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SOME GEOMETRIC PROPERTIES RELATED TO THE FIXED POINT THEORY FOR NONEXPANSIVE MAPPINGS

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The main result of this paper asserts that if a Banach space admits a sequentially weakly continuous duality function, then a condition introduced by Opial to characterize weak limits by means of the norm is satisfied and the space has normal structure in the sense of Brodskii-Milman. This result of geometric nature allows some unification in the fixed point theory for both single-valued and multi-valued non-expansive mappings.

Let K be a nonempty weakly compact convex subset of a real Banach space X and let T be a nonexpansive mapping of K into its nonempty compact subsets (i.e., $D(Tx, Ty) \leqq ||x - y||$ for all $x, y \in K$, where $D(,)$ denotes the Hausdorff metric). While the question of the existence of a fixed point for T remains open, several positive results were proved recently under various conditions of geometric type on the norm of X. We list here the conditions we have in mind:

(I) (Browder [5]) X admits a sequentially continuous duality function $F_\phi: X, \sigma(X, X^*) \rightarrow X^*, \sigma(X^*, X)$ (i.e., a function F_ϕ such that $\langle x, F_\phi(x)\rangle = ||x||\,||F_\phi(x)||$ and $||F_\phi(x)|| = \phi(||x||)$ for all $x \in X$, where $\phi: \boldsymbol{R}^+ \rightarrow \boldsymbol{R}^+$ is continuous strictly increasing with $\phi(0) = 0$ and $\phi(+\infty) = +\infty$).

(II) (Opial [17]) If a sequence $\{x_n\}$ converges weakly in X to x_0, then $\liminf ||x_n - x|| > \liminf ||x_n - x_0||$ for all $x \neq x_0$.

(III) (Brodskii-Milman [4]) Every weakly compact convex subset H of X has normal structure (i.e., for each convex subset L of H which contains more than one point there exists $x \in L$ such that $\sup\{||x - y||; y \in L\} < \sup\{||u - v||; u, v \in L\}$).

When T is single-valued, the existence of a fixed point for T in K was proved by Browder [5] if X satisfies (I) and if T can be extended outside K in a nonexpansive way, and by Kirk [12] if X satisfies (III). A similar situation occurs in the multivalued case where one also encounters two different approaches: one by Browder [6] who proved a fixed point theorem under condition (I) and some additional assumptions, and another by the second author [14] who obtained the same conclusion under condition (II).

It is a consequence of our main theorem that in both cases the second approach is more general than the first[1]:

THEOREM 1. (I) *implies* (II) *and* (II) *implies* (III). *No converse implication holds, not even when* X *and* X^* *are supposed to be uniformly convex.*

Although (II) does not imply (I), there is some result in that direction, which supports the feeling that the gap between (II) and (III) is much deeper than that between (I) and (II). To state this result we define (I_0) as (I) except that F_ϕ is only required to be sequentially continuous at zero and (II_0) as (II) except that $>$ is replaced by $\geqq$.

THEOREM 2. (I_0) *implies* (II_0). *The converse implication holds when the norm of* X *is uniformly Gâteaux differentiable.*

Theorems 1 and 2 are proved in §2 and §3 respectively. Some related results and several examples are presented in §4 which shed more light on the connections between these geometric properties.

In §5 we show that the space c_0 endowed with Day's norm, which is locally uniformly convex (cf. [18]), does not satisfy (III). This example should be connected with the well-known facts that all uniformly convex spaces, as well as the spaces showed by Day and Lovaglia to be locally uniformly convex but not isomorphic to any uniformly convex space, satisfy (III) (cf. [3, 6, 11]).

2. The main result. To prove Theorem 1 we need two lemmas about the duality map $J_\phi: X \to 2^{X^*}$ defined by

$$J_\phi(x) = \{x^* \in X^*; \langle x, x^*\rangle = ||x||\,||x^*|| \text{ and } ||x^*|| = \phi(||x||)\}$$

for all $x \in X$. In this context condition (I) asserts the existence for some gauge ϕ of a sequentially weakly continuous selection for J_ϕ. The first lemma follows from the monotonicity of J_ϕ; it has been extended in [10] to general monotone operators.

