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LATTICES OF LOWER SEMI-CONTINUOUS FUNCTIONS AND ASSOCIATED TOPOLOGICAL SPACES

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In this paper the lattice of all real-valued lower semi-continuous functions on a topological space is studied. It is first shown that there is no essential loss if attention is restricted to T_0-spaces. By suitably topologizing a certain set of equivalence classes of prime ideals, it is shown that a topological space is determined by the lattice. This topological space is homeomorphic with the original space X whenever X has the property that every non-empty irreducible closed set is a point closure. The sublattices of functions taking values only in intervals of the form $(a, b]$ and $[a, b]$ are compared. Relations between the above function lattices and the lattice of all closed subsets are also discussed.

Preliminaries. Let $L_R(X)$ (or briefly L_R) denote the lattice of all lower semi-continuous functions defined on the topological space X into the real line R. It is well known that L_R is a conditionally complete distributive lattice under the usual order relation $f \leqq g$ which means $f(x) \leqq g(x)$ for all $x \in X$ (except where otherwise indicated, lattice—theoretic terminology will follow [1]). For an arbitrary bounded non-empty set $F \subset L_R$ the join $\bigvee F$ satisfies $\bigvee F(x) = \sup\{f(x): f \in F\}$; the meet $\bigwedge F$ is defined as $\bigvee\{g \in L_R: g \leqq f \text{ for all } f \in F\}$ and it should be noted that $\bigwedge F(x) = \inf\{f(x): f \in F\}$ need not hold when F is infinite. The constant function with value s will be written $\mathbf{s}$.

The elements of L_R can be regarded more conveniently as continuous functions on X into R_l where R_l is the T_0-space obtained by giving the real line the topology having as non-empty closed sets those of the form $\{x: x \leqq r\}$ $(r \in R)$.

Some other function lattices will also be considered towards the end of the paper. Let H, I denote the real intervals $(0, 1]$, $[0, 1]$ respectively and $L_H(X)$, $L_I(X)$ the sublattices of $L_R(X)$ consisting of those functions which take values only in H, I respectively (no essential difference will arise if any extended real intervals $(a, b]$, $[a, b]$ are taken for H, I).

We will use $\mathscr{C}(X)$ to denote the lattice of closed subsets of the topological space X. The set of nonzero *irreducible* elements of $\mathscr{C}$ will be denoted by $\mathscr{A}(X)$; thus $\mathscr{A}$ consists of the nonempty closed sets A which cannot be expressed as the union of two properly smaller closed sets. Closures will be written $\mathrm{cl}\, A$ with $\mathrm{cl}\, x = \mathrm{cl}\{x\}$ for point closures.

Relations between X and the lattice $\mathscr{C}(X)$ have been studied by several authors notably Thron [3] and Blanksma [2]. We now give a summary of relevant facts from these two papers in a form suitable for our needs. We restrict attention to T_0-spaces as this entails no essential loss of generality.

The set $\mathscr{A}$ can be topologized by taking as closed sets those of the form $\{A \in \mathscr{A}\colon A \subset F\}$ where $F \in \mathscr{C}$ (see [3, proof of 3.1] and [2, I, ch 2]. We will denote the topological space thus obtained by πX. Since every point closure $\mathrm{cl}\, x$ is irreducible, the mapping $\gamma(x) = \mathrm{cl}\, x$ is an embedding of X into the set $\mathscr{A}$ and moreover it is a topological embedding of X into πX (see [2, I, 3.4]). An important class of spaces are those in which every $A \in \mathscr{A}$ is a point closure (see [2, I, 2.2]). Such spaces will be called *pc-spaces*; T_D-spaces (see [3]) are defined to be those for which $\mathrm{cl}\, x - \{x\}$ is always a closed set. It is perhaps worth pointing out that these two types of spaces can be regarded as the extreme cases of a certain situation. If we use $\mathscr{A}^*$ to denote the set of strongly irreducible elements of $\mathscr{A}$ (i.e. those $A \in \mathscr{A}$ which cannot be written $A = \mathrm{cl} \bigcup_{s \in S} B_s$ for any family $(B_s) \subset \mathscr{C}$ with $B_s \subsetneqq A$), then we have for any T_0-space X

