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ON $L(S)$-TUPLES AND l-PAIRS OF MATRICES

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In this paper we study $L(S)$-tuples of matrices, a class of k-tuples that includes as special cases the L-pairs studied by Motzkin and Taussky, and the l-pairs defined by Taussky for complex elements and diagonable first matrix. Some light on these concepts and a few relevant results are produced by linking them to properties of algebraic hypersurfaces and especially curves with respect to an exterior point.

1. Characteristic hypersurfaces. Let I (the unit matrix), $A_1, \cdots, A_k$ $(k \geqq 2)$ be n by n matrices $(n \geqq 2)$ with elements in a field F of characteristic f and algebraic closure $\bar{F}$. Let $\sigma_0, \cdots, \sigma_k$ be the homogeneous coordinates of a variable point (σ_0, s) in projective k-dimensional space over $\bar{F}$.

DEFINITION 1. The characteristic determinant and hypersurface of the k-tuple $A_1, \cdots, A_k$ are defined by $D \equiv D(\sigma_0, s) \equiv |\sigma_0 I - \sigma_1 A_1 - \cdots - \sigma_k A_k|$ and $D = 0$.

The point $s = 0$, called origin, is not on $D = 0$. Two k-tuples related by similarity have the same characteristic determinant and hypersurface. For two linearly independent k-tuples defining the same linear family, the characteristic hypersurfaces are projectively related.

2. Generality of characteristic hypersurfaces. (1) For $f = 0$, every curve of degree n not through the origin is characteristic curve of at least one (almost always of, but for an orthogonal similarity, only one) pair of symmetric matrices with elements in $\bar{F}$. For no $k \geqq 3$ and $n \geqq 2$ does the corresponding assertion hold.

"Almost always," as in the sequel, means that all exceptions, if any, fulfill some nonidentical polynomial relation between their defining constants.

The first part of (1) follows from a result of Grace [4] (Dixon [5], credited with the proof by Room [6, pp. 126–127], admits that one case escapes him). The second part follows from a simple count of constants, viz. $\binom{n+k}{k} - 1$ for hypersurfaces vs. $n + k\binom{n+1}{2}$ for not orthogonally similar k-tuples.

(2) For $f = 0$, every quadric and cubic surface not through the

origin is characteristic surface of at least one (almost always of, up to a similarity, only one) triple of matrices with elements in $\bar{F}$. For no values $k = 3, n \geqq 4$ and no $k \geqq 4, n \geqq 2$ does the corresponding assertion hold.

For statements equivalent to the first part cf. [6, p. XXI] Lasker [7, p. 439] and Wakeford [8, p. 409]. For the second part the count of constants rests on their number for nonsimilar k-tuples, which is $1 + (k-1)n^2$ for $k \geqq 2$.

3. **$L(S)$-tuples.** DEFINITION 2. For a k-tuple $A_1, \cdots, A_k$ and prescribed $s = (\sigma_1, \cdots, \sigma_k)$, the roots of $sA \equiv \sigma_1 A_1 + \cdots + \sigma_k A_k$ are defined as the points $(\sigma_0, \sigma_1, \cdots, \sigma_k)$ for which $D = 0$.

Let S be a set of points $s = (\sigma_1, \cdots, \sigma_k)$ in projective $(k-1)$-space S_0 over $\bar{F}$. The space S_0 will be thought of as the hyperplane at infinity $\sigma_0 = 0$ of k-space. If s^* is the straight line connecting the point $(0, s)$ and the origin $(1, 0)$ then the roots of sA, counted with the proper multiplicities as points of intersection of $D = 0$ and s^*, form a set D_s of n points.

DEFINITION 3. The k-tuple $(A_1, \cdots, A_k)$ is said to be $L(S)$ (or an $L(S)$-k-tuple or $L(S)$-tuple, or to have property $L(S)$, or to split linearly over S) if there exist $\lambda_{11}, \cdots, \lambda_{kn}$ in $\bar{F}$, such that, identically in σ_0, $D \equiv l_1 \cdot \cdots \cdot l_n$ for all $(\sigma_1, \cdots, \sigma_k)$ in S, with $l_1 \equiv \sigma_0 - \lambda_{11}\sigma_1 - \cdots - \lambda_{k1}\sigma_k, \cdots, l_n \equiv \sigma_0 - \lambda_{1n}\sigma_1 - \cdots - \lambda_{kn}\sigma_k$.

This expresses a property of the characteristic hypersurface with respect to the origin: the roots D_s, for s in S, lie on n hyperplanes $l_1 = 0, \cdots, l_n = 0$ that do not pass through the origin, and ν-fold roots lie on ν of these hyperplanes.

