SIMPLE POINTS IN PSEUDOLINE ARRANGEMENTS

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A finite collection of simple closed curves in the real projective plane each two of which have exactly one point in common at which point they "cross" is called a pseudoline arrangement. Such arrangements seem to have been first systematically studied by Levi. Recently B. Grünbaum has called attention to the desirability for a better understanding of the differences as well as the similarities in the behavior of arrangements of lines and the arrangements of pseudolines.

Among other things, Grünbaum asks if every pseudoline arrangement, not all curves of which intersect in a single point, must have a simple vertex (a point on exactly two of the curves of the arrangement) as is the case in line arrangements. In fact Kelly and Moser have shown that, in general, an n-line arrangement in an ordered projective plane has at least $3/7n$ simple vertices. It is shown here that their reasoning can be nearly dualized to prove the analogous result for pseudoline arrangements.

However we make heavy use of a lemma of Levi proved in [4]. Examples show that it is possible to realize pseudoline arrangements which are not combinatorially equivalent to any line arrangement.

2. Preliminaries, definitions, notation.

DEFINITION 2.1. A finite collection, F, of simple closed curves in the real projective plane, each two of which have exactly one point in common, at which point they cross, is called a *pseudoline arrangement*. Each curve is called a *pseudoline* or, where there is no danger of confusion merely a line. The set of points of intersection of the lines of F is denoted $I(F)$ and each point of $I(F)$ is a *vertex* of F. If exactly two lines of F pass through a vertex it is *simple* or *ordinary*.

LEMMA OF LEVI. *If F is a pseudoline arrangement in the real projective plane $\mathscr{P}$ and A and B any two points of $\mathscr{P}$, then there exists a simple closed curve, g, containing A and B such that $F \cup \{g\}$ is a pseudoline arrangement.*

By successive applications of this lemma we obtain a pseudoline arrangement containing F such that each two points of $I(F)$ are on exactly one line of the arrangement. Each line of F is intersected by the lines of this extended arrangement in a finite number of points

partitioning the line into a finite number of open arcs. Let S be a set of points contained in the union of such arcs with exactly one point on each arc.

We now further extend the arrangement so that each point of S is joined by a pseudoline to each point of $I(F)$. This enlarged arrangement is denoted F^*. We are now able to confine our attention to this arrangement F^*.

We need some of the analogues of the order and separation properties of projective lines in the pseudolines of our arrangement F.

If A_1, A_2, A_3, A_4 are four points on a simple closed curve Γ and if A_1 and A_2 are in different components of $\Gamma - \{A_3, A_4\}$ we say that A_1 and A_2 *separate* A_3 and A_4 and write $A_1, A_2 \| A_3, A_4$. The usual separation axioms for separation on the real projective line are well known to be satisfied e.g. $A_1, A_2 \| A_3, A_4 \Rightarrow A_3, A_4 \| A_1, A_2$.

If g_1, g_2, g_3, g_4 are four pseudolines through a point P in an arrangement F and if $g_1 - \{P\}$ and $g_2 - \{P\}$ are in different components of $\mathscr{P} - (g_3 \cup g_4)$ then g_1 and g_2 are said to *separate* g_3 and g_4. This separation is similarly symbolized $g_1, g_2 \| g_3, g_4$.

If l is a line of F not through P and $g_i \cap l = A_i$ it follows by conventional arguments that $g_1, g_2 \| g_3, g_4$ iff $A_1, A_2 \| A_3, A_4$. This implies that if l' is a second line of F not through p with $l' \cap g_i = A_i$ then $A_1, A_2 \| A_3, A_4$ iff $A_1', A_2' \| A_3', A_4'$ i.e., separation on pseudolines is a "perspective invariant" in F.

Two points of $I(F) \cup S$ are joined by at most one line of F^*. If P and Q are two points so joined we denote this unique line of F^* by PQ. If A is a point of S and K a subset of $I(F)$ then the pencil $\{AX \mid X \in K\}$ is denoted $[A:K]$. A set of lines all but one of which go through a point is a *near pencil*.

