PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 3, 1972

OPERATORS SATISFYING CONDITION (G_1) LOCALLY

GLENN R. LUECKE

The class of operators that satisfy condition (G_1) locally is studied. For operators in this class, conditions on the spectra which will insure normality are investigated.

An operator (continuous linear transformation from H into H) T on the complex Hilbert space H satisfies condition (G_1) if $\|(T-zI)^{-1}\| = 1/d(z, \sigma(T))$ for all $z \in \rho(T)$, where $\rho(T)$ is the resolvent set of T and $d(z, \sigma(T))$ is the distance from z to $\sigma(T)$, the spectrum of T. T satisfies (G_1) locally if T satisfies (G_1) in an open neighborhood of $\sigma(T)$, i.e. $\|(T - zI)^{-1}\| = 1/d(z, \sigma(T))$ for all $z \in U - \sigma(T)$ where U is some open set containing $\sigma(T)$. Let $\mathscr{G}$ and $\mathscr{G}_{loc}$ be all operators on H satisfying (G_1) and (G_1) locally, respectively. First it is shown how to construct nontrivial examples of operators in $\mathscr{G}$ and $\mathscr{G}_{loc}$. When $\dim H < \infty$, it is well-known that $\mathscr{G}_{loc} = \mathscr{G} = \mathscr{N}$, the set of all normal operators on H. However, when $\dim H = \infty$ then $\mathscr{N}$ is a proper subset of $\mathscr{G}$ and $\mathscr{G}$ is a proper subset of $\mathscr{G}_{loc}$. Next, for $T \in \mathscr{G}_{loc}$ having $\sigma(T)$ countable, conditions on $\sigma(T)$ are investigated to guarantee that T be normal.

1. Properties of $\mathscr{G}$ and $\mathscr{G}_{loc}$. First we show how to construct nontrivial operators in $\mathscr{G}$ and $\mathscr{G}_{loc}$. Let A be any operator on H. Then $A \oplus N \in \mathscr{G}$ on the Hilbert space $H \oplus K$ (the orthogonal direct sum of H and K), whenever N is a normal operator on K with $\sigma(N) \supseteq W(A)$, the numerical range of A [see 8]. The following is an analogous way to construct operators in $\mathscr{G}_{loc}$.

THEOREM 1. *If A is an operator on H, then $A \oplus N \in \mathscr{G}_{loc}$ on $H \oplus K$ whenever N is a normal operator on K with $\sigma(N) \supseteq U$, where U is an open set containing $\sigma(A)$.*

Proof. Let $T = A \oplus N$ where A and N are as above. Then $\sigma(T) = \sigma(A) \cup \sigma(N) = \sigma(N)$. Let $R(S, z) = (S - zI)^{-1}$ denote the resolvent of S at z. Then for $z \in \rho(T)$ [see 11],

$$\begin{aligned} \|R(T, z)\| &= \text{Max}\{\|R(A, z)\|, \|R(N, z)\|\} \\ &= \text{Max}\{\|R(A, z)\|, 1/d(z, \sigma(T))\} . \end{aligned}$$

The last equality holds since N is a normal operator and thus $\|R(N, z)\| = 1/d(z, \sigma(N)) = 1/d(z, \sigma(T))$. Since there is an open set U such that $\sigma(N) \supseteq U \supseteq \sigma(A)$, there exists an open set $V \supseteq \sigma(N) = \sigma(T)$

such that for each $z \in V - \sigma(T)$, $\|R(A, z)\| \leqq 1/d(z, \sigma(T))$. Thus $\|R(T, z)\| = 1/d(z, \sigma(T))$ for all $z \in V - \sigma(T)$, and hence $T \in \mathscr{G}_{loc}$.

It is well-known [13, Th. 1] that $\mathscr{G}$ contains $\mathscr{N}$, the set of all normal operators on H. It is immediate that $\mathscr{G} \subseteqq \mathscr{G}_{loc}$. Putnam [10] has shown that for $T \in \mathscr{G}_{loc}$ the isolated points of $\sigma(T)$ are normal eigenvalues ($z \in \sigma(T)$ is a normal eigenvalue of T if z is an eigenvalue of T and $\{x \in H: Tx = zx\} = \{x \in H: T^*x = z^*x\}$ where z^* is the complex conjugate of z). Thus for $\dim H < \infty$, $\mathscr{G}_{loc} = \mathscr{N}$; and consequently $\mathscr{G}_{loc} = \mathscr{G} = \mathscr{N}$ [see 7].

