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TWO BRIDGE KNOTS ARE ALTERNATING KNOTS

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H. Schubert introduced a numerical knot invariant called the bridge number of a knot. In particular, he classified the two-bridge knots and proved that they were prime knots. Later, Murasugi showed that if K is an alternating knot then the matrix of K is alternating. The latter is true of two-bridge knots. The purpose of the following is to give a somewhat unusual geometric presentation of two-bridge knots from which it will be seen that they are alternating knots.

By a knot we will mean a polygonal simple closed curve in E^3. Let C denote the unit circle in the xy plane and f a homeomorphism from C to a knot K. We will assume that K is in a regular position with respect to a projection into the $y = 0$ plane [1] and that those points of K which do not have unique images will be the crossing points of K. Let $f^{-1}(a_1), f^{-1}(a_2), \cdots, f^{-1}(a_{2n})$ be the points of C ordered clockwise where a_1 are the crossing points of K. If K has a presentation with an associated f such that a_i is an overcrossing point if and only if i is odd, then K is said to be an alternating knot. By a two-bridge knot we mean a nontrivial knot in E^3 which can be represented by two linear segments through a convex cell and two arcs on the boundary of the cell.

THEOREM 1. *If K is a two-bridge knot, then K is an alternating knot.*

Proof. We will start with K in a two-bridge representation (Fig. 1a) and apply several space homeomorphisms to E^3, so that the resulting representation of K is described by an arc 'monotonely' approaching the center of the cube and four linear segments (Fig. 1b). The proof

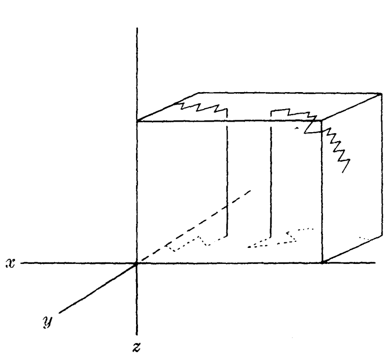

Figure 1a.

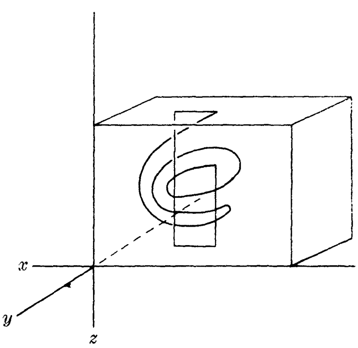

Figure 1b

will be completed by proving a lemma that shows that this representation is an alternating representation.

First assume that the knot K is respresented by two arcs $A_i = \{(x, y, z) \mid x = i/3, y = 1/2, 0 \leqq z \leqq 1\}, i = 1, 2,$ through the cube $I = \{(x, y, z) \mid 0 \leqq x \leqq 1, 0 \leqq y \leqq 1, 0 \leqq z \leqq 1\}$ and two connecting arcs on the boundary of I, i.e. B_1 and B_2. Furthermore, we can assume that $B_1 \cup B_2$ does not intersect the planes $y = 0$ and $y = 1$ (Fig. 2).

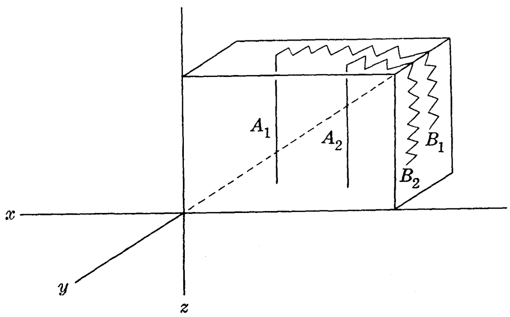

Figure 2.

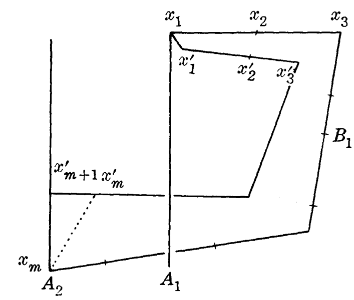

Figure 3.