DEFINITION 2.2. A point $P \in I(F)$ is a neighboring point (or simply a neighbor) of $l \in F$ iff there exists a point $A \in S$ on l such that AP and l are distinct and separate no two lines in the pencil $[A: I(F)]$. P is said to be a neighbor of l relative to A in F^*.

3. The principal theorem.

THEOREM 3.1. *If F is not a near pencil then each line of F has at least three neighbors in $I(F)$.*

Proof. Let l be any line of F and $A \in S \cap l$. The pencil $[A: I(F)]$ contains a pair of lines AP and AQ separating l from any other line in the pencil. P and Q are, by definition, neighbors of l relative to A.

Since F is not a near pencil, P and Q are distinct and we denote the line PQ by t. Furthermore, for the same reason, there exists a

point $R \in I(F)$ not on $t \cup l$. Let $RP \cap l = R'$, $t \cap l = Q'$ and B a point of S on the arc $R'Q'$ which does not contain A.

Let $AR \cap t = V$ and $RB \cap t = M$. By the definition of neighbor it follows that V and Q lie together on the same segment of t with end point P and Q'. On the other hand $A, B \| Q'R' \Rightarrow V, M \| Q', P$ which means that V and M are on complementary segments of t with end points P and Q'. Thus M and Q are on complementary segments with end points P and Q', i.e., $M, Q \| P, Q'$ or $BP, BQ \| BP, l$. Hence P is not a neighbor of l relative to B.

But, as above, there are two neighbors of l relative to B and it follows that l has at least three neighbors.

THEOREM 3.2. *If $P \in I(F)$ is a neighbor of three lines l, m, n of F then any line of F through P goes through one of the points $l \cap m$, $l \cap n$, $m \cap n$.*

Proof. Suppose t a line of F through P, with $t \cap l = A$ $t \cap m = B$, $t \cap n = C$. Suppose that A, B, C are distinct and that P and A separate B and C. Since P is a neighbor of l, there is a point $X \in S$ on l such that XP and l separate no pair of lines of the pencil $[X: I(F)]$. But XP and l separate XB and XC for any $X \in S$ on l. Hence t must pass through one of the points $l \cap m$, $l \cap n$, $m \cap n$.

COROLLARY 3.2.1. *Three or more lines of F meeting at one point cannot have a common neighbor.*

THEOREM 3.3. *If $P \in I(F)$ is a neighbor of a line l of F, then at most two of the points of intersection of l with lines of F through P are not simple.*

Proof. The theorem is obvious if P is simple. Suppose, then, that $m_1, m_2, \cdots, m_k$, $k > 2$, are lines of F through P and that $l \cap m_i = M_i$ for each i. If P is a neighbor of l relative to the point X we may assume that M_1 and M_2 separate X from each of the other M_i $i = 3, 4, \cdots, k$. Assume that M_j, $j > 2$, is not simple and that t is a third line (in addition to m_j and l) through M_j. If $X' = t \cap XP$ $M_1' = m_1 \cap t$ and $M_2' = m_2 \cap t$ then $M_1', M_2' \| X', M_j$ which implies that XM_1' and XM_2' separate XP and l. Thus P is not a neighbor of l relative to X. This contradiction means that each M_j, $j > 2$, is simple and proves the theorem.

COROLLARY 3.3.1. *If P is a neighbor of four lines of F, then P is a vertex of the diagonal triangle of the complete quadrilateral defined by the four lines. Furthermore P is simple, the two lines of F through P being the diagonal lines of the quadrilateral.*

Proof. If P is a neighbor of the four lines l_1, l_2, l_3, l_4 it follows from Corollary 3.2.1 that no three of these lines can be concurrent and the four lines form a complete quadrilateral, K.

