PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 3, 1972

CARDINALITY OF $\mathfrak{k}$-COMPLETE BOOLEAN ALGEBRAS

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An infinite complete Boolean algebra satisfies $|B|^{\aleph_0} = |B|$ (where $|\ |$ denotes cardinality). This is a theorem of R. S. Pierce, derived in consequence of his general decomposition theorem [9]. It is here shown (directly) that $|B|^{\aleph_0} = |B|$ for B merely countably complete; this has the corollary (actually, equivalent) that if A is an algebra of measurable functions modulo null functions, and D is a subset of A which is dense in the uniform topology, then $|D| = |A|$. The relation $|B|^{\mathfrak{k}} = |B|$ for $\mathfrak{k}$-complete Boolean algebras B is considered; the main result is a structure theorem for the nontrivial counterexamples (which are shown to exist abundantly).

The contribution of the referee deserves special mention. In detail, he translated our original paper from topology into Boolean algebras, simplifying the results and their poofs, and he removed our use of the Generalized Continuum Hypothesis in the theorem on countably complete algebras. (We announced the latter theorem, with GCH, in [3]. Subsequently, Monk and Sparks announced the result, with no mention of GCH, in [8]; this was our first knowledge that the result was obtainable with GCH. We do not know how our methods and those of Monk and Sparks compare.)

Following the referee's advice, our setting is Boolean algebras, but we indicate the translation to topology. The means is Stone duality, of course, whereby the Boolean algebra B is isomorphic to the algebra of open-and-closed subsets of the Stone space $S(B)$ [7, 11]. It results that $|B| = wS(B)$ (where w is the weight, or least cardinal of an open basis), and that B is $\mathfrak{k}$-complete if and only if $S(B)$ has the property that the closure of $\bigcup\mathscr{U}$ is open whenever $\mathscr{U}$ is a family of clopen sets with $|\mathscr{U}| \leqq \mathfrak{k}$. Thus, the dual form of the theorem on countably complete algebras is that $(wX)^{\aleph_0} = wX$ whenever X is infinite, compact, and has the above property, commonly called basic disconnectivity. (The Stone spaces of *complete* algebras are said to be extremally disconnected.)

For B a Boolean algebra and $b \in B$, we write $(b) = \{a \in B\colon a \leqq b\}$ and we set $|b| = |(b)|$. A subset D of B is *disjointed* if $d_1 \neq d_2$ in D implies $d_1 \wedge d_2 = 0$.

1. LEMMA. *Let $\mathfrak{k}$ be infinite, B a $\mathfrak{k}$-complete Boolean algebra and D a disjointed subset of B with $|D| \geqq \mathfrak{k}$. Let $b \equiv lub\, D$. Then*

$$(b) \cong \prod_{a \in D} (a) \quad \textit{and} \quad |b| = \prod_{a \in D} |a| .$$

Proof. Define the Boolean homomorphism φ: $(b) \to \prod_{a \in D} (a)$ by the rule $(\varphi(c))_a = a \wedge c$. Clearly, φ is one-to-one. If $p \in \prod_{a \in D}(a)$, then $p_a \leqq a$ for each $a \in D$, and $p = \varphi(\text{lub}\{p_a \colon a \in D\})$ results from $\mathfrak{k}$-completeness.

2. THEOREM. *If B is an infinite, countably complete Boolean algebra, then* $|B|^{\aleph_0} = |B|$.

Proof. Suppose there is a counterexample, and choose one, B, of minimal cardinal $\mathfrak{m}$. Let $J \equiv \{b \in B \colon |b| < \mathfrak{m}\}$; if $a \in J$, and $|a|$ is infinite, then $|a|^{\aleph_0} = |a|$ by minimality of $\mathfrak{m}$. We assert that

(1) J is a σ-ideal of B; and

(2) B/J is finite, so that $|J| = \mathfrak{m}$.

Surely J is an ideal. Given a countably infinite subset D of J let $a = \text{lub}\, D$ and let D' be a countably infinite, disjointed subset of J with $b = \text{lub}\, D'$. Then

$$|b| = \textstyle\prod_{a \in D'} |a| \geqq 2^{\aleph_0}$$

by the lemma, so

$$|b|^{\aleph_0} = \textstyle\prod_{a \in D'} |a|^{\aleph_0} = \prod_{a \in D'} |a| = |b|;$$

thus $|b| \neq \mathfrak{m}$, so $|b| < \mathfrak{m}$ and $b \in J$ and (1) is proved. If (2) fails there is a disjointed sequence $\{b_n\}_{n<\omega}$ of elements of $B \backslash J$, so that (again from the lemma) one has

$$\mathfrak{m} = |B| \geqq |\text{lub}\{b_n \colon n < \omega\}| = \textstyle\prod_{n<\omega} |b_n| = \mathfrak{m}^{\aleph_0} .$$