THEOREM 2. *$\mathscr{G} \neq \mathscr{G}_{loc}$ when* $\dim H = \infty$.

Proof. Let M be a two dimensional subspace of H and let $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ on M. Let N be normal operator on $M^\perp$ with $\sigma(N)$ equal to the closed disc of radius 1/2 about the origin (this requires that $\dim M^\perp = \infty$). From Theorem 1, $T = A \oplus N \in \mathscr{G}_{loc}$. However $T \notin \mathscr{G}$ since upon calculation one finds that $\|R(T, z)\| > 1/d(z, \sigma(T))$ when, for example, $z = 1$.

From [9] we know that for each $T \in \mathscr{G}$, $\text{co}\,\sigma(T) = \text{Cl}\, W(T)$, where $\text{co}\,\sigma(T)$ denotes the convex hull of $\sigma(T)$ However, from the example in the proof of Theorem 2 we see that not all $T \in \mathscr{G}_{loc}$ satisfy $\text{co}\,\sigma(T) = \text{Cl}\, W(T)$.

Let $B(H)$ denote the set of all operators on H and give $B(H)$ the norm topology. When $\dim H < \infty$, then $\mathscr{G}_{loc} = \mathscr{G} = \mathscr{N}$ is a closed subset of $B(H)$. When $\dim H = \infty$, then $\mathscr{G}$ and $\mathscr{N}$ are closed subsets of $B(H)$ [8].

THEOREM 3. *$\mathscr{G}_{loc}$ is niether an open nor closed subset of $B(H)$ when* $\dim H = \infty$.

Proof. To see that $\mathscr{G}_{loc}$ is not open, it suffices to observe that (1) the zero operator is in $\mathscr{G}_{loc}$, (2) $T \in \mathscr{G}_{loc}$ and α a complex number implies $\alpha T \in \mathscr{G}_{loc}$, and (3) $\mathscr{G}_{loc} \neq B(H)$.

Let H, M, and A be as in the proof of Theorem 2. Let N_n be a normal operator on $M^\perp$ whose spectrum is the closed disc of radius $1/n$ about the origin. Let $T_n = A \oplus N_n$. By Theorem 1, $T_n \in \mathscr{G}_{loc}$. Let Z be the zero operator on $M^\perp$. Then $T_n \to A \oplus Z$ in norm and since $A \oplus Z \notin \mathscr{G}_{loc}$, $\mathscr{G}_{loc}$ is not closed.

For a detailed discussion of the topological properties of $\mathscr{G}$ see

[8].

II. Operators in $\mathscr{G}_{loc}$ with countable spectra. In general an operator $T \in \mathscr{G}_{loc}$ with countable spectrum need not be normal. However, such a non-normal operator can always be decomposed as the orthogonal direct sum of a normal operator and another operator:

THEOREM 4. *If $T \in \mathscr{G}_{loc}$ has countable spectrum, then either T is normal or $T = A \oplus N$ where N is a normal operator with $\sigma(N) = \sigma(T)$ and A is an operator with $\sigma(A)$ a subset of the derived set of $\sigma(T)$.*

Proof. If z is an isolated point of $\sigma(T)$, then by [10] z is a normal eigenvalue of T; let $E(z)$ be the eigenspace of z. Let $\sigma_0(T)$ denote the isolated points of $\sigma(T)$ and let

$$M = \text{closed span} \bigcup_{z \in \sigma_0(T)} E(z) .$$

Since each $E(z)$, $z \in \sigma_0(T)$, reduces T, T is normal on $E(z)$; and consequently M reduces T and T is normal on M. Since $\sigma(T)$ must have at least one isolated point, $M \neq (0)$. If $M = H$, then T is normal.

