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A QUASI-KUMMER FUNCTION

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A particular integral of Kummer's inhomogeneous differential equation is obtained when the right hand member belongs to a general class of multiform functions. A few basic properties of the solution function are established.

1. Introduction. Let σ be a complex constant but not negative real. We denote by $\mathscr{R}_\sigma$ the Riemann surface of z^σ. Suppose $f(z)$ to be analytic everywhere on the disc K_R: $|z| < R$, and let

$$\mathscr{K}_R{}^\sigma \equiv \{\phi(z) \,|\, \phi = z^\sigma f(z);\ f \text{ analytic on } K_R\} .$$

Then, to each element of the multiplier space $\mathscr{R}_\sigma$, there corresponds one and only one element of the product space $\mathscr{K}_R{}^\sigma$ which is regular everywhere in the domain K_R of the analytic component $f(z)$ slit and screwed in the usual way, if necessary. A few subspaces of $\mathscr{K}_R{}^\sigma$ are:

$\mathscr{K}^\sigma_{R(\rho)}$: $\{\phi \,|\, K_R,\ R \leqq \rho < \infty\}$

$\mathscr{K}^\sigma_{\pi(p)}$: $\{\phi \,|$ analytic component a polynomial of degree $p\}$

and

$\mathscr{K}^\sigma_{\infty(k)}$: $\{\phi \,|\, |f^{(m)}(0)| \leqq Bk^m,\ (B, k) > 0\}$.

Now consider the equation

$$z\,\frac{d^2W}{dz^2} + (b - z)\,\frac{dW}{dz} - aw = \phi(z) . \tag{1.1}$$

The associated homogeneous problem leads to Kummer's confluent hypergeometric and other well-known transcendental functions. But the properties of the particular integral of the inhomogeneous equation have been studied in detail only recently by Babister [1] who has considered a few particular cases. In this paper we take the general classes $\mathscr{K}^\sigma_{R(\rho)}$, $\mathscr{K}^\sigma_{\infty(e)}$, $\mathscr{K}^\sigma_{\pi(p)}$ and use Frobenius's method to show that in each case a particular integral of (1.1) exists and belongs to some similar subspace of $\mathscr{K}^{\sigma+1}_R$. We also give some basic properties of the solution-function which we have called quasi-Kummer.

As $f(z)$ is analytic on K_R ,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n,\ |z| < R .$$

Accordingly, a formal series solution of (1.1) is given by

$$(1.2)\qquad {}_1A_1\left(\begin{matrix}\sigma \\ z\end{matrix}\middle|\begin{matrix}a \\ b\end{matrix};\ f(z)\right) = \sum_{n=0}^{\infty} \frac{(\sigma + a + 1)_n P_n(\sigma; a, b; f)}{(\sigma + 1)_{n+1}(\sigma + b)_{n+1}}\, z^{n+\sigma+1},$$

where

$$(1.3)\qquad P_n(\sigma; a, b; f) = \sum_{m=0}^{n} \frac{(\sigma + 1)_m(\sigma + b)_m}{(\sigma + a + 1)_m} \frac{f^{(m)}(0)}{m!}$$

and $(\nu)_n$ denotes the Pochhammer product $\nu(\nu + 1) \cdots (\nu + n - 1)$.

2. Some subsidiary results. In order to establish our main results, we require some formulae which will be stated in the form of lemmas. For convenience we write:

$$\begin{aligned} \alpha &= |\sigma + a + 1| & \lambda &= \sec(1/2 \arg(\sigma + a + 1)) \\ \beta &= |\sigma + b| & \mu &= \sec(1/2 \arg(\sigma + b)) \\ \gamma &= |\sigma + 1| & \nu &= \sec(1/2 \arg(\sigma + 1)). \end{aligned}$$

Also, it is assumed that α, β, γ are all finite and positive.