As an example, let $k = 2, l_1, \cdots, l_n$ and a set S consisting of n points given, with $S^* = 0$ the product of the equations of the n lines s^*. Then any matrix pair having $\alpha S^* + \beta l_1 \cdot \cdots \cdot l_n = 0$ $(\alpha, \beta \neq 0, 0)$ as characteristic curve is $L(S)$.

4. **Special sets S.** For $S = S_0$, property $L(S)$ does not involve the origin and amounts to demanding that the characteristic hypersurface $D = 0$ split into n, not necessarily distinct, hyperplanes $l_1 = 0$, $l_n = 0$. This is exactly property L of [1], geometrically characterized in [2], in its obvious generalization from $k = 2$ to general $k \geqq 2$:

(3) $L(S_0) = L$.

At the other extreme we have:

(4) If the points of S are linearly independent then every k-tuple

of matrices is $L(S)$.

In fact, in whichever way we partition the points D_s, s in S, into n sets $D_1, \cdots, D_n$ each having exactly one point on each line s^*, there exists at least one hyperplane through all points of each set.

5. Unindependent S. The next smallest sets S are the unindependent sets:

DEFINITION 4. A set of points is called unindependent if there is, except for a factor, exactly one linear relation, with coefficients not all zero, between its points.

(5) If the points $s_1, \cdots, s_r$ $(r \geqq 2)$ are linearly independent and span the projective $(r-1)$-space S_1 then either $(A_1, \cdots, A_k)$ is $L(S_1)$ or there exist at most $(n-r)(n^r-1)$ points s_0 such that $S = (s_0, s_1, \cdots, s_r)$ is unindependent and that $(A_1, \cdots, A_k)$ is $L(S)$.

Indeed suppose that $(A_1, \cdots, A_k)$ is not $L(S_1)$ and therefore $r \leqq n$, and that exactly n_1 among the at most n^r projective $(r-1)$-spaces connecting points of $D = 0$ corresponding to $s_1, \cdots, s_r$ belong to $D = 0$. Then to each s_0 there corresponds at least one point of $D = 0$ on one of the $n^r - n_1$ other $(r-1)$-spaces, while each of the latter contains at most $n - r$ points of $D = 0$ that do not correspond to $s_1, \cdots, s_r$. Hence there exist at most $(n-r)(n^r - n_1)$ points s_0; but for $n_1 = 0$ each s_0 consumes even n of the $(r-1)$-spaces.

From (5) follows:

(6) For $n = 2$ an $L(S)$-tuple is also $L(S_1)$, where S_1 is the linear space spanned by S.

Thus, for $n = 2$, no property $L(S)$ essentially different from L (for a linear subfamily) exists.

6. $L(S)$ for infinitely many S. No s_0 as in (5) need exist for any $s_1, \cdots, s_r$ (see (16). On the other hand:

(7) If $D - \sigma_0^n$ is free of λ then an s_0 exists for almost all $s_1, \cdots, s_r$.

This follows from:

(8) If $D - \sigma_1^n$ is free of σ_0 then, for arbitrarily prescribed $\lambda_1, \cdots, \lambda_k$ in $\bar{F}$, $(A_1, \cdots, A_k)$ is $L(S)$ where S is the set of all points s for which $D(\lambda_1\sigma_1 + \cdots + \lambda_k\sigma_k, s) = 0$.

Indeed, for s in S, we have $D \equiv (\lambda - \varepsilon_1(\alpha_1\mu_1 + \cdots + \alpha_k\mu_k)) \cdot \cdots \cdot (\lambda - \varepsilon_n(\alpha_1\mu_1 + \cdots + \alpha_k\mu_k))$, with $\lambda^n - 1 \equiv (\lambda - \varepsilon_1) \cdot \cdots \cdot (\lambda - \varepsilon_n)$.

The linear families with σ_0-free $D - \sigma_0^n$ are those contained in the locus: $|\sigma_0 I - A| - \sigma_0^n$ free of σ_0. The number of ensuing conditions leaves $n^2 - n + 1$ degrees of freedom for A_1 (for $f = 0$, A_1 is, but for a scalar factor, similar to a diagonal matrix with the nth roots of unity in the diagonal), at least $\binom{n+1}{2}$ for A_2, $2 - 1$ for A_3, so that such families, with linearly independent $A_1, \cdots, A_k$, exist for every $n \geqq 2$ and $k = 2$ or 3. For A_4 the number of degrees of freedom is, for $n = 5, 6, \cdots$, at least $3, 1, \cdots$, which leaves the existence of an A_4 independent of A_1, A_2, A_3 in doubt; but for $n = 3$ and $n = 4$, their number is 4, so that such families exist also for $k = 4$ and $n = 3$ or 4. For A_5 and $n = 4$ the number is 1, so that an independent A_5 may not exist; for A_5 and $n = 3$ it is 3, and since the conditions for $n = 3$ are all but one linear and easily seen to be independent, $n = 3$ and $k = 5$ is impossible. For $n = 2$ the matrices A such that $(|\sigma_0 I - A| - \sigma_0^n)$ is free of σ_0 are those of trace 0 and form a linear family with $k = 3$.