To complete the proof let

$$Q = \{S \subset J \colon |S| \leqq \aleph_0\}$$

and for $S \in Q$ define $\varphi(S) = \text{lub}\, S$. Then $\varphi(S) \in J$ whenever $S \in Q$ (by (1)), and for $b \in J$ one has

$$\varphi^{-1}(b) \subset \{S \in Q \colon S \subset (b)\} .$$

Thus for each b in J either (b) is finite or

$$|\varphi^{-1}(b)| \leqq |b|^{\aleph_0} = |b| ,$$

so from (2) we have

$$\mathfrak{m} < \mathfrak{m}^{\aleph_0} = |J|^{\aleph_0} = |Q| \leqq \textstyle\sum_{b \in J} \aleph_0 \cdot |b| < \mathfrak{m} \cdot \aleph_0 \cdot \mathfrak{m} = \mathfrak{m} .$$

This contradiction completes the proof.

We derive as a corollary the result mentioned earlier on algeblas of measurable functions perhaps modulo an ideal of null functions. Let T be a set and $\mathscr{A} \subset 2^T$ a σ-field; let M be the real functions f with $f^{-1}(\theta) \in \mathscr{A}$ if θ is open. Let $\mathscr{N}$ be a σ-ideal of $\mathscr{A}$ and N those $f \in M$ with support in $\mathscr{N}$. Endow M/N with the metric of uniform convergence except on a member of $\mathscr{N}$. (In this metric, M/N is complete.)

3. COROLLARY. *Any dense subset of M/N has cardinality $|M/N|$.*

Proof. If X is a topological space, let $D(X)$ be the almost-finite extended real-valued functions on X. Taking X the space of maximal ideals in M/N, X is basically disconnected and M/N is isomorphic and isometric to $D(X)$ [6] (where $D(X)$ has the metric of uniform convergence on all of X). With δ denoting minimum cardinal of a dense subset, $\delta D(X) = \delta C(X)$ [4]. Since X is compact, $wX = \delta C(X)$ [11]. By the dual form of the preceding theorem, $(wX)^{\aleph_0} = wX$. But of course, $\delta C(X) \leqq |C(X)| \leqq (\delta C(X))^{\aleph_0}$, since sequences from any dense subset of $C(X)$ determine $C(X)$ (as with any metric space). The proof is complete.

The corollary applies, of course, to Lebesgue measurable functions on the reals, modulo or not the usual null functions, and to Baire and Borel functions on any space modulo or not various ideals of null functions, e.g., the ones vanishing except on meager sets. See [7, 11].

We next consider the possibility of generalizing the result on countably complete Boolean algebras. As a point of reference, consider the statement: if B is infinite and $\mathfrak{k}$-complete, and contains a disjointed family of cardinal $\mathfrak{k}$, then $|B|^{\mathfrak{k}} = |B|$. (The last hypothesis prevents the choice of complete B and relatively huge $\mathfrak{k}$.) We present a positive result under additional hypotheses, a class of counterexamples, and a theorem describing all counterexamples. For these, recall that the cofinality of the cardinal $\mathfrak{m}$, $cf(\mathfrak{m})$, is the least $\mathfrak{k}$ for which a set of $\mathfrak{m}$ points is the union of $\mathfrak{k}$ sets each of $< \mathfrak{m}$ points. $\mathfrak{m}$ is called regular if $cf(\mathfrak{m}) = \mathfrak{m}$, and otherwise singular. We encounter $cf(\mathfrak{m})$ in these considerations because $\mathfrak{m}^{cf(\mathfrak{m})} > \mathfrak{m}$ always, and with *GCH*, $cf(\mathfrak{m})$ is the least such exponent [1]. Thus, we are forced, essentially, to consider algebras B which are $cf(|B|)$-complete, and the structure theorem below (6.) is for these.

4. THEOREM. *Suppose that B is infinite, $\mathfrak{k}$-complete, has a disjointed family of cardinal $\mathfrak{k}$, and that $\mathfrak{p} < |B|$ implies $2^{\mathfrak{p}} \leqq |B|$. If either $|B|$ is regular, or for each $b \in B$ there is $a \leqq b$ with $|a| < |B|$, then $|B|^{\mathfrak{k}} = |B|$.*

This is from the original version of the present paper titled the relation $\mathfrak{m}^{\mathfrak{k}} = \mathfrak{m}$ in Stonian spaces. The proof can be found in the first author's survey [2] (in topological dual).

5. EXAMPLES. Let $\mathfrak{m}$ be singular with $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$. Let $\mathfrak{k}$ and $\mathfrak{l}$ satisfy $cf(\mathfrak{m}) \leqq \mathfrak{k} \leqq \mathfrak{l} \leqq \mathfrak{l}^{\mathfrak{k}} < \mathfrak{m}$. Let A_1 be the completion by cuts of the free Boolean algebra on $\mathfrak{m}$ generators (or equivalently, the algebra of open-and-closed subsets of the projective cover [5] of the topological space $2^{\mathfrak{m}}$); so $|A_1| = \mathfrak{m}$. Let A_2 be the algebra of $\mathfrak{k}$-sets (i.e., sets of cardinal at most $\mathfrak{k}$) and co-$\mathfrak{k}$-sets in a set of cardinal $\mathfrak{l}$; evidently, A_2 is $\mathfrak{k}$-complete, but not $\mathfrak{k}^+$-complete unless $\mathfrak{k} = \mathfrak{l}$ (in which case A_2 is complete).