LEMMA 1. *For* $0 < R \leqq \lambda < \infty \operatorname{Max}_{|z|=R} |f(z)| = M(R)$

$$|P_n(\sigma; a, b; f)| \leqq \begin{cases} \text{(i)}\quad \dfrac{(\gamma)_n\lambda^{n-1}}{|\beta - \alpha + 1|}\left|\dfrac{(\beta)_{n+1}}{(\alpha)_n} + 1 - \alpha\right|\dfrac{M(R)}{R^n}, & \beta - \alpha + 1 \neq 0 \\ \text{(ii)}\quad \dfrac{(\beta)_n\lambda^{n-1}}{|\gamma - \alpha + 1|}\left|\dfrac{(\gamma)_{n+1}}{(\alpha)_n} + 1 - \alpha\right|\dfrac{M(R)}{R^n}, & \gamma - \alpha + 1 \neq 0 \\ \text{(iii)}\quad (\gamma)_n(n + 1)\lambda^{n-1}\dfrac{M(R)}{R^n}, & \beta = \gamma = \alpha - 1 > 0 \end{cases}$$

Proof. By Cauchy's inequality

$$|P_n(\sigma; a, b; f)| \leqq \sum_{m=0}^{n} \left|\frac{(\sigma + 1)_m(\sigma + b)_m}{(\sigma + a + 1)_m}\right| \frac{M(R)}{R^m},$$

which on applying Erber's estimate [3]:

$$\frac{1}{|(\delta)_n|} \leqq \frac{\sec^{n-1}(1/2 \arg \delta)}{(|\delta|)_n},\quad |\arg \delta| < \pi$$

and after simplification proves the first part of the lemma. The second part follows mutatis mutandis on replacing β by γ. Also in the third case $|P_n(\sigma; a, b; f)| \leqq (\beta)_n\lambda^{n-1}\gamma \sum_{m=0}^{n} 1/\gamma + m$. Hence the result.

LEMMA 2. *If* $|f^{(m)}(0)| \leqq Bk^m$, $B > 0$, $k > 0$, *then*

$$|P_n(\sigma; a, b; f)| \leqq \begin{cases} \dfrac{B}{\lambda}\, {}_2F_1\left[\begin{matrix}\gamma, \beta \\ \alpha\end{matrix}; k\lambda\right] & k\lambda < 1 \\ \dfrac{Bl^n}{\lambda}\, {}_2F_1\left[\begin{matrix}\gamma, \beta \\ \alpha\end{matrix}; \dfrac{k\lambda}{l}\right] & 1 \leqq k\lambda < l, \end{cases}$$

Proof.

$$|P_n(\sigma; a, b; f)| \leqq B\left|\sum_{m=0}^{n}\frac{(\sigma+1)_m(\sigma+b)_m}{(\sigma+a+1)_m}\frac{k^m}{m!}\right|$$
$$\leqq \frac{Bl^n}{\lambda}\sum_{m=0}^{n}\frac{(\gamma)_m(\beta)_m}{(\alpha)_m m!}\left(\frac{k\lambda}{l}\right)^m, \ 1 \leqq k\lambda < l$$

which leads to the second part. The proof of the first part is very straightforward.

3. Main Theorems. By Lemma 1 (i), the modulus of the general coefficient in the power-series (1.2) can be majorised by

$$\frac{(\alpha)_n(\gamma)_n}{(\gamma)_{n+1}(\beta)_{n+1}}\left(\frac{\lambda\mu\nu}{R}\right)^n\left|\frac{(\beta)_{n+1}}{(\alpha)_n}+1-\alpha\right|\frac{M(R)}{\lambda|\beta-\alpha+1|}.$$

Hence the series converges absolutely and uniformly to an analytic function for all $|z| < R/\lambda\mu\nu$. Another majorant is provided by Lemma 1 (ii) both leading to

THEOREM 1. *If* $\phi \in \mathscr{K}^{\sigma}_{R(\lambda)}$, $(\lambda, \mu, \nu) < \infty$, *then* ${}_1A_1\left(\begin{matrix}\sigma\\ z\end{matrix}\middle|\begin{matrix}a\\ b\end{matrix}; f(z)\right) \in \mathscr{K}^{\sigma+1}_{R(\rho)}$, $\rho\lambda\mu\nu \leqq R$ *and* $\left|{}_1A_1\left(\begin{matrix}\sigma\\ z\end{matrix}\middle|\begin{matrix}a\\ b\end{matrix} f(z)\right)\right|$ *never exceeds*

$$\frac{M(R)|z^{\sigma+1}|}{\beta\gamma\lambda|\beta-\alpha+1|}\left\{\beta\,{}_2F_1\left[\begin{matrix}\gamma,\ 1\\ \gamma+1\end{matrix};\frac{\lambda\mu\nu|z|}{R}\right]+{}_3F_2\left[\begin{matrix}\alpha,\ \gamma,\ 1\\ \beta+1,\ \gamma+1\end{matrix};\frac{\lambda\mu\nu|z|}{R}\right]\right\},$$
$$\beta-\alpha+1\neq 0\,,$$