Note that $(A_1, \cdots, A_k)$ is not L for σ_0-free $D - \sigma_0^n$ except if $D - \sigma_0^n$ is an nth power. In this case the number of degrees of freedom becomes at least $1 + (n/2)$ for A_2 and 1 for A_3. For n = 2 these are the same as before, in harmony with (6). But e.g., for $n = 3$, $k = 2$, $f = 0$, a pair not L but in infinitely many ways $L(S)$ is given by $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$, where $1 + \varepsilon + \varepsilon^2 = 0$. The characteristic curves of such pairs are also mentioned in the proof of (15).

7. $L(S)$**-pairs.** Evidently:

(9) For $k = 2$ and infinite S, $L(S) = L$.

This follows also from (6) or (10).

For $k = 2$ and finite sets S we consider also sets S with finite multiplicities attached to their points. We write $A, B, \lambda_1, \cdots, \lambda_n, \mu_1, \cdots, \mu_n$ for $A_1, A_2, \lambda_{11}, \cdots, \lambda_{1n}, \lambda_{21}, \cdots, \lambda_{2n}$.

DEFINITION 5. The pair (A, B) is said to be $L(S)$ if there exist $\lambda_1, \cdots, \lambda_n, \mu_1, \cdots, \mu_n$ in $\bar{F}$ such that, identically in σ_0, $D \equiv (\sigma_0 - \lambda_1\sigma_1 - \mu_1\sigma_2) \cdot \cdots \cdot (\sigma_0 - \lambda_n\sigma_1 - \mu_n\sigma^2)$ for all (σ_1, σ_2) in S, with corresponding identities derived by differentiation up to $\rho - 1$ times along the branches of $D = 0$ for each ρ-fold point of S.

That is, the n straight lines are required not only to pass (severally when needed) through the points $(\sigma_0, \sigma_1, \sigma_2)$ of $D = 0$ with (σ_1, σ_2) in S, but for $\rho > 1$ to have $(\rho - 1)$-order contact with the branches at these points. Then:

(10) If the sum r of multiplicities of points of S surpasses n, $L(S) = L$.

Indeed, each of the n straight lines has (taking account of multiplicities) more than n points in common with $D = 0$ and thus is part of $D = 0$.

8. **l-pairs.** DEFINITION 6. Property l is defined as $L(S)$, $S = ((1, 0), (1, 0), (0, 1))$.

From (10) follows (extending the corresponding result in [2, Theorem 3] and strengthening (6)):

(11) For $n = 2, l = L$.

For $n \geqq 3$, (A, B) can be l without (B, A) being l; but even if both are l-pairs, (A, B) need not be L. Such double-l-not-L pairs exist already for $n = 3$; the three roots $D_{1,0}$ of A (i.e., the points $D = 0, \sigma_2 = 0$) and those $D_{0,1}$ of B are connected by six tangents at the roots. For $n \geqq 4$ there can be $n, n+1, \cdots, 2n$ connecting tangents.

The coefficients of σ_0^{n-1} in D and in $l_1, \cdot \cdots \cdot l_n$ are linear and thus coincide identically if they coincide for two values of σ_2/σ_1, or for one value and the derivative there. Hence we obtain:

(12) If $n - 1$ branch tangents to the characteristic curve $D = 0$ of the pencil $\sigma_1 A + \sigma_2 B$ at the roots of A pass through the roots of B (but for one root, as often as the multiplicity of the roots of B indicates) then (A, B) is l.

9. **The l-pairs in a pencil.** Every pair $(\gamma_1 A, \gamma_2 A)$ is trivially L and a fortiori l. For $n = 2$ a pencil $\sigma_1 A + \sigma_2 B$, where (A, B) is not L, cannot contain any l-pair except these trivial (linearly dependent) pairs; for $n \geqq 4$ it need not, though as we shall see (15) this is for $n = 3$ and $f = 0$ still an exceptional case.

On the other hand, for (A, B) not L, the pencil cannot contain more than $n - 2$ nontrivial l-pairs with given first matrix. Even so:

(13) If $D - \sigma_0^n$ is free of σ_0 then the pencil contains infinitely many nontrivial l-pairs.

The proof is analogous to that of (7).

10. **Characteristic cubics.** Obviously:

(14) For $n = 3$ and reducible $D = 0$, no or all nontrivial pairs in a pencil are l according as its characteristic cubic splits into a conic and a line or into three (not necessarily distinct) lines.

