x or y in place of z, then x, y, x^{-1}, and y^{-1} may be deleted from F.

Principal results. The first theorem guarantees that if any sequence of admissible operations is applied to the formula F of a knot K, then the resulting formula is the knot formula of some knot isomorphic to K (the knot L is said to be isomorphic to K provided that there exists a homeomorphism of E^3 onto itself carrying L onto K).

THEOREM 1. *Let K be a polygonal knot in regular position in E^3 with formula F, and let G be a formula obtained from F by a single application of an admissible operation. Then there exists a polygonal knot L in regular position in E^3 such that G is the formula of L and L is isomorphic to K.*

The proof of Theorem 1 presents no intuitive difficulties and few technical ones. The details of the cases for the first seven admissible operations are available in the author's doctoral dissertation [3]; alternatively, most of the techniques are similar to those of Graeub [1]. Hence we omit the proof here. It is worth noting that the effect of the first four admissible operations is to allow one, when given a presentation of a knot, to select an arbitrary initial point and direction, and to label the crossings with any distinct symbols whatsoever. In addition, in only the third admissible operation is the constructed homeomorphism between K and L not orientation-preserving.

LEMMA 1. *Let the polygonal knot K in general position in E^3 be the image of the mapping g of* $[0, 1)$ *into E^3, let F denote the formula of K, and let C_1 $C_2, \cdots, C_n$ be the complementary domains in E^2* (as $\pi(E^3)$) *of $\pi(K)$. Suppose that C_1 is the unbounded complementary domain of $\pi(K)$ and that* $\mathrm{Cl}(C_1) \cap \mathrm{Cl}(C_2)$ *contains an arc. Then there exists a polygonal knot L in E^3, the image of the mapping h on* $[0, 1)$, *in general position, such that*

(a) *The formula of L is also F;*

(b) *πg and πh have the same set of double points $a_1, a_2, \cdots, a_k$ in* $[0, 1)$;

(c) *If $b_1, b_2, \cdots, b_j$ are the components of $(\pi g)^{-1}(\mathrm{Bdry}\, C_2)$, then there is a complementary* domain D_2 of *$\pi(L)$ such that if $B = b_1 \cup b_2 \cup \cdots \cup b_j$, then $\pi h(B) =$* Bdry (D_2);

(d) *D_2 is the unbounded complementary* domain *of $\pi(L)$; and*

(e) *L is isomorphic to K.*

This lemma just says that if one of the complementary domains of $\pi(K)$ is adjacent to the unbounded one, then the part of K that projects onto their common boundary arc can be lifted and moved to the "other side" of K, without disturbing the rest of K or its formula

F, so that the first-mentioned complementary domain "becomes" the unbounded one. The same comments on this proof apply as they did in the comments on the proof of Theorem 1. And by successive applications of this lemma we can "make" any of the complementary domains of $\pi(K)$ "become" the unbounded complementary domain.

Now suppose that K and L are polygonal knots in general position in E^3, the images of the mappings g and h on $[0, 1)$ respectively, and suppose that $\pi(K) = \pi(L)$. We define next what it means for the crossings of K to correspond to the crossings of L in the natural sense.

Let $R = \pi(K) = \pi(L)$, and let q be any double point of R; that is, $q = \pi g(a_i)$ where a_i is a double point of g. We suppose that the mapping h is reparametrized if necessary so that the double points of h in $[0, 1)$ are the same, in the same order, as the double points of g. Let α and β be two closed subarcs of R that contain no double points of R other than q and such that α crosses β at q (in the sense of the definition on page 182 of [2]). Let x_1 and x_2 be the endpoints of α, and x_3 and x_4 the endpoints of β. Let $y_i = K \cap \pi^{-1}(x_i)$ for $1 \leqq i \leqq 4$ and $w_i = L \cap \pi^{-1}(x_i)$ for $1 \leqq i \leqq 4$.

Let α_K be the subarc of K with endpoints y_1 and y_2 such that $\pi(\alpha_K) = \alpha$. Let β_K be the subarc of K with endpoints y_3 and y_4 such that $\pi(\beta_K) = \beta$. We similarly define α_L and β_L. Let z_1 denote the z-coordinate of $\alpha_K \cap \pi^{-1}(q)$, let z_2 denote the z-coordinate of $\beta_K \cap \pi^{-1}(q)$, let z_3 denote the z-coordinate of $\alpha_L \cap \pi^{-1}(q)$, and let z_4 denote the z-coordinate of $\beta_L \cap \pi^{-1}(q)$.

To say that the crossings of K and L correspond in the natural sense means that if q is any crossing of R, and the z_i are defined as above, then $z_1 < z_2$ if and only if $z_3 < z_4$. Of course, all this means is that when one subarc of K is above another, then the corresponding subarc of L is above the other corresponding one, the correspondence determined by use of the common projection R of K and L.