or

$$\frac{M(R)|z^{\sigma+1}|}{\beta\gamma\lambda|\gamma-\alpha+1|}\left\{\gamma\,{}_2F_1\left[\begin{matrix}\beta,\ 1\\ \beta+1\end{matrix};\frac{\lambda\mu\nu|z|}{R}\right]+{}_3F_2\left[\begin{matrix}\alpha,\ \beta,\ 1\\ \beta+1,\ \gamma+1\end{matrix};\frac{\lambda\mu\nu|z|}{R}\right]\right\},$$
$$\gamma-\alpha+1\neq 0$$

or

$$\frac{1}{\beta\gamma\lambda}\,{}_2F_2\left[\begin{matrix}\gamma,\ \ 2\\ \gamma+1,\ 1\end{matrix};\frac{\lambda\mu\nu|z|}{R}\right],\ \beta=\gamma=\alpha-1>0\,.$$

Similarly from Lemma 2, we easily obtain

THEOREM 2. *If* $\phi \in \mathscr{K}^{\sigma}_{\infty(k)}$, $(\lambda, \mu, \nu) < \infty$, *then* ${}_1A_1\left(\begin{matrix}\sigma\\ z\end{matrix}\middle|\begin{matrix}a\\ b\end{matrix}; f(z)\right) \in \mathscr{K}^{\sigma+1}_{\infty}$ *and* $\left|{}_1A_1\left(\begin{matrix}\sigma\\ z\end{matrix}\middle|\begin{matrix}a\\ b\end{matrix}; f(z)\right)\right|$ *is dominated by*

$$\frac{B|z^{\sigma+1}|}{\beta\gamma\lambda}\ {}_2F_1\left[\begin{matrix}\gamma, \beta\\ \alpha\end{matrix}; k\lambda\right]{}_2F_2\left[\begin{matrix}\alpha, 1\\ \gamma+1, \beta+1\end{matrix}; \mu\nu|z|\right], k<\frac{1}{\lambda}$$

$$\frac{B|z^{\sigma+1}|}{\beta\gamma\lambda}\ {}_2F_1\left[\begin{matrix}\gamma, \beta\\ \alpha\end{matrix}; \frac{k\lambda}{l}\right]{}_2F_2\left[\begin{matrix}\alpha, 1\\ \gamma+1, \beta+1\end{matrix}; \mu\nu l|z|\right], 1\leqq k\lambda<l\,.$$

Now, if $f(z)$ is a polynomial of degree p, then for all nonnegative integers r, $P_{p+r}(\sigma; a, b; f) = P_p(\sigma; a, b; f)$. Hence, denoting the set of nonpositive integers by Z^{0-} we have

THEOREM 3. *If* $\phi \in \mathscr{K}^{\sigma}_{\pi(p)}$, $(\sigma+1, \sigma+a+1, \sigma+b) \notin Z^{0-}$, *then*

$$ {}_1A_1\left(\begin{matrix}\sigma\\ z\end{matrix}\middle|\begin{matrix}a\\ b\end{matrix}; f(z)\right) = \sum_{n=0}^{p-1}\frac{(\sigma+a+1)_n P_n(\sigma; a, b; f)}{(\sigma+1)_{n+1}(\sigma+b)_{n+1}}\, z^{\sigma+1+n}$$
$$+\frac{(\sigma+a+1)_p P_p(\sigma; a, b; f)}{(\sigma+1)_{p+1}(\sigma+b)_{p+1}}\, z^{\sigma+1+p}\, {}_2F_2\left[\begin{matrix}\sigma+a+1+p, 1\\ \sigma+b+1+p, \sigma+2+p\end{matrix}; z\right].$$

4. **Contiguity relations.** As the generalized power series (1.2) is uniformly convergent, a number of interesting contiguity relations can be obtained by applying the operator d/dz or $\delta(\equiv z\, d/dz)$ termwise. For example

$$\text{(4.1)}\quad \begin{aligned}\frac{d}{dz}\,{}_1A_1\left(\begin{matrix}\sigma\\ z\end{matrix}\middle|\begin{matrix}a\\ b\end{matrix}; f(z)\right) &= \sigma\ {}_1A_1\left(\begin{matrix}\sigma-1\\ z\end{matrix}\middle|\begin{matrix}a+1\\ b+1\end{matrix}; f(z)\right)\\ &+ {}_1A_1\left(\begin{matrix}\sigma\\ z\end{matrix}\middle|\begin{matrix}a+1\\ b+1\end{matrix}; \frac{df}{dz}\right).\end{aligned}$$