In the next result "almost always" means "except possibly for values of the coefficients satisfying an additional polynomial relation." For such values the number of pairs could conceivably be infinity, by indeterminacy, smaller than indicated, by coincidence of pairs, and even 0, by coincidence of elements of a pair. (We are careful not to use nonhomogeneous formulas.)

(15) For $n = 3$ and $f = 0$, a pencil whose characteristic cubic is of class 3, 4, or 6, contains almost always 1, 2, or 4 l-pairs.

"Almost always" refers equivalently to cubics or pencils; this follows without difficulty from (1).

First consider a cubic of class 3; after a projective transformation it can be parametrically represented by $(1, \tau, \tau^3)$. Three distinct points τ_1, τ_2, τ_3 on it are collinear if and only if

$$\begin{vmatrix} 1 & \tau_1 & \tau_1^3 \\ 1 & \tau_2 & \tau_2^3 \\ 1 & \tau_3 & \tau_3^3 \end{vmatrix} \Big/ (\tau_2 - \tau_1)(\tau_3 - \tau_1)(\tau_3 - \tau_2) \equiv \tau_1 + \tau_2 + \tau_3 = 0 .$$

The coordinates $\gamma_0, \gamma_1, \gamma_2$ of the line c joining them are $-\tau_1\tau_2\tau_3, \tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3, 1$. The tangents at the three points meet the cubic in $-2\tau_1, -2\tau_2, -2\tau_3$, points on $c' = (-8\gamma_0, 4\gamma_1, \gamma_2)$. If both lines c and c' are to pass through a given point $d = (\delta_0, \delta_1, \delta_2)$ then c is uniquely determined if $\delta_0\delta_1\delta_2 \neq 0$; if one of $\delta_0, \delta_1, \delta_2$ is 0 no c with $c' \neq c$ exists; if two are 0 each line through d, with two exceptions, will do (special case of (13)). In the application to pencils of matrices d is given by $\sigma_1 = 0, \sigma_2 = 0$ and can be any point outside the cubic.

For the cubic $(1, \tau^2, \tau^3 + \tau)$, of class 4, the collinearity condition is $\tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3 = 1$. Here $\gamma_0, \gamma_1, \gamma_2$ are $-\tau_1\tau_2\tau_3, -\tau_1 - \tau_2 - \tau_3, 1$. The tangents meet the cubic in $1/2(1/\tau_1 - \tau_1), 1/2(1/\tau_2 - \tau_2), 1/2(1/\tau_3 - \tau_3)$, points on $c' = ((\gamma_0 + \gamma_1)^2 - 4\gamma_2^2, 4(\gamma_2^2 - \gamma_0\gamma_1), 8\gamma_0\gamma_2)$. This gives two c for general d.

For a cubic $(\eta_0(\tau), \eta_1(\tau), \eta_2(\tau))$ of class 6, where η_0, η_1, η_2 are elliptic functions of periods π_1 and π_2, collinearity is expressed by $\tau_1 + \tau_2 + \tau_3 \equiv 0 \pmod{\pi_1, \pi_2}$. The tangents meet the cubic in $-2\tau_1$, $-2\tau_2$, $-2\tau_3$. Hence every c' belongs to 16 lines

$$c = (\tau_1 + \psi_1, \tau_2 + \psi_2, \tau_3 + \psi_3) ,$$

where $\psi_3 = -\psi_1 - \psi_2$ and $2\psi_1 \equiv 0, 2\psi_2 \equiv 0$. Since no c' belongs to infinitely many c the polynomials expressing c' in terms of c must be of degree 4.

Another proof uses the Chasles correspondence principle.

11. No-L-pencils. DEFINITION 6. A pencil $\sigma_1 A + \sigma_2 B$ is called no-L if there is no 3-point S for which (A, B) is $L(S)$.

This excludes trivial sets S (with at most two points) and trivial pencils (generated by a trivial pair). Property $L(S)$ for any S with more than three (not necessarily distinct) points implies $L(S')$ for every 3-point subset S'.

(16) Almost all pencils are no-L for $n = 2$ and $n \geqq 5$, but not for $n = 3$ and $n = 4$.

Indeed, for given A and B, choose two values of (σ_1, σ_2). There are then, for irreducible characteristic curve and $n \geqq 3$, finitely many candidates for a third point of s. By (12), $n - 2$ conditions have to be fullfilled to ensure $L(S)$-ness. This is, if and only if $n \geqq 5$, too much for an appropriate choice of the first two points.

Pencils with a characteristic curve consisting of a conic and $n - 2$ straight lines are always no-L. Whether other no-L pencils exist for $n = 3$ and $n = 4$ remains open.

For $n \geqq 7$ almost all pencils are no-L with respect to every point outside the characteristic curve.

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