LEMMA 1. *If* X *satisfies* (I) (*for some gauge* ϕ), *then* J_ϕ *is single-valued* (*for any gauge* ϕ).

Proof. The monotone operator F_ϕ is hemicontinuous by (I), thus maximal monotone by Minty's classical argument [15]. Since J_ϕ is

[1] It should be noticed that the existence theorems of [**12**] and [**14**] have been recently extended in [**13**] and [**2**] where it is only required that T maps the boundary of K into K.

also monotone we must have $J_\phi = F_\phi$. That J_ψ is single-valued for another gauge ψ follows from the equality $J_\psi(x) = \psi(\|x\|)/\phi(\|x\|)J_\phi(x)$.

The second lemma uses the observation of Asplund [1] that J_ϕ is the subdifferential of the convex function $\Phi(\|x\|)$ where

$$\Phi(t) = \int_0^t \phi(s)ds,$$

i.e.,

$$J_\phi(x) = \{x^* \in X^*;\ \Phi(\|y\|) \geqq \Phi(\|x\|) + <y - x, x^*> \forall y \in X\}\ .$$

LEMMA 2. *If J_ϕ is single-valued then*

$$\Phi(\|x + y\|) = \Phi(\|x\|) + \int_0^1 <y, J_\phi(x + ty)> dt$$

for all $x, y \in X$.

Proof. If J_ϕ is single-valued then J_ϕ is the Gâteaux gradient of $\Phi(\|x\|)$; this follows from a general result in the theory of convex functions (see [16; p. 66]). But J_ϕ is easily verified to be hemicontinuous. Consequently the lemma just expresses the fact that a function of a real variable is the integral of its continuous derivative.

Proof of Theorem 1. Assume (I) and let $x_n \rightharpoonup x_0$ ($\rightharpoonup$ will denote weak convergence, $\rightarrow$ norm convergence). By Lemmas 1 and 2,

$$\Phi(\|x_n - x\|) = \Phi(\|x_n - x_0\|) + \int_0^1 <x_0 - x, J_\phi((x_n - x_0) + t(x_0 - x))> dt$$

for all $x \in X$, so

$$\liminf \Phi(\|x_n - x\|) \geqq \liminf \Phi(\|x_n - x_0\|) + \liminf \int_0^1 \cdots dt\ .$$

The sequential weak continuity of J_ϕ and the dominated convergence theorem give

$$\liminf \Phi(\|x_n - x\|) \geqq \liminf \Phi(\|x_n - x_0\|) + \int_0^1 \|x_0 - x\| \Phi(t\|x_0 - x\|) dt\ , \tag{1}$$

an inequality which clearly implies condition (II).

The proof that (II) implies (III) relies upon a characterization of normal structure given in [11]: X satisfies (III) if and only if X does not contain a diametral sequence $\{x_n\}$ weakly converging to zero (i.e., a nonconstant sequence with

$$d(x_n; \operatorname{co}\{x_1, \cdots, x_{n-1}\}) \rightarrow \delta(\{x_n\})\ , \tag{2}$$

where $d(x_n; \mathrm{co}\{x_1, \cdots, x_{n-1}\})$ denotes the distance of x_n to the convex hull of $\{x_1, \cdots, x_{n-1}\}$ and $\delta(\{x_n\})$ the diameter of $\{x_n\}$).

Assume that (III) does not hold and take such a sequence. It follows from (2) that

$$\lim \|x_n - y\| = \delta(\{x_n\})$$

for all $y \in \mathrm{co}\{x_n\}$, hence for all $y \in \overline{\mathrm{co}}\{x_n\}$. Taking $y = 0$, we get $\lim \|x_n\| = \delta(\{x_n\})$, but for each $y = x_{n_0}$ we obtain $\lim \|x_n - x_{n_0}\| = \delta(\{x_n\})$. This contradicts (II).