The algebra $B \equiv A_1 \times A_2$ is $\mathfrak{k}$-complete and has a disjointed family of power $\mathfrak{k}$ (indeed, $\mathfrak{l}$), but

$$|B| = \mathfrak{m} < \mathfrak{m}^{cf(\mathfrak{m})} \leqq \mathfrak{m}^{\mathfrak{k}} .$$

And all examples are like this.

6. THEOREM. *Let $\mathfrak{m}$ be infinite and singular, with the property that for each $\mathfrak{p} < \mathfrak{m}$ either $2^{\mathfrak{p}} < \mathfrak{m}$ or $2^{\mathfrak{p}} = \mathfrak{p}^+$. If B is a $cf(\mathfrak{m})$-complete Boolean algebra with $|B| = \mathfrak{m}$, then $B = A_1 \times A_2$, where*

(a) $|A_1| = \mathfrak{m}$ *and* $|A_2| < \mathfrak{m}$;
(b) *if* $0 \neq b \in A_1$, *then* $|b| = \mathfrak{m}$;
(c) *if* $D \subset A_1$ *and* D *is disjointed, the* $|D| < cf(\mathfrak{m})$;
(d) A_1 *is complete.*

Proof. Define

$$J_1 = \{a \in B\colon\ |c| = \mathfrak{m} \quad \text{whenever } 0 \neq c \in (a)\}$$

and $J_2 = \{b \in B\colon |b| < \mathfrak{m}\}$, so that J_1 and J_2 are disjoint ideals in B.

Given a disjointed subset D of J_1 with $|D| \leqq cf(\mathfrak{m})$ we have from the lemma

$$\mathfrak{m} = |B| = |\operatorname{lub} D| = \prod_{a \in D} |a| = \mathfrak{m}^{|D|} ,$$

so that $|D| < cf(\mathfrak{m})$. If follows that $J_1 = (a_1)$ for some element a_1 of J_1: given a maximal disjointed subset D of J_1 we have $|D| < cf(\mathfrak{m})$ from the computation above, so that $\operatorname{lub} D = a_1$ exists in B; evidently $a_1 \in J_1$, and $J_1 = (a_1)$ by maximality. Another consequence is that (a_1) is a complete Boolean algebra: given a subset S of (a_1) some disjointed subset D of (a_1) is maximal with respect to the property that it refines S (i.e., for each d in D, $d \in (s)$ for some s in S), so that $\operatorname{lub} D$ exists in (a_1).

Let a_2 be the complement in B of (a_1), and set $A_1 = (a_1)$ and $A_2 = (a_2)$. The other assertions being obvious, it remains only to show that $|A_2| < \mathfrak{m}$.
We claim first:

(*) if E is a disjointed subset of J_2 and $\mathfrak{p} = \sum_{b \in E} |b|$, then $\mathfrak{p} < \mathfrak{m}$. If $\mathfrak{p} \geqq \mathfrak{m}$ then either $|E| = \mathfrak{m}$ or $\sup\{|b|: b \in E\} = \mathfrak{m}$. In the former case we would have

$$\mathfrak{m} = |B| = |\{S \subset E: |S| = cf(\mathfrak{m})\}| = \mathfrak{m}^{cf(\mathfrak{m})} > \mathfrak{m} ,$$

and in the latter, replacing E if necessary by a subset E' for which $|E'| = cf(\mathfrak{m})$ and $\sup\{|b|: b \in E'\} = \mathfrak{m}$,

$$\mathfrak{m} \geqq |B| \geqq |\operatorname{lub} E'| = \prod_{b \in E'} |b| > \mathfrak{m} .$$

In either case a contradiction is achieved and (*) is proved.

Now let E be a maximal, disjointed subset of (a_2), let $|E| = \mathfrak{p} < \mathfrak{m}$, and set $K = \bigcup_{b \in E}(b)$. Because E is maximal the map $c \to K \bigcap (c)$ is one-to-one from (a_2) to the power set of K. Thus $|a_2| \leqq 2^{\mathfrak{p}}$ (and $\mathfrak{p} < \mathfrak{m}$ by (*)). The assertion $|a_2| < \mathfrak{m}$ now follows from the cardinality hypothesis of this theorem together with the fact that $\mathfrak{p}^+$ is a regular cardinal.

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Received November 2, 1970. The authors gratefully acknowledge partial support received from the National Science Foundation (U.S.A.) under grants NSF-GP-8357 and NSF-GP-18825.

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