$$\text{(4.2)}\quad \begin{aligned}(b-a-1)\,{}_1A_1\left(\begin{matrix}\sigma\\ z\end{matrix}\middle|\begin{matrix}a\\ b\end{matrix}; f(z)\right) &= (\sigma+b-1)\,{}_1A_1\left(\begin{matrix}\sigma\\ z\end{matrix}\middle|\begin{matrix}a\\ b-1\end{matrix}; f(z)\right)\\ &-(\sigma+a+1)\,{}_1A_1\left(\begin{matrix}\sigma\\ z\end{matrix}\middle|\begin{matrix}a+1\\ b\end{matrix}; f(z)\right)+{}_1A_1\left(\begin{matrix}\sigma+1\\ z\end{matrix}\middle|\begin{matrix}a\\ b-1\end{matrix}\frac{df}{dz}\right)\\ &-{}_1A_1\left(\begin{matrix}\sigma+1\\ z\end{matrix}\middle|\begin{matrix}a+1\\ b\end{matrix}\frac{df}{dz}\right).\end{aligned}$$

Now, as ${}_1A_1$ can be written as

$$\frac{z^{\sigma+1}f(0)}{(\sigma+1)(\sigma+b)}+\frac{(\sigma+a+1)z^{\sigma+2}}{(\sigma+1)(\sigma+b)}\sum_{n=0}^{\infty}\frac{(\sigma+a+2)_n z^n}{(\sigma+2)_{n+1}(\sigma+b+1)_{n+1}}\Bigg\{f(0)$$
$$+\sum_{m=1}^{n-1}\frac{(\sigma+1)_m(\sigma+b)_m}{(\sigma+a+1)_m}\,\frac{f^{(m)}(0)}{m!}\Bigg\},$$

we have on simplification

$$\text{(4.3)}\quad {}_1A_1\left(\begin{matrix}\sigma\\ z\end{matrix}\middle|\begin{matrix}a\\ b\end{matrix}; f(z)\right)-{}_1A_1\left(\begin{matrix}\sigma+1\\ z\end{matrix}\middle|\begin{matrix}a\\ b\end{matrix}; \frac{f(z)-f(0)}{z}\right)$$

$$= \frac{f(0)z^{\sigma+1}}{(\sigma+1)(\sigma+b)} + \frac{(\sigma+a+1)f(0)z^{\sigma+2}}{(\sigma+1)_2(\sigma+b)_2}\,{}_2F_2\left[\begin{matrix}\sigma+a+2, & 1\\ \sigma+b+2, & \sigma+3\end{matrix}; z\right].$$

As particular cases, we see that

$${}_1A_1\left(\begin{matrix}\sigma-1\\ z\end{matrix}\middle|\begin{matrix}a\\ b\end{matrix}; e^{\rho z}\right) = \Lambda_{\rho,\sigma}(a, b; z) \text{ and } {}_1A_1\left(\begin{matrix}\sigma-1\\ z\end{matrix}\middle|\begin{matrix}a\\ b\end{matrix}; 1\right) = \theta_\sigma(a, b; z)$$

where Λ and θ are Babister's nonhomogeneous confluent functions, so (4.1) and (4.3) reduce to known results [1], (4.236), (4.189). Also from (4.3)

$$\begin{aligned}
&{}_1A_1\left(\begin{matrix}\sigma\\ z\end{matrix}\middle|\begin{matrix}a\\ b\end{matrix}; {}_pF_q\left[\begin{matrix}a_1, \cdots, a_p\\ b_1, \cdots, b_q\end{matrix}; z\right]\right)\\
&\quad - \frac{a_1\cdots a_p}{b_1\cdots b_q}\,{}_1A_1\left(\begin{matrix}\sigma\\ z\end{matrix}\middle|\begin{matrix}a\\ b\end{matrix}; {}_{p+1}F_{q+1}\left[\begin{matrix}a_1+1, \cdots, a_p+1 & 1\\ b_1+1, \cdots, b_q+1 & 2\end{matrix}; z\right]\right)\\
&= \frac{z^{\sigma+1}}{(\sigma+1)(\sigma+b)} + \frac{(\sigma+a+1)z^{\sigma+2}}{(\sigma+1)_2(\sigma+b)_2}\,{}_2F_2\left[\begin{matrix}\sigma+a+2, & 1\\ \sigma+b+2, & \sigma+3\end{matrix}; z\right]
\end{aligned} \tag{4.4}$$

with the usual restriction on the parameters.