We now turn to the last part of Theorem 1. When $1 < p < \infty$ $p \neq 2$, $L^p(0, 2\Pi)$ satisfies (III) since it is uniformly convex, but Opial [17] showed that even (II_0) does not hold. When $1 < p \neq q < \infty$ the Hilbert product of l^p and l^q satisfies (II) (cf. [14]; since it is easily verified that (I_0) holds, (II) also follows from Theorem 2 of §3 and Proposition 1 of §4), but Bruck [7] showed that (I) does not hold.

REMARKS 1. A finite dimensional space whose norm is not differentiable provides another example of a space satisfying (I_0), (II) but not (I) by Lemma 1.

2. In the Hilbert space case when $\phi(t) = t$, estimation (1) reduces to an estimation obtained by Opial [17; p. 592].

3. A partial converse.

The following simple lemma whose proof proceeds by taking subsequences will be needed.

LEMMA 3. *Conditions* (II) *and* (II_0) *are respectively equivalent to the analogous conditions obtained by replacing* lim inf *by* lim sup.

Proof of Theorem 2. Assume (I_0) and let $x_n \rightharpoonup x_0$. As J_ϕ is the subdifferential of $\Phi(\|x\|)$ we have

$$\Phi(\|x_n - x\|) \geqq \Phi(\|x_n - x_0\|) + \langle x_0 - x, F_\phi(x_n - x_0)\rangle$$

for all $x \in X$, so

$$\liminf \Phi(\|x_n - x\|) \geqq \liminf \Phi(\|x_n - x_0\|)\ ,$$

an inequality which clearly implies condition (II_0).

In the second part of Theorem 2, the assumption of uniform Gâteaux differentiability is equivalent to the condition that J_ϕ be singlevalued and continuous on $X, \|\ \ \|$ into $X^*, \sigma(X^*, X)$, uniformly on bounded sets; this follows easily from a result of Cudia [8; p. 302]. Under this assumption we will show that if (II_0) holds then for any ϕ, J_ϕ is sequentially weakly continuous at zero.

Let $x_n \rightharpoonup 0$ and suppose that $J_\phi(x_n)$ does not converge to zero for

$\sigma(X^*, X)$. Then there exist $z \in X$, a subsequence $\{x_m\}$ and $x^* \in X^*$ such that $< z, J_\phi(x_m) > \to < z, x^* > \neq 0$. Define

$$f(x) = \limsup \Phi(||x_m - x||) ;$$

f is a continuous convex function on X which assumes its minimum at $x = 0$ by condition (II_0) and Lemma 3. Hence

$$\frac{1}{\lambda}(f(-\lambda y) - f(0) \geqq 0 \qquad \forall\, y \in x, \forall\, \lambda > 0 ,$$

and thus

$$\limsup \frac{1}{\lambda}(\Phi(||x_m + \lambda y||) - \Phi(||x_m||)) \geqq 0 ;$$

as J_ϕ is the subdifferential of $\Phi(||x||)$,

$$\limsup < y, J_\phi(x_m + \lambda y) > \geqq 0 .$$

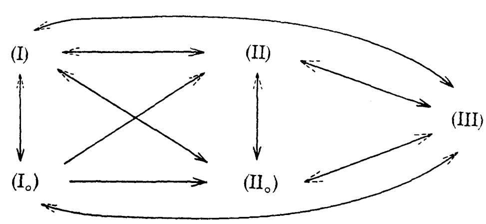
Letting $\lambda \downarrow 0$, limits can be interchanged by the uniform continuity of J_ϕ, and consequently

$$\limsup < y, J_\phi(x_m) > \geqq 0 \qquad \forall y \in X .$$

In particular $<z, x^*> = 0$, a contradiction.

REMARK. We do not know whether the differentiability hypothesis in Theorem 2 is essential.

