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A COMBINATORIAL PROBLEM; STABILITY AND ORDER FOR MODELS AND THEORIES IN INFINITARY LANGUAGES

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Some infinite combinatorial problems of Erdös and Makkai are solved, and we use them to investigate the connection between unstability and the existence of ordered sets; we also prove the existence of indiscernible sets under suitable conditions.

0. Introduction. In § 1 we deal with combinatorial problems raised by Erdös and Makkai in [5] (they appear later in Erdös and Hajnal [3], [18] Problem 71).

Let us define: $P2(\lambda, \mu, \alpha)$ holds when for every set A of cardinality μ, and family S of subsets of A of cardinality λ, there are $a_k \in A$, $X_k \in S$ for $k < \alpha$, such that either $k, l < \alpha$ implies $a_k \in X_l \Leftrightarrow k < l$ or $k, l < \alpha$ implies $a_k \in X_l \Leftrightarrow l \leqq k$.

Erdös and Makkai proved in [5] that if $\lambda > \mu \geqq \aleph_0$, then $P2(\lambda, \mu, \omega)$ holds. Assuming G.C.H. for simlicity only, our theorems imply $P2(\aleph_{\beta+2}, \aleph_{\beta+1}, \aleph_\beta)$ holds for every β.

In § 2 we mainly generalize results on stability from Morley [9] and Shelah [12] to models, and theories of infinitary languages. We first deal with stable models. Let M be a model, L the first-order language associated with it, Δ a set of formulas of $L_{\lambda^+,\omega}$ (for any λ) each with finite number of free variables. We shall assume Δ is closed under some simple operations. M is (Δ, λ)-stable, if for each $A \subset |M|$, $|A| \leqq \lambda$, the elements of M realize over A no more than λ different Δ-types. Let $\lambda \in Od_\Delta(M)$ if there is $\varphi(\bar{x}, \bar{y}) \in \Delta$ and sequences $\bar{a}^k$, $k < \lambda$, of elements of M such that for every $k, l < \lambda$, $M \models \varphi[\bar{a}^k, \bar{a}^l]$ if and only if $k < l$.

By Theorem 2.1, if M is not (Δ, κ)-stable $\kappa^{|\Delta|} = \kappa$, $\kappa = \sum_{\mu<\lambda}(\kappa^\mu + 2^{2^\mu})$, then $\lambda \in Od_\Delta(M)$. Theorem 2.2 says that if M is (Δ, λ)-stable, $\lambda \notin Od_\Delta(M)$, $\|M\| > \lambda$, $A \subset |M|$, $|A| \leqq \lambda$, and the cofinality of λ is $> |\Delta|$, then in M there is an indiscernible set over A of cardinality $> \lambda$. This generalizes Theorem 4.6 of Morley [9] for models of totally transcendental theories.

A theory T, $T \subset L_{\lambda^+,\omega}$ for some λ, is (Δ, μ)-stable, if every model of T is (Δ, μ)-stable. By Theorem 2.4, if $T, \Delta \subset L_{\lambda^+,\omega}$ $|T| \leqq \lambda$, and $\mu(\lambda) \in Od_\Delta(M)$ for some model M of T, *then* for every κ, T is not (Δ, κ)-stable. This is a converse of Theorem 2.1. (Morley [9] proved a particular case of this theorem (3.9) that if T is a first-order, countable, complete, totally trancendental theory, (i.e., T is $(\Delta, \aleph_0)$-stable, where Δ is the set of all formulas of L), then $\aleph_0 \notin Od_\Delta(M)$ for any model M of T. (In fact he used a little stronger definition for $\aleph_0 \in Od_\Delta(M)$.)

By Theorem 2.5, if $T \subset L_{\lambda^+,\omega}$, and Δ is arbitrary, and for every κ, T is not (Δ, κ)-stable, then for some $\Delta_1 \subset L_{\lambda^+,\omega}$, $|\Delta_1| \leqq \lambda$, T is (Δ_1, κ)-unstable for every κ. By Shelah [16], we deduce that for every $\kappa > |T| + \lambda$, T has 2^κ nonisomorphic models of cardinality κ.