$$\mathscr{A}^* \subset \gamma(X) \subset \mathscr{A} .$$

(It is easily verified that each $A \in \mathscr{A}^*$ must be a point closure). The T_D-spaces can now be described as those for which $\mathscr{A}^* = \gamma(X)$ while the point closure spaces are those for which $\gamma(X) = \mathscr{A}$. The specific results concerning πX and $\mathscr{C}$ which will be needed in this paper can now be stated as follows. (When we say X *is determined as a space with property P by the lattice $C(X)$ (resp. $L_R(X)$)* we mean that if Y is also a space with property P then X and Y are homeomorphic iff $C(X)$ and $\mathscr{C}(Y)$ (resp. $L_R(X)$, $L_R(Y)$) are isomorphic.)

1. Known facts.

For any T_0-space X we have

(a) *πX is a pc-space.*

(b) *If X is a pc-space, then X and πX are homeomorphic.*

(c) *Every $f \in L_H(X)$ has a unique extension $f^\pi \in L_H(\pi X)$ (here we have identified X with a dense subspace of πX, as may be done).*

(d) *The lattices $\mathscr{C}(X)$ and $\mathscr{C}(\pi X)$ are isomorphic.*

(e) *The space πX is determined as a pc-space by the lattice $\mathscr{C}(X)$ (hence if X is a pc-space, it is determined as such by the lattice $\mathscr{C}(X)$).*

(f) *If X is a T_D-space, it is determined as such by the lattice $\mathscr{C}(X)$.*

For (a) through (e), see [2, I, chapters 2, 3]; (c) is not stated explicitly, but H. Herrlich has pointed out in his review of [2] (MR

37, 5851) that the *pc*-spaces form an epireflective subcategory of the T_0-spaces and (c) follows at once from the fact that πX is the epireflection of X. See [3] for a proof of (f).

THEOREM 2. *Let T be any topological space and X its T_0-identification. Then the lattices $L_R(T)$ and $L_R(X)$ are isomorphic.*

Proof. X is the quotient space T/ρ, where the relation $x\rho y$ means $\mathrm{cl}\, x = \mathrm{cl}\, y$. Let c denote the canonical mapping of T onto T/ρ. Notice that $\mathrm{cl}\, x \neq \mathrm{cl}\, y$ iff $f(x) \neq f(y)$ for some $f \in L_R(T)$. Hence for each $f \in L_R(T)$ there is a unique function f^* on T/ρ such that $f^* \circ c = f$. Since f^* is defined on a quotient space, its continuity follows from the continuity of f. The proof is completed with the simple verification that $f \rightarrow f^*$ is an isomorphism of $L_R(T)$ onto $L_R(T/\rho)$.

In view of this theorem all spaces X under discussion will from now on supposed to be T_0-spaces.

Closed prime ideals in $L_R(X)$. By an *ideal* in L_R will be meant a nonempty proper subset J of L_R such that $f \wedge g \in J$ whenever $f \in J$, $g \in L_R$ and $f \vee g \in J$ whenever $f, g \in J$ (here we differ from [1] where an ideal in a lattice need not be a proper subset). An ideal J will be called *closed* if for any $G \subset J$ such that $\bigvee G$ exists in L_R we have $\bigvee G \in J$. As usual, *prime* ideal will mean an ideal which contains $f \wedge g$ only if it contains f or g.