4. Further connections. We begin this section by showing that the following situation holds in general:



where $(A) \to (B)[(A) \not\to (B)]$ means that (A) implies [does not imply] (B). Taking into account §2 and 3, it suffices to exhibit a space satisfying (I_0) but not (III). Consider the space l^2 endowed with the norm

$$||x|| = \max\left\{\frac{1}{2}||x||_{l^2}; ||x||_{l^\infty}\right\} .$$

James [3] proved that (III) does not hold; however the application

$$x = (x(1), x(2), \cdots) \longrightarrow \begin{cases} \dfrac{x}{4} \text{ if } \dfrac{1}{2}\|x\|_{l^2} \geqq \|x\|_{l^\infty}, \\ (0, \cdots, 0, x(m_0), 0, \cdots) \text{ if} \\ \qquad \dfrac{1}{2}\|x\|_{l^2} < \|x\|_{l^\infty}, \end{cases}$$

where $m_0 = \inf\{m; x(m) = \|x\|_{l^\infty}\}$ and $x(m_0)$ stands at the m_0-th place, defines a duality function which is sequentially weakly continuous at zero. Another example of a space satisfying (I_0) but not (III) will be given in §5.

Although (II_0) does not generally imply (II), one can prove the following proposition which includes Opial's result of [17; lemma 3].

PROPOSITION 1. (II_0) *implies* (II) *when X is uniformly convex.*

This proposition as well as Opial's result are no longer true if the assumption of uniform convexity is weakened to that of local uniform convexity (see the example in §5).

Proof of Proposition 1. Let $x_n \rightharpoonup x_0$ and suppose that

$$\liminf \|x_n - x\| = \liminf \|x_n - x_0\|$$

with $x \neq x_0$. Then

$$\liminf \left\|x_n - \frac{x + x_0}{2}\right\| < \liminf \|x_n - x_0\|$$

by uniform convexity, which contradicts (II_0).

We conclude this section with two results connecting (I) with two classical rotundity conditions.

PROPOSITION 2. *If* (I) *holds then X has property* (A), *i.e.*, $x_n \to x$ *whenever* $x_n \rightharpoonup x$ *and* $\|x_n\| \to \|x\|$.

Proof. By Lemmas 1 and 2,

$$\Phi(\|x_n - x\|) = \Phi(\|x_n\|) + \int_0^1 < -x, J_\phi(x_n - tx) > dt,$$

so

$$\lim \Phi(\|x_n - x\|) = \Phi(\|x\|) + \int_0^1 < -x, J_\phi(x - tx) > dt.$$

Applying Lemma 2 to the last integral, we get $\lim \Phi(\|x_n - x\|) = 0$.

COROLLARY. *If* (I) *holds and if X is strictly convex, then X has property* (H), *i.e., X has property* (A) *and is strictly convex.*

PROPOSITION 3. *If* (I) *holds and if X is strictly convex and reflexive, then X is locally uniformly convex.*

Proof. We have to prove that $x_n \to x$ whenever $\|x_n\| \to \|x\|$ and $\|x + x_n\| \to 2\|x\|$. Since the balls in X are weakly sequentially compact, it suffices to see that any weakly convergent subsequence $\{x_m\}$ converges in norm to x. Let $x_m \rightharpoonup y$. We will show that $y = x$, and the proof will follow from Proposition 2.

By Lemmas 1 and 2,

$$\Phi(\|x_m + x\|) - \Phi(\|x_m\|) = \int_0^1 \langle x, J_\phi(x_m + tx) \rangle \, dt \; ;$$

going to the limit and applying Lemma 2 to the first member, we get

$$\int_0^1 \langle x, J_\phi(x + tx) \rangle \, dt = \int_0^1 \langle x, J_\phi(y + tx) \rangle \, dt \; . \tag{3}$$

The first integral equals $\int_0^1 \|x\| \phi(\|x + tx\|) dt$ and the second is majorized by

$$\int_0^1 \|x\| \phi(\|y + tx\|) dt \leqq \int_0^1 \|x\| \phi(\|x + tx\|) dt$$

since $\|y\| \leqq \|x\|$; consequently

$$\langle x, J_\phi(y + tx) \rangle = \|x\| \phi(\|y + tx\|) \qquad \forall t \in [0, 1] \; .$$