NOTATIONS. Let $\lambda, \kappa, \mu, \chi$ denote cardinals (infinite, if not clear otherwise). Let $\alpha, \beta, \gamma, i, j, k, l$ denote ordinals and m, n denote natural numbers. We shall indentify cardinals with initial ordinals, and $\aleph_\alpha$ will be the αth infinite cardinal ($\aleph_0$-the first). The first infinite ordinal is denoted by ω. λ^+ is the first cardinal greater than λ. $|A|$ is the cardinality of the set A.

1. Combinatorial problems.

Let A denote a set, S a family of subsets of A. Let $A\ (-)\ S$ be the family $\{A - B : B \in S\}$. A^α is the set of sequences of length α of A; and if $\bar{a} \in A^\alpha$, $l(\bar{a}) = \alpha$ and $\bar{a}_\beta$ is the βth element in the sequence. After Erdös and Makkai [5], $\bar{a}$ if strongly cut by S if for every $\beta < \alpha$, there is $X_\beta \in S$ such that $a_\gamma \in X_\beta \Leftrightarrow \gamma < \beta$ for every $\gamma, \beta < \alpha$. Erdös and Makkai [5] proved that is $|S| > |A| \geqq \aleph_0$, then there is a sequence $\bar{a} \in A^\omega$ which is strongly cut by S or by $A\ (-)\ S$. They asked several questions ([5] p. 159 and [3] problem 71 p. 45). We shall here answer some of their questions.

Let us define

DEFINITION 1.1. $P1(\lambda, \mu, \alpha)$ holds, if $|S| = \lambda, |A| = \mu$ implies there are $\bar{a}, \bar{b} \in A^\alpha$, $\bar{X} \in S^\alpha$ such that: for every $\beta, \gamma < \alpha$,

$$\bar{a}_\beta \in \bar{X}_\gamma \Leftrightarrow \bar{b}_\beta \in \bar{X}_\gamma \text{ if and only if } \gamma < \beta.$$

DEFINITION 1.2. $P2(\lambda, \mu, \alpha)$ holds, if $|S| = \lambda, |A| = \mu$ implies there are $\bar{a} \in A^\alpha$, $\bar{X} \in S^\alpha$ such that:

$$\text{either } \beta, \gamma < \alpha \text{ implies } \bar{a}_\beta \in \bar{X}_\gamma \Longleftrightarrow \beta < \gamma$$

or

$$\beta, \gamma < \alpha \text{ implies } \bar{a}_\beta \in \bar{X}_\gamma \Longleftrightarrow \gamma \leqq \beta\ .$$

REMARK. This means that $\bar{a}$ is strongly cut by S or by $A\ (-)\ S$.

DEFINITION 1.3. $P3(\lambda, \mu, \alpha)$ holds if $|S| = \lambda, |A| = \mu$ implies there are $\bar{a} \in A^{\alpha}$, $\bar{X} \in S^{\alpha}$ such that for every $\beta, \gamma < \alpha$, $\bar{a}_{\beta} \in \bar{X}_{\gamma} \Leftrightarrow \beta < \gamma$.

REMARK. This means $\bar{a}$ is strongly cut by S.

NOTATION. In each of $P1, P2, P3$ we shall always implicitly assume $2^{\mu} \geqq \lambda > \mu$. For otherwise, those relations are not interesting.

Clearly, the theorem of [5] is by our notation, that $P2(\lambda^{+}, \lambda, \omega)$ holds. Let us now list the results proved here about those three properties.

THEOREM 1.1. *For every* λ, $P3(\lambda^{+}, \lambda, \omega)$ *does not hold.* (This solves negatively problem 1 in [5], which is the same as problem 71A, in [3] p. 45.) (In fact, we prove a stronger result.)

THEOREM 1.2. *If* $\lambda > \sum_{0 \leqq \kappa < \chi} (\mu^{\kappa} + 2^{2^{\kappa}})$ *then* $P1(\lambda, \mu, \chi)$ *holds.*

THEOREM 1.3. *If* $\lambda > \mu^{2^{\chi}}$ *then* $P2(\lambda, \mu, \chi^{+})$ *holds. Moreover if* $\chi^{0} = \sum_{0 \leqq \kappa < \chi} 2^{\kappa}$, $\lambda > \mu^{\chi^{0}}$ *then* $P2(\lambda, \mu, \chi)$ *holds.*

THEOREM 1.4. *If* $P1(\lambda, \mu, \chi)$ *and* $\chi \rightarrow (\kappa)^{2}_{4}$ *holds, then* $P2(\lambda, \mu, \kappa)$ *holds.*

REMARK. (1) $\chi \rightarrow (\kappa)^{2}_{4}$ is defined in Erdös, Hajnal and Rado [4]. As the proof is straightforward, we leave it to the reader.