PROPOSITION 3. *The set $I(r, A) = \{f \in L_R(X)\colon f(x) \leqq r$ when $x \in A\}$ is a closed prime ideal, where $r \in R$ and $A \in \mathscr{A}$. Every closed prime ideal in L_R is of this form.*

Proof. If $f_1 \wedge f_2 \in I(r, A)$, then the closed sets $A_i = \{x \in A\colon f_i(x) \leqq r\}$ $(i = 1, 2)$ have A as their union. Since A is irreducible we conclude that $A = A_i$ and $f_i(x) \leqq r$ when $x \in A$ for some i. Hence $I(r, A)$, which is clearly a closed ideal, is prime. Let us now consider any closed prime ideal P in L_R and let B denote the set of all $x \in X$ for which the number

$$m(x) = \sup\{p(x)\colon p \in P\}$$

exists. We show that $m(x)$ is the same number for all $x \in B$. If $m(y) < m(z)$ holds we can chose $s, t \in R$ such that

$$m(y) < s < m(z) < t\,.$$

Given any $g \in P$ we define elements $u, v \in L_R$ as follows: $u = \boldsymbol{s} \vee g$ and

$$v(x) = \begin{cases} g(x) \text{ when } g(x) \leqq s \\ t \vee g(x) \text{ when } g(x) > s \,. \end{cases}$$

Then $g = u \wedge v$, $u \notin P$ and $v \notin P$ which is absurd. We conclude that B is of the form $\{x \in X: m(x) = r\}$ for some $r \in R$. The function $e = \bigvee\{p \wedge s: p \in P\}$ belongs to P (where $r < s$) and so $\{x: e(x) \leqq r\} = B$ is a closed set. Moreover, B is irreducible for if it is the union of closed proper subsets B_1, B_2 then

$$f_i(x) = \begin{cases} r \text{ if } x \in B_i \\ s \text{ if } x \notin B_i \end{cases} \qquad (i = 1, 2)$$

satisfy $f_1 \wedge f_2 = e \in P$, $f_1 \notin P$, $f_2 \notin P$. Finally, if $f \in I(r, B)$, then $f = \bigvee\{f \wedge p: p \in P\} \in P$ and we conclude $P = I(r, B)$.

The symbol $\mathscr{K}$ will be used from now on to denote the set of closed prime ideals in L_R. For a given $P \in \mathscr{K}$ the irreducible set A such that $P = I(r, A)$ will be called the *carrier* of P and for elements $P, Q \in \mathscr{K}$ we define $P \sim Q$ to hold iff P and Q have the same carrier. The relation thus defined is evidently an equivalence relation and it will be of importance to know that this relation can be characterized in terms of the lattice structure of $L_R(X)$ without reference to X. For this purpose we make the following definition. An ideal $I(r, A) \in \mathscr{K}$ is called *quasi-minimal* if $\{P \in K: P \subset I(r, A)\}$ forms a chain under the relation $\subset$.

LEMMA 4.

(a) *An ideal* $I(r, A) \in \mathscr{K}$ *is quasi-minimal iff* A *is maximal irreducible.*

(b) *For quasi-minimal ideals the relation* $I(r, A) \subset I(s, B)$ *holds iff* $r \leqq s$ *and* $A = B$.

Proof. The elements $P \in \mathscr{K}$ with $P \subset I(r, A)$ are those of the form $I(r', B)$ where $r' \leqq r$ and $B \supset A$. These elements form a chain iff $B = A$ holds. Thus (a) follows and (b) is an immediate consequence.

Notice that $P \sim Q$ can hold only if $P \subset Q$ or $Q \subset P$. So in order to obtain the desired characterization of the relation $\sim$ it is enough to consider comparable ideals.

LEMMA 5. *Let* $P, Q \in \mathscr{K}$ *satisfy* $P \subset Q$. *Then* $P \sim Q$ *holds iff there exists a pair of quasi-minimal ideals* $J_1, J_2 \in \mathscr{K}$ *such that*

$$J_1 \subset J_2,\ J_1 \subset P \cap J_2 \text{ and } P \vee J_2 = Q\,,$$

where $P \vee J_2 = \{f \vee g: f \in P, g \in J_2\}$.