But the strict convexity of X means that every nonzero $x^* \in X^*$ assumes its norm at most one point of the unit sphere of X. Thus, eliminating the trivial case $x = 0$,

$$\frac{x}{\|x\|} = \frac{y + tx}{\|y + tx\|}$$

for all $t \in [0, 1]$ with $y + tx \neq 0$, an equality which gives $y = kx$ with $k \geqq 0$. It then follows from (3) that $k = 1$, which completes the proof.

5. An example.

Consider the space c_0. The formula

$$\|x\| = \sup \left[\sum_{i=1}^{\infty} 2^{-2i} x^2(\alpha_i) \right]^{1/2} ,$$

where the supremum is taken over all permutations α of N, defines on c_0 an equivalent norm [9] which is known to be locally uniformly convex [18]. We will show that c_0 endowed with this norm satisfies (I_0) but not (III).

First recall an equivalent definition of $\|\ \|$ by means of the decreasing rearrangement map D of c_0 into l^2 (cf. [9]). Given $x \in c_0$, N can be enumerated in a sequence $\{\beta_i\}$ in such a way that $|x(\beta_i)| \geqq |x(\beta_{i+1})|$ for all i and that $\beta_i \leqq \beta_j$ if $|x(\beta_i)| = x(\beta_j)|$ and $i \leqq j$. Define $(Dx)(\beta_i) = 2^{-i}x(\beta_i)$. Then $\|x\| = \|Dx\|_{l^2}$.

We assert that $D^2\colon c_0 \to l^1$ is a duality function with gauge $\phi(r) = r$ which is sequentially continuous at zero for $\sigma(c_0, l^1)$ and $\sigma(l^1, c_0)$. Indeed D^2x is defined by $(D^2x)(\beta_i) = 2^{-2i}x(\beta_i)$, and we have

$$<x, D^2x> = \sum_{i=1}^{\infty} 2^{-2i}x^2(\beta_i) = \|Dx\|_{l^2}^2 = \|x\|^2 ,$$

while for all $y \in c_0$

$$\begin{aligned} <y, D^2x> &= \sum_{i=1}^{\infty} 2^{-2i}y(\beta_i)x(\beta_i) \\ &\leqq \left[\sum_{i=1}^{\infty} 2^{-2i}y^2(\beta_i)\right]^{1/2}\left[\sum_{i=1}^{\infty} 2^{-2i}x^2(\beta_i)\right]^{1/2} \leqq \|y\|\,\|x\| \end{aligned}$$

by definition of the norm. Thus D^2 is a duality function. The continuity requirement is clearly satisfied.

To prove that (III) does not hold, we construct a diametral sequence $x_n \rightharpoonup 0$ (see the characterization of normal structure mentioned in the proof of Theorem 1). Take $x_1 = 0$ and, for $n > 1$,

$$x_n(i) = \begin{cases} 0 \text{ if } i \leqq 2^{-1}n(n-1), \\ 1 \text{ if } 2^{-1}n(n-1) < i \leqq 2^{-1}n(n-1) + n, \\ 0 \text{ if } i > 2^{-1}n(n-1) + n . \end{cases}$$

It is immediate that $<x_n, x^*> \to 0$ for each $x^* \in l^1$, thus $x_n \rightharpoonup 0$. On the other hand

$$\delta(\{x_n\})^2 = \sup_{n,m} \|x_n - x_m\|^2 = \sup_{n,m} \sum_{i=1}^{n+m} 2^{-2i} = \sum_{i=1}^{\infty} 2^{-2i}$$

but

$$d(x_{n+1}, \mathrm{co}\,\{x_1, \cdots, x_n\})^2 \geqq \sum_{i=1}^{n+1} 2^{-2i} \to \sum_{i=1}^{\infty} 2^{-2}i ,$$

and consequently the sequence is diametral.

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