LEMMA 2. *Suppose that K and L are polygonal knots in general position in E^3 such that $\pi(K) = \pi(L)$, and the crossings of K correspond to the crossings of L in the natural sense. Then K is isomorphic to L.*

Proof. Using the natural correspondence, we map appropriate double points of K to the corresponding double points of L. This function moves a finite number of points vertically. Using a triangulation of E^3 in which both K and L are subcomplexes, this function may be extended to a homeomorphism of all of E^3 onto itself taking K onto L.

Closely related, but not equivalent, to a knot's being prime is the property of having a prime formula, which we next define.

The knot formula $F = x_1x_2x_3 \cdots x_n$ is said to be prime if there is no pair of integers j and k such that: (a) $1 \leqq j < k \leqq n$; (b) $k - j \leqq n - 2$; and (c) for each p with $j \leqq p \leqq k$, there exists q such that $k \leqq q \leqq k$ and $(x_p)^{-1} = x_q$.

THEOREM 2. *Suppose that K and L are tame polygonal knots in general position in E^3 such that K and L have the same formula F. If F is prime then K is isomorphic to L.*

Proof. Let K be the image of the mapping g on $[0, 1)$ and L, similarly, the image of h. Let $R = \pi(K)$ and $S = \pi(L)$. Then πg and πh are prime mappings in the sense of Treybig [4] because F is prime. Let $\{a_1, a_2, \cdots, a_n\}$ be the set of double points of πg in $[0, 1)$. Then F has length n, and so since F is also the formula of L, then πh also has n double points $b_1, b_2, \cdots, b_n$ in $[0, 1)$. We reparametrize h so that $b_i = a_i$ for $1 \leqq i \leqq n$.

Since K and L have the same formula, the double points then double up in the same order; that is, if $a_i \neq a_j$ but $\pi g(a_i) = \pi g(a_j)$, then also $\pi h(a_i) = \pi h(a_j)$, and conversely. Moreover, as F is the same for K and L, it follows that K and L have the same overcrossing structure in the sense that if $a_i \neq a_j$ but $\pi g(a_i) = \pi g(a_j)$, then the z-coordinate of $g(a_i)$ exceeds that of $g(a_j)$ if and only if the z-coordinate of $h(a_i)$ exceeds that of $h(a_j)$.

Let D be a complementary domain of $R = \pi(K)$, and let $c_1, c_2, \cdots, c_j$ be the components of $(\pi g)^{-1}(\text{Bdry } D)$. By Theorem 1 of [4], there is a unique complementary domain E of $S = \pi(L)$ such that the components of $(\pi h)^{-1}(\text{Bdry } E)$ are exactly $c_1, c_2, \cdots, c_j$. Moreover, by Lemma 1 of this paper, we may assume that D is unbounded if and only if E is unbounded.

By Theorem 3 of [4], there is homeomorphism f_1 from E^2 (as $\pi(E^3)$) onto itself such that $h = f_1 g$ on $[0, 1)$. We extend f_1 to E^3 by defining $f_2(x, y, z) = (f_1(x, y), z)$. Then $f_2(K)$ and L have the same regular projection S, and as f_2 is constant in the third coordinate, the crossings of $f_2(K)$ and the crossings of L correspond in the natural sense. By Lemma 2 of this paper there is a homeomorphism f_3 of E^3 onto itself such that $f_3(f_2(K)) = L$.

Define f from E^3 to itself by $f = f_3 f_2$. Then f is a homeomorphism of E^3 onto itself such that $f(K) = L$. Hence K is isomorphic to L.

Our last result is also the principal result of this paper.

THEOREM 3. *If K and L are tame polygonal knots with formulas F and G respectively, G is prime, and G can be obtained by the application of a finite number of admissible operations to F, then K is isomorphic to L.*

Proof. It of course suffices to demonstrate the conclusion of the theorem in the case that only one admissible operation is applied to F. Suppose then that this is the case. By Theorem 1 there exists a knot L', polygonal, and in general position in E^3, such that L' has formula G and L' is isomorphic to K. But G is prime. Hence, by Theorem 2, L' is isomorphic to L. Therefore K is isomorphic to L.

Concluding remarks. The converse of Theorem 3 has been established by Treybig in [6], and in [7] he has partial results for the equally interesting question of the existence of a bound on the number of admissible operations required. Some of this work is based on his earlier research in [5], in which, among other things, he characterizes those "formular" which are knot formulas. A complete answer to the bound problem would permit an algorithmic approach for the construction of knot tables, no doubt with the use of electronic computers for reasons of practicality.

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