(2) We can combine theorems 1.2 and 1.4 to get results about $P2(\lambda, \mu, \alpha)$. For example by Ramsey [11], $\aleph_0 \rightarrow (\aleph_0)^{2}_{4}$, hence $P2(\lambda, \mu, \omega)$ holds (which is the result of [5]). (Here, as usual, we implicitly assume $\lambda > \mu \geqq \aleph_0$.)

(3) Theorems 1.2, 1.3, 1.4 give partial answer to a question which naturally arises from [5], and problem 2, [5], and 71B [3] are the most simple cases of it.

THEOREM 1.5. $P2(\lambda, \mu, \omega + 1)$ *holds. Moreover, if* $\lambda > \mu = \mu^{\aleph_0}$, $n < \omega$, *then* $P2(\lambda, \mu, \omega + n)$ *holds.*

REMARK. This answers problem 3 of [5] (in fact even stronger) and partially answer problem 2 of [5] (= 71B of [3]). The proof gives several more results of this kind.

To clarify our results let us assume G.C.H.

COROLLARY 1.6. (G.C.H.) *For every regular cardinality* μ, *and any cardinal* $\chi < \mu$, $P2(\mu^{+}, \mu, \chi)$ *holds. Moreover, if* μ *is singular,* χ *is less than the cofinality of* μ, *then* $P2(\mu^{+}, \mu, \chi)$ *holds. If* χ *is* *not greater than the cofinality of* μ, $P1(\mu^+, \mu, \chi)$ *holds.*

Proof. Immediate from Theorems 1.2, 1.3, 1.4, and by [4], $(2^\lambda)^+ \to (\lambda^+)^2_4$ holds.

The question naturally arises whether those are the best possible results. Prikry essentially proved this. See [18] Problem. 72.

THEOREM 1.7. *Suppose* $\lambda = \mu^\chi > \sum_{0 \leqq \kappa < \chi} \mu^\kappa = \mu_0$ *then* $P2(\lambda, \mu_0, \chi + 2)$ *does not holds.* ($\chi + 2$—*this is an ordinal addition*). *Moreover* $P1(\lambda, \mu_0, \chi + 2)$ *does not holds.*

In [5], not $P2(\aleph_1, \aleph_0, \omega + 2)$ was proved; and as the proof is similar and straightforward we leave it to the reader.

The most simple open problems are: (for simplicity only we assume G.C.H.)

PROBLEM 1. If $\aleph_\alpha$ is regular, does $P1(\aleph_{\alpha+1}, \aleph_\alpha, \aleph_\alpha)$ hold? Does $P2(\aleph_{\alpha+1}, \aleph_\alpha, \aleph_\alpha)$ hold?

PROBLEM 2. If $\aleph_\alpha$ singular, $\aleph_\beta$ is the cofinality of $\aleph_\alpha$, does $P2(\aleph_{\alpha+1}, \aleph_\alpha, \aleph_\beta)$ hold?

Maybe the answers are independent of $ZF + AC$.

Let us summarize the trivial facts about our properties.

LEMMA 1.8. (A) *If* $\lambda_1 \geqq \lambda$, $\mu_1 \leqq \mu$, $\alpha_1 \leqq \alpha$ *and* $P1(\lambda, \mu, \alpha)$ *hold, then* $P1(\lambda_1, \mu_1, \alpha_1)$ *holds. The same is ture for* $P2$ *and* $P3$.

(B) $P3(\lambda, \mu, \alpha)$ *implies* $P2(\lambda, \mu, \alpha)$; $P2(\lambda, \mu, \alpha)$ *implies* $P1(\lambda, \mu, \alpha)$, *where* α *is a limit ordinal; and* $P2(\lambda, \mu, \alpha + 1)$ *implies* $P1(\lambda, \mu, \alpha)$.

(