5. Illustration. The above results are of particular advantage when the analytic component of ϕ involves functions of hypergeometric type because these (for that matter, almost all) special functions belong to one of the classes considered.
For example: (See Table on next page).

The first four are Babister's nonhomogeneous confluent functions, the next three are obtained via a result due to Carlitz [2]:

$${}_5F_4\left[\begin{matrix}a, 1+a/2, b, c, d;\\ a/2, 1+a-b, 1+a-c, 1+a-d\end{matrix}\right]_n = \frac{(1+a)_n(1+b)_n(1+c)_n(1+d)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n n!}$$

provided that $a = b + c + d$. In the last two cases $P_n(\sigma; a, b; f) = ((\sigma+1)_n)/n!$ or $n+1$ respectively yielding the results with the usual restriction on the parameters.

Some other properties of ${}_1A_1\left(\begin{matrix}\sigma\\ z\end{matrix}\middle|\begin{matrix}a\\ b\end{matrix}; f(z)\right)$ will be discussed in another communication.

σ	$f(z)$	${}_1A_1\left(\begin{matrix}\sigma \\ z\end{matrix}\middle\vert\begin{matrix}a \\ b\end{matrix}; f(z)\right)$
0	$\dfrac{2^{1-b}\Gamma(b)}{\Gamma(a)\Gamma(b-a)}e^{z/1}$	$\Omega(a, b; z)$
$1-b$	$\dfrac{2^{b-1}\Gamma(2-b)e^{z/2}}{\Gamma(a-b+1)\Gamma(1-a)}$	$\overline{\Omega}(a, b; z)$
$\sigma-1$	1	$\theta_\sigma(a, b; z)$
$\sigma-1$	$e^{\rho z}$	$\Lambda_{\rho,\sigma}(a, b; z)$
σ	${}_3F_3\left[\begin{matrix}2\sigma+a+b, \sigma+1+\frac{1}{2}(a+b), a-1 \\ 2\sigma+b+2, \sigma+\frac{1}{2}(a+b), \sigma+a+b\end{matrix}; z\right]$	$\dfrac{z^{\sigma+1}}{(\sigma+1)(\sigma+b)}\,{}_2F_2\left[\begin{matrix}2\sigma+a+b+1, a \\ 2\sigma+b+2, \sigma+a+b\end{matrix}; z\right]$
σ	${}_3F_3\left[\begin{matrix}2\sigma+a+1, \sigma+\frac{1}{2}(a+3), a-b \\ 2\sigma+b+2, \sigma+\frac{1}{2}(a+1), \sigma+a-b+2\end{matrix}; z\right]$	$\dfrac{z^{\sigma+1}}{(\sigma+1)(\sigma+b)}\,{}_2F_2\left[\begin{matrix}2\sigma+a+2, 1+a-b \\ 2\sigma+b+2, \sigma-a-b+2\end{matrix}; z\right]$
σ	${}_3F_3\left[\begin{matrix}2\sigma+2a+2, \sigma+a+2, 2a-b+1 \\ 2\sigma+b+2, 2a+\sigma+2, 2a-b+\sigma+2\end{matrix}; z\right]$	$\dfrac{z^{\sigma+1}}{(\sigma+1)(\sigma+b)}\,{}_3F_3\left[\begin{matrix}\sigma+a+1, 2\sigma+2a+3, 2a-b+2 \\ 2\sigma+b+2, 2a+\sigma+2, \sigma+2a-b+2\end{matrix}; z\right]$
σ	${}_1F_1\left[\begin{matrix}\sigma+a+1 \\ \sigma+b\end{matrix}; z\right]$	$\dfrac{z^{\sigma+1}}{(\sigma+1)(\sigma+b)}\,{}_1F_1\left[\begin{matrix}\sigma+a+1 \\ \sigma+b+1\end{matrix}; z\right]$
σ	${}_2F_2\left[\begin{matrix}\sigma+a+1, 1 \\ \sigma+b, \sigma+1\end{matrix}; z\right]$	$\dfrac{(\sigma+a+1)z^{\sigma+2}}{(\sigma+1)_2(\sigma+b)_2}\,{}_2F_2\left[\begin{matrix}\sigma+a+2, 2 \\ \sigma+b+2, 3\end{matrix}; z\right].$

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