If $M \neq H$, then write $H = K \oplus M$ and $T = A \oplus N$ where A is T restricted to K and N is T restricted to M. Clearly $\sigma(N) = \sigma(T)$ and N is normal. Suppose to the contrary that $\sigma(A)$ is not a subset of the derived set of $\sigma(T)$. Then there exists $w \in \sigma(A)$ such that w is an isolated point of $\sigma(T)$. Therefore w is an isolated point of $\sigma(A)$, so there exists a circle C about w such that if $z \in C$, then $|z - w| = d(z, \sigma(T)) = d(z, \sigma(A))$. Then for $z \in C$

$$\begin{aligned} \|R(A, z)\| &\leqq \operatorname{Max}\{\|R(A, z)\|, \|R(N, z)\|\} = \|R(T, z)\| \\ &= 1/d(z, \sigma(T)) = 1/d(z, \sigma(A)) . \end{aligned}$$

Then since $\|(z - w)R(A, z)\| \leqq 1$ as $z \to w$, $(z - w)R(A, z)$ is a vector-valued analytic function of z at $z = w$. Therefore $(z - w)R(A, z)$ is analytic on an open disc containing C. Let

$$P = -\frac{1}{2\pi i}\int_C R(A, z)dz .$$

then

$$AP - wP = -\frac{1}{2\pi i}\int_C (z - w)R(A, z)dz = 0$$

so that $AP = wP$. Since $P \neq 0$ [12, p. 421], w is an eigenvalue of A

and hence of T. Since w is isolated point of $\sigma(T)$, w is a normal eigenvalue of T. Hence $K \cap M \neq (0)$. Contradiction.

With Theorem 4 we can easily classify all compact operators in $\mathscr{G}_{loc}$.

COROLLARY. *If* $T \in \mathscr{G}_{loc}$ *is compact, then either* T *is normal or* $T = A \oplus N$ *where* N *is compact and normal, and* A *is compact and quasi-nilpotent.*

Proof. The spectrum of a compact operator is countable with zero the only possible point of accumulation.

The existence of a non-normal $T \in \mathscr{G}_{loc}$ follows immediately from the following:

THEOREM 5. *If* A *is any operator, then there exists a normal operator* N *such that*

1. $A \oplus N \in \mathscr{G}_{loc}$
2. $\sigma(N) \supseteqq \sigma(A)$, *and*
3. $\sigma(N) - \sigma(A)$ *is a countable set whose points of accumulation are contained in* $\sigma(A)$.

Proof. Assume $\|A\| = 1$. We would like to find a normal operator N so that $\sigma(N)$ is the disjoint union of $\sigma(A)$ and some countable set $X \subseteqq \{z: |z| \leqq 2\}$ such that the following properties hold:

(i) the accumulation points of X are contained in $\sigma(A)$,

(ii) for $|z| \geqq 2$, $d(z, \sigma(N) \leqq d(z, W(A))$, and

(iii) for $|z| < 2$ and $z \in \rho(N)$, $\|R(A, z)\| \leqq 1/d(z, \sigma(N))$.

Property (i) guarantees that $\sigma(A) \cup X$ is a compact set so that there does exist a normal operator N with $\sigma(N) = \sigma(A) \cup X$. Let $T = A \oplus N$. Then for $|z| > 2$ property (ii) implies

$$\|R(A, z)\| \leqq 1/d(z, W(A)) = 1/d(z, \sigma(T)) .$$

Combining this with property (iii) we see that for every $z \in \rho(T)$, $\|R(A, z)\| \leqq 1/d(z, \sigma(T))$. Consequently $T = A \oplus N \in \mathscr{G} \subseteqq \mathscr{G}_{loc}$. Thus, it sufficies to construct such a set X.