Proof. If $P \sim Q$ with $P \subset Q$, then $P = I(r, A)$, $Q = I(s, A)$ for some $r < s$. A standard application of Zorn's lemma shows that there is a maximal set $M \in \mathscr{A}$ such that $M \supset A$. Put $J_1 = I(r, M)$, $J_2 = I(s, M)$. It is easy to verify that J_1, J_2 satisfy the requirements of the lemma. Conversely, let us start with $P = I(r, A) \subset I(s, B) = Q$ and $J_1 = I(m, M)$, $J_2 = I(n, N)$ as stated. From $J_1 \subset P$ and $P \vee J_2 = Q$ we deduce respectively $M \supset A$ and $A \cap N = B$. Using $J_1 \subset J_2$ and Lemma 4 we conclude that $M = N$ and hence $A = B$. This completes the proof.

In view of Lemma 5 and the remark preceding it we have the following important fact.

COROLLARY. *The set $\Omega(L_R)$ (or briefly Ω) of equivalence classes $\omega(P) = \{Q: P \sim Q\}$ $(P \in \mathscr{K})$ is determined by the lattice structure of L_R.*

The topological space $\Omega(L_R)$. Our next undertaking is to introduce a topology in the set Ω. We do this by specifying a subset $\Sigma \subset \Omega$ to be closed iff it has the following property: if $P, Q_t \in \mathscr{K}$ $(t \in T)$ are such that $P \supset \bigcap_{t \in T} Q_t \neq \emptyset$ holds where each Q_t belongs to some $\sigma \in \Sigma$, then $\omega(P) \in \Sigma$. This is reminiscent of the hull-kernel topology encountered in commutative ring theory.

THEOREM 6. *The topological space $\Omega(L_R)$ is determined by the lattice structure of $L_R(X)$. It is a pc-space, homeomorphic to the space πX.*

Proof. It will be shown that Ω can be put in a $1-1$ correspondence with the *pc*-space πX in such a way that the sets called closed above correspond to the closed subsets of πX. We note first of all that, by definition, a class $\omega(P) \in \Omega$ consists of all ideals $Q \in \mathscr{K}$ which have a common carrier. Thus by putting

$$I(A) = \{I(r, A): r \in R\}$$

we obtain a $1-1$ correspondence between the elements $A \in \pi X$ and the elements $I(A) \in \Omega$. Let us now consider an arbitrary subset $\Sigma = \{I(A): A \in \mathscr{S}\}(\mathscr{S} \subset \pi X)$ of Ω and any $I(s, B) \in \mathscr{K}$. Then $I(s, B) \supset \bigcap_{t \in T} I(r_t, A_t) \neq \emptyset$ holds for some family $\{I(r_t, A_t): t \in T\}$ with all $A_t \in \mathscr{S}$ iff $B \subset \operatorname{cl} \cup \mathscr{S}$. Indeed, if $B \not\subset \operatorname{cl} \cup \mathscr{S}$ and $f \in \cap_{t \in T} I(r_t, A_t)$ holds with all $A_t \in \mathscr{S}$, then either $f \notin I(s, B)$ or any function g such that $g(x) = s' \vee f(x)$ $(s' > s)$ when $x \in X \backslash \operatorname{cl} \bigcup_{t \in T} A_t$ and which agrees with f on $\operatorname{cl} \bigcup_{t \in T} A_t$ satisfies $g \in \bigcap_t I(r_t, A_t)$ and $g \notin I(s, B)$; and if $B \subset \operatorname{cl} \cup \mathscr{S}$ holds then $I(s, B) \supset \cap \{I(s, A): A \in \mathscr{S}\} \neq \emptyset$ clearly holds. We conclude that Σ is closed in Ω iff it is the image of some closed set $\mathscr{S} \subset \pi X$ under the mapping $A \rightarrow I(A)$. Hence Ω is homeomorphic to πX. That the topological space Ω is determined by the lattice L_R is clear from the corollary to Lemma 5 and the definition of closed sets in Ω: only the lattice structure of L_R is used to define it.

COROLLARY 7. *If X is a pc-space, then X is homeomorphic to $\Omega(L_R)$ and is therefore determined by the lattice structure of $L_R(X)$.*

This follows at once in view of Theorem 6 and the known fact 1(*b*).