The first homeomorphism h_1 will move the arc B_1 to an arc starting at the boundary and monotonely approaching the center of I so that it will not cross itself (in the y direction). h_1 will be constructed by the following five steps:

(1) Move B_1 on the boundary of I, leaving the A_i fixed, so that no segment of B_1 lies on the simple closed curve defined by (boundary of I) $\cap$ (the plane $y = 1/2$).

(2) Define L to be the cone from the center of I to B_1 and define O_t to be the annulus $\{(x, y, z) \mid \max(x - 1/2, z - 1/2) = 1/2 - t, 0 \leqq y \leqq 1\}, 0 \leqq t \leqq 1/2$.

(3) From (1) we have $L \cap (A_1 \cup A_2)$ equal to a finite set of points. Hence define ε so that the interior of $\bigcup_0^\varepsilon O_t \cap L$ contains no point of $A_1 \cup A_2$.

(4) Let $x_1, \cdots, x_m$ be the vertices of B_1 ordered from A_1 to A_2. If $1 \leqq k \leqq m$, let x'_k be the point common to $O_{k\varepsilon/m+1}$ and the linear segment joining x_k to the center of I and let $x'_{m+1} = O_\varepsilon \cap A_2$.

(5) $L \cap \bigcup_0^\varepsilon O_t$ is a disk whose intersection with K is B_1. Hence the vertices $x'_1, x'_2, \cdots, x'_m, x'_m, \cdots, x_1$ determine a simple closed curve which bounds a disk in $\bigcup_0^\varepsilon O_t$ whose intersection with K is B_1. Move B_1 to $x_1, x'_1, \cdots, x'_m, x_m$ without moving $A_1 \cup A_2 \cup B_2$. Then move $x'_{m+1}x_m x'_m$ to the segment $x'_{m+1}x'_m$ without moving the rest of K (Fig. 3).

The points of $h_1(B_1)$ approach the center of I in the sense that if x'_i, x'_j are vertices of $h_1(B_1)$ such that $i < j$ and $x'_i \varepsilon O_{t_i}, x'_j \varepsilon O_{t_j}$, then $t_i < t_j$. Hence if $h_1(K)$ is projected in the y direction, $h_1(B_1)$ will not cross itself.

As $h_1(K) \cap$ (boundary of I) $= B_2 \cup |x_1|$, we can find a homeomorphism h_2 such that h_2 is fixed on $A_1 \cup \{A_2 - |x'_{m+1}, x_m|\} \cup h_1(B_1)$ and h_2 takes B_2 to an arc on the simple closed curve formed by (boundary of I) $\cap$ (plane $y = 1/2$).

Next, we will define a homeomorphism h_3 which will move $h_1(B_1)$ so that the crossings of $h_3(h_1(B_1))$ will alternate with respect to a projection in the $y = 0$ plane and $h_3(h_1(B_1))$ will still approach the center of I monotonely. Let $b_1, b_2, \cdots, b_r$, be the crossing points of $h_1(B_1)$ ordered from A_1 and let $E_1 = A_1 \cap \{(x, y, z) | z \geqq 1/2\}$, $E_2 = A_1 \cap \{(x, y, z) | z \leqq 1/2\}$, and $E_3 = A_2 - [x_m, x_{m+1}]$. A two valued function g may be defined on $\{b_i\}$ so that $g(b_i) = 0$ if b_i is an over-crossing and $g(b_i) = u$ if b_i is an undercrossing (in the y-direction). Assume that two successive values of g are equal and then there are essentially two cases; i.e., case a, b_i and b_{i+1} both lie above (or below) E_1, E_2, or E_3, and case b, b_i lies above (or below) E_l and b_{i+1} lies above (or below) E_k with $l \neq k$.