By Theorem 3.2 any line of F through P must go through a pair of opposite vertices of K. Thus since there are certainly two (distinct) such lines, P is a vertex of the diagonal triangle of K.

If there were a third line of F through P it would have to be the third side of the diagonal triangle. But since the vertices of this triangle are certainly not simple we would have a contradiction of Theorem 3.3 and it follows that P must be simple.

COROLLARY 3.3.2. *A point P of $I(F)$ is a neighbor of at most four lines of F.*

DEFINITION 3.1. The number of simple points of $I(F)$ on a line l of F is the *order* of l. The number of neighbors of l which are simple is the rank of l. The order plus the rank is the *index*.

THEOREM 3.4. *The index of each line of F which is not of order 2 is at least 3.*

Proof. First observe that the theorem is true for a near pencil and dismiss this case from further consideration.

Case 1. The order of l is zero. By Theorem 3.3 every neighbor of l must be simple and by Theorem 3.1 the rank, hence the index, of l is at least 3.

Case 2. The order of l is 1.

Suppose P_1 and P_2 two nonsimple neighbors of l, M the line simple point on l and m the second line of F through M. It follows from Theorem 3.3 that there are exactly three lines of F through each of the points P_i one of which must go through M and the other two of which meet l in the distinct points A_i and B_i.

Since M is simple $P_1M = P_2M = m$. Now if $P_3 \neq P_1, P_2$ or M is a point on m then for some permutation (i, j, k) of $(1, 2, 3)$, $P_i, M \| P_j, P_k$ from which if follows that P_3 cannot be a neighbor of l. We can now conclude that there are at most two nonsimple neighbors of l. It is easy to see that there exists a point $Y \in S$ on l such that, subject to proper labelling of A_i and B_i, $Y, A_i \| M, B_i$. Since A_i is nonsimple there is a line $n \in F$ distinct from P_iA_i and l. Let $n \cap P_iY = X$, $n \cap m = Z$, $n \cap P_iB_i = W$.

$$Y, A_i \| M, B_i \Rightarrow X, A_i \| Z, W \Rightarrow YP_i, l \| YZ, YW .$$
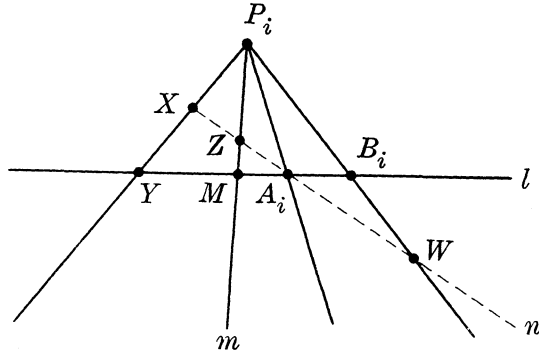
FIGURE 1

Thus P_i is not a neighbor of l relative to point Y.

But as in Theorem 3.1 there are two distinct neighbors of l relative to Y, both of which are simple. Thus the index of l is at least 3.

Case 3. The order of l is at least 3. The index of l is, by definition, at least 3.

THEOREM 3.5. *If λ_i is the index of line $l_i \in F$, then $\lambda \geqq 1/6 \sum_1^n \lambda_i$ where λ is the number of simple points of $I(F)$.*

Proof. We count the number of simple points by observing the index for each of the n lines of F. In this count a particular simple point may be counted as many as six times, four as a neighbor and two because it lies on two lines of F.

THEOREM 3.6.

$$\lambda \geqq \frac{3}{7}n \, .$$

Proof. Suppose that there are k lines of order 2. Clearly $\lambda \geqq k$. By Theorem 3.4 and 3.5,

$$\lambda \geqq \frac{3(n-k)+2k}{6} = \frac{3n-k}{6} \, .$$

Elimination of k gives the desired result.

REMARK. That the inequality is sharp is shown by the seven lines forming a complete quadrilateral and its diagonal triangle. The three vertices of the diagonal triangle are the only simple points of the configuration.

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