Remarks on the lattice $L_H(X)$. We will now turn our attention to the sublattice $L_H(X)$ of $L_R(X)$. The results proved above for $L_R(X)$ all remain valid if L_H is substituted for L_R and some minor adaptations are made, the most important of which is to restrict the variable r in the ideals $I(r, A)$ to the interval $0 < r < 1$. As a matter of fact, the theory for $L_H(X)$ can be simplified by using prime elements (i.e. those $g \in L_H$ such that $g < 1$ and $g = u \wedge v$ only if $g = u$ or $g = v$) rather than closed prime ideals. This will be clear from the following fact.

PROPOSITION 8.
(a) *$g \in L_H$ is prime iff g is of the form e_{rA} where $e_{rA}(x) = r$ when $x \in A$ and $e_{rA}(x) = 1$ when $x \in X \backslash A$* ($0 < r < 1$ and $A \in \mathscr{A}$).
(b) *Every closed prime ideal in L_H is of the form $\{f: f \leqq e_{rA}\}$.*

The proof is very much like that of Proposition 3 and is therefore omitted.

By using equivalence classes of primes in L_H one can now prove as for L_R that the lattice L_H determines a topological space $\Omega(L_H)$ which is homeomorphic to πX and hence to X whenever X is a *pc*-space.

We now consider some properties of L_H which are not included in the theory presented for L_R.

PROPOSITION 9.
(a) *The lattices $L_H(X)$ and $L_H(\pi X)$ are isomorphic.*
(b) *For a given X the lattices $L(X)$ and $\mathscr{C}(X)$ determine each other.*
(c) *If X is a T_D-space, then it is determined as such by the lattice $L_H(X)$.*

Proofs. (a) The mapping $f \rightarrow f^{\pi}$ (see 1(c)) is easily seen to be an isomorphism of $L_H(X)$ onto $L_H(\pi X)$.

(b) If the lattice $L_H(X)$ (resp. $\mathscr{C}(X)$) is known then πX is known and thus also the lattice $\mathscr{C}(\pi X) = \mathscr{C}(X)$ (resp. $L_H(\pi X) = L_H(X)$).

(c) The lattice $L_H(X)$ determines the lattice $\mathscr{C}(X)$ which determines X as a T_D-space.

Let us give an example to show that 9(a) is not valid for $L_R(X)$. If X is the subspace $(0, 1)$ of R_l, then πX is the subspace $(0, 1]$ of R_l. In the case of both X and πX there is just one maximal irreducible set namely the whole space. Thus the quasi-minimal ideals in both $L_R(X)$ and $L_R(\pi X)$ are the principal ideals $\{f: f \leqq r\}$ of the constant functions (see Lemma 4(a)). Now every f in $L_R(\pi X)$ attains a maximum value $f(1)$ and so every $f \in L_R(\pi X)$ belongs to some quasi-minimal ideal. This cannot hold for $L_R(X)$ which clearly contains functions unbounded above. The two lattices are therefore not isomorphic.

There is in fact a class wider than the *pc*-spaces such that the lattice $L_R(X)$ determines X whenever X belongs to this class. This will be discussed in a later paper. L. D. Nel and R. G. Wilson, *Epireflections in the category of T_0-spaces* (to appear in Fund. Math.).

Remarks on the lattice $L_I(X)$. The method used above to prove that the lattices $L_R(X)$ and $L_H(X)$ determine *pc*-spaces cannot be applied to $L_I(X)$. Theorem 2, Propositions 3 and 8 remain valid for $L_I(X)$ but Lemma 4(a) fails (and therefore all further results based on it). It fails because in the case of $L_I(X)$ there are two types of quasi-minimal ideals, namely those of the form $I(0, A)$ for non-maximal $A \in \mathscr{A}$ and those of the form $I(r, A)$ for $0 < r < 1$ and maximal $A \in \mathscr{A}$. There appears to be no lattice theoretic method of distinguishing between these types.

It seems plausible nevertheless that $L_I(X)$ should determine X whenever it is a *pc*-space. A settlement of this open question should be interesting.

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