Let

$$S_n = \{z: |z| \leqq 2 \text{ and } 3/(n+1) \leqq d(z, \sigma(A)) \leqq 3/n\}$$

for $n = 1, 2, 3, \cdots$. Since $\|R(A, z)\|$ is bounded on each compact set

S_n, there exists a finite set of points $X_n \subseteqq S_n$ such that $d(z, X_n) \leqq \|R(A, z)\|^{-1}$ for all $z \in S_n$. Let

$$X = \bigcup_{n=1}^{\infty} X_n .$$

Since $\|R(A, z)\| \geqq 1/d(z, \sigma(A))$ [see 4, p. 566], X has all of its accumulation points in $\sigma(A)$, and hence property (i) is satisfied. To see that property (iii) is satisfied, let $z \in S_n \cap \rho(N)$. Then

$$d(z, \sigma(N)) = d(z, X) \leqq d(z, X_n) \leqq \|R(A, z)\|^{-1} .$$

Thus $\|R(A, z)\| \leqq 1/d(z, \sigma(N))$. Since $W(A)$ is a subset of the closed unit disc, property (ii) can be satisfied, for example, by making sure taht X contains the points $2 \exp(n\pi i/4)$, for $n = 0, 1, \cdots, 7$.

One can further require in Theorem 5 that $T = A \oplus N \notin \mathscr{G}$. This can be done, in essentially the same manner as above, by choosing $\sigma(N) = \sigma(A) \cup X$ where X is as above only instead of satisfying properties (ii) and (iii) X satisfies the following: for $x \in \rho(N)$ $\|R(A, z)\| \leqq 1/d(z, \sigma(N)$ only for z contained in a sufficiently small neighborhood of $\sigma(A)$ instead of for all $z \in \{z \in \rho(N): |z| < 2\}$. This can be done by choosing m sufficiently large and then letting

$$X = \bigcup_{n=m}^{\infty} X_n .$$

To show that there exists a non-normal $T \in \mathscr{G}_{loc}$ with countable spectrum, let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and choose a normal operator N as in Theorem 5.

Stampfli [17] has shown that if $T \in \mathscr{G}_{loc}$ has $\sigma(T)$ lying on a C^2-smooth rectifiable Jordon curve C, then T is normal. The following question now arises: If $T \in \mathscr{G}_{loc}$ has countable spectrum, then can we weaken the assumption that $\sigma(T) \subseteqq C$ and still conclude that T must be normal? The answer is not fully known, but the following material gives a partial answer.

If S is a countable compact subset of the complex plane, then S satisfies *condition* (A) if for each $p \in S$ there exists $q \notin S$ such that $|q - p| = d(q, S)$.

To show that S satisfying condition (A) is weaker than $S \subseteqq C$, let S be the following countable, compact set of complex numbers:

$$S = \{0\} \cup \{1/n + i(\sin n)/n: n = 1, 2, 3, \cdots\} .$$

Then S does not lie on a C^2-smooth rectifiable Jordon arc, but S does satisfy condition (A).

THEOREM 6. *If T is a scalar operator in $\mathscr{S}_{loc}$ whose spectrum is countable and satisfies condition (A), then T is normal.*

Proof. Let $u \in \sigma(T)$, then there exists a sequence $\{u_n\} \subseteqq \rho(T)$ such that $u_n \rightarrow u$ and $|u_n - u| = d(u_n, \sigma(T))$. Since T is scalar

$$T = \int_{\sigma(T)} z dE_z .$$

Therefore

$$(u - u_n)R(T, u_n) = \int_{\sigma(T)} \frac{u - u_n}{z - u_n} dE_z .$$

Let $x, y \in H$ be fixed and define m to be the complex Borel measure $m(S) = (E(S)x, y)$ for each Borel set S in $\sigma(T)$. For each $z \in \sigma(T)$ let

$$f_n(z) = \frac{u - u_n}{z - u_n} \text{ and } f(z) = \begin{cases} 1 \text{ if } z = u \\ 0 \text{ if } z \neq u \end{cases} .$$

Then $|f_n(z)| \leqq 1$ and $f_n(z) \rightarrow f(z)$. Therefore we may apply the Lebesgue dominated convergence theorem:

$$\begin{aligned} |m(\{u\})| &= \left| \int_{\sigma(T)} f(z) \, \mathrm{dm}\,(z) \right| \\ &= \lim_{n\rightarrow\infty} \left| \int_{\sigma(T)} f_n(z) \, \mathrm{dm}\,(z) \right| \\ &= \lim_{n\rightarrow\infty} |((u - u_n)R(T, u_n)x, y)| \\ &\leqq |u - u_n| \, ||R(T, u_n)|| \, ||x|| \, ||y|| = ||x|| \, ||y|| . \end{aligned}$$

Since $m(\{u\}) = (E(\{u\})x, y)$, we have that

$$|(E(\{u\})x, y)| \leqq ||x|| \, ||y|| .$$

Letting $y = E(\{u\})x$, we obtain $||E(\{u\})x|| \leqq ||x||$, and hence $||E(\{u\})|| \leqq 1$. Therefore $E(\{u\})$ is an orthogonal projection for each $u \in \sigma(T)$.

Let $S \subseteqq \sigma(T)$ be a Borel set, then S is a countable set so write $S = \{z_1, z_2, z_3, \cdots\}$. Then for each $x, y \in H$, we have

$$\begin{aligned} (E(S)x, y) &= \sum_{n=1}^{\infty} (E(\{z_n\})x, y) = \sum_{n=1}^{\infty} (x, E(\{z_n\})y) \\ &= \operatorname{conj} \sum_{n=1}^{\infty} (E(\{z_n\})y, x) = \operatorname{conj}\,(E(S)y, x) = (x, E(S)y) . \end{aligned}$$

Therefore $E(S) = E(S)^*$, the adjoint of $E(S)$, and hence $E(S)$ is an orthogonal projection. Consequently, T is a scalar operator with a resolution of the identity of orthogonal projections; and thus T is normal.

In light of Theorem 6 it seems reasonable to conjecture the following theorem: If $T \in \mathscr{G}_{loc}$ has countable spectrum satisfying condition (A), then T is normal. The following thorem shows that this conjecture is false.

THEOREM 7. *There exists $T \in \mathscr{G}_{loc}$ with $\sigma(T)$ satisfying condition A such that*

(i) *$\sigma(T)$ is countable with zero the only point of accumulation,*

(ii) *if $z \in \sigma(T)$, then $|z-2| \leqq 2$, and*

(iii) *T is not normal.*

Proof. Let D_n be the closed disc of radius n about n, for $n = 1, 2$. Let V be the Volterra integration operator. Let $B = (I+V)^{-1}$, and let $A = I - B$. By [6, problem 150], $\sigma(B) = \{1\}$ and $\|B\| = 1$. Hence $\sigma(A) = \{0\}$ and $W(B)$ is contained in the closed disc about the origin of radius $\|B\| = 1$. Therefore $W(A) \subseteqq D_1$. We now proceed to fill up D_2 with enough points, X so that if N is a normal operator with $\sigma(N) = X \cup \{0\}$, then $A \oplus N \in \mathscr{G}_{loc}$ and $\sigma(A \oplus N)$ is a countable set with zero the only point of accumulation. The procedure is similar to that used in the proof of Theorem 5 only the details are a little more involved. For $n = 1, 2, \cdots$, let

1. $F_n = \{z \in D_2 \colon 4/(n+1) \leqq |z| \leqq 4/n\}$.
2. $M_n = \sup\{\|R(A, z)\| \colon z \in F_n\}$,
3. $d_n = \inf\{d(z, W(A)) \colon z \in (\partial D_2) \cap F_n\} > 0$
4. $P_n = \operatorname{Max}\{M_n, 1/d_n\}$, and
5. $B(z, r)$ be the open disc of radius r about z.

Then

$$F_n \subseteqq \bigcup_{z \in F_n} B(z, 1/P_n) .$$

Since F_n is compact, there exists $z_{n_i} \in F_n$, $1 \leqq i \leqq m_n$, such that

$$F_n \subseteqq \bigcup_{i=1}^{m_n} B(z_{u_i}, 1/P_n) .$$

Let N be a normal operator with $\sigma(N) = \{0\} \cup \{z_{n_i} \colon 1 \leqq i \leqq m_n,\ n = 1, 2, 3, \cdots\}$, then $\sigma(N)$ is a countable set with zero the only point of accumulation. Let $T = A \oplus N$, then $\sigma(T) = \sigma(N)$. We now verify that $T \in \mathscr{G}_{loc}$.

If $z \in D_2$, $z \neq 0$, then there exists n and i such that $z \in F_n \cap B(z_{n_i}, 1/P_n)$. Then

$$\begin{aligned} d(z, \sigma(N))\|R(A, z)\| &\leqq |z - z_{n_i}|\,\|R(A, z)\| \\ &\leqq (1/P_n)\|R(A, z)\| \\ &\leqq (1/M_n)\|R(A, z)\| \leqq 1 . \end{aligned}$$

If z is real and negative, then

$$d(z, \sigma(N)) \| R(A, z) \| \leqq |z| / d(z, W(A)) = 1 .$$

Suppose $z \notin D_2$ and that z is not real and negative. Let x be the point of intersection of ∂D_2 with the shortest line segment connecting z and Cl $W(A)$. Observe that $x \neq 0$. Then $d(z, W(A)) = |z - x| + d(x, W(A))$. There exists n and i such that $x \in F_n \cap B(z_{n_i}, 1/P_n)$. Then $|x - z_{n_i}| \leqq 1/P_n$, and so

$$\begin{aligned} |z - z_{n_i}| &\leqq |z - x| + 1/P_n \leqq |z - x| + d_n \\ &\leqq |z - x| + d(x, W(A)) = d(z, W(A)) . \end{aligned}$$

Therefore,

$$\begin{aligned} d(z, \sigma(N)) \| R(A, z) \| &\leqq |z - z_{n_i}| \, \| R(A, z) \| \\ &\leqq d(z, W(A)) / d(z, W(A)) = 1 . \end{aligned}$$

Therefore, for each complex number $z \neq 0$, $d(z, \sigma(N)) \| R(A, z) \| \leqq 1$. Since N is normal, for each $z \in \rho(T) = \rho(N)$,

$$\| R(N, z) \| = 1/d(z, \sigma(N)) = 1/d(z, \sigma(T)) .$$

Hence, for $z \in \rho(T)$

$$\| R(T, z) \| = \text{Max} \{ \| R(A, z) \|, \| R(N, z) \| \} = 1/d(z, \sigma(T))) .$$

Therefore $T \in \mathscr{G}_{loc}$.

REFERENCES

1. S. K. Berberian, *Some conditions on an operator implying normality*, Math. Ann., **184** (1970). 188-192.
2. ———, *Some conditions on an operator implying normality II*, Proc. Amer. Math. Soc., (to appear).
3. W. F. Donoghue, *On the numerical range of a bounded operator*, Michigan Math. J., **4** (1957), 261-263.
4. N. Dunford and J. T. Schwartz, *Linear Operators Part I and II*, Interscience Publishers, New York, 1957.
5. N. Dunford, *A survey of the theory of spectral operators*, Bull. Amer. Math. Soc., **64** (1958), 217-274.
6. P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand Company, Princeton, New Jersey, 1967.
7. S. Hildebrandt, *Uber den Numerischen Wertebereich eines Operators*, Math. Annalen, **163** (1966), 230-247.
8. G. R. Luecke, *Topological properties of paranormal operators*, Trans. Amer. Math. Soc., (to appear).
9. G. Orland, *On a class of operators*, Proc. Amer. Math. Soc., **15** (1964), 75-79.
10. C. R. Putnam, *Eigenvalues and boundary spectra*, Illinois J. Math., **12** (1968), 278-282.
11. ———, *The spectra of operators having resolvents of first-order growth*, Trans.

Amer. Math. Soc., **133** (1968), 505-510.
12. F. Riesz and B. Sz-Nagy, *Functional Analysis*, F. Ungar Publishing Company, New York, 1965.
13. J. G. Stampfli, *Hyponormal operators*, Pacific J. Math., **12** (1962), 1453-1458.
14. J. F. Stampfli, *Hyponormal operators and spectral density*, Trans. Amer. Math. Soc., **117** (1966), 469-476.
15. J. G. Stampfli, *Analytic extensions and spectral localization*, J. Math. Mech., **19** (1966), 287-296.
16. ———, *Minimal range theorems for operators with thin spectra*, Pacific J. Math., **23** (1967), 601-612.
17. ———, *A local spectral theorey for operators*, J. Functional Anal., **4** (1969), 1-10.
18. ———, *A local spectral theory for operators II*, Bull. Amer. Math. Soc., **75** (1969), 803-806.

Received September 21, 1970.

IOWA STATE UNIVERSITY