If case a holds, then there exists t' and t'' such that $\bigcup_{t' \leqq t \leqq t''} O_t$ contains only b_i and b_{i+1} as crossings of $h_2h_1(K)$. There is an arc α, such that (1) $\alpha \subset \bigcup_{t' \leqq t \leqq t''} O_t$ (2) α has endpoints $h_1(B_1) \cap O_{t'}$ and $h_1(B_1) \cap O_{t''}$ 3) α does not cross E_1, E_2 or E_3 and (4) α monotonely approaches the center of I. Let f_i be a space homeomorphism moving $h_1(B_1) \cap \bigcup_{t' \leqq t \leqq t''} O_t$ to α and leaving $E_1 \cup E_2 \cup E_3$ and $E^3 - [\bigcup_{t \leqq t \leqq t''} O_t]$ fixed (Fig. 4).

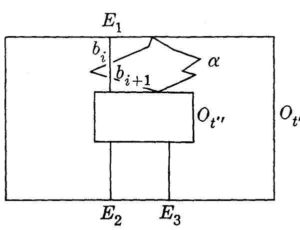

Figure 4.

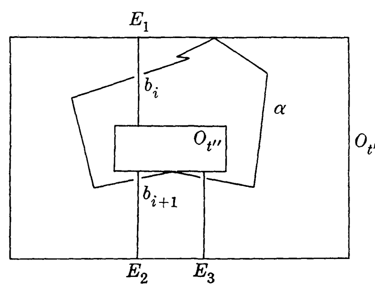

Figure 5.

If case b holds, define t', t'', and α as above, except α will cross the third E segment once in the same way that $h_1(B_1)$ crosses the other two. Define f_1 as a space homeomorphism taking $h_1(B_1) \cap \bigcup_{t' \leqq t \leqq t''} O_t$ to α and leaving $E_1 \cup E_2 \cup E_3$ and $E^3 - [\bigcup_{t' \leqq t \leqq t''} O_t]$ fixed (Fig. 4).

Hence if $h_2h_1(B_1)$ is not alternating then there exists a sequence of $\{f_i\}$ such that $f_{i_1}f_{i_2} \cdots f_{i_k}h_2h_1(B_1)$ is alternating. Let $h_3 = f_{i_1}f_{i_2} \cdots f_{i_k}$. Then $h_3h_2h_1(K)$ is alternating by the following lemma.

LEMMA 1. *Let K be a knot in regular position with respect to* *the $y = 0$ plane, and B a subarc of K such that* (1) *B does not cross itself,* (2) *every crossing of K has exactly one crossing point in B, and* (3) *the crossings of B alternate, then K is an alternating knot.*

Proof. It can be assumed that $B = \{(x, y, z) \mid 0 \leqq x \leqq 1, y = 0, z = 0\}$ and B satisfies conditions (1) through (3). If K is not an alternating knot, then there are two successive crossings of K, b_1, b_2, such that both b_1 and b_2 are overcrossings (or undercrossings). Let A be the arc joining b_1 and b_2 which has no crossings in its interior (Fig. 6). As the crossings of B alternate, A cannot lie in B.

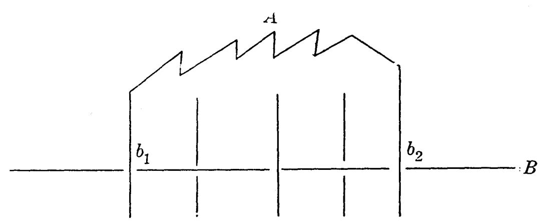

Figure 6.

A cannot contain both endpoints of B. If A contains neither endpoint of B, define C to be the simple closed curve containing A, the subarc B' of B with endpoints below (above) b_1 and b_2, and the two vertical segments joining b_1 and b_2 to their respective undercrossing (overcrossing) points. If K contains a single endpoint of B, define C to be the simple closed curve containing A, the subarc B' of B containing one of b_1 or b_2 in its interior and having as endpoints the other b_i and the endpoint of B in A, and the vertical segment joining the b_i endpoint of B' to A.

As the crossings of B alternate and b_1 and b_2 are both overcrossing points, there is an odd number of crossings on B' between b_1 and b_2, and hence an odd number of crossings on C. $C \cup K$ is the union of three simple closed curves, C, C_1, and C_2(C_2 is possibly degenerate). But $C_1 \cup C_2$ must cross C an even number of times, contradicting the fact that C is crossed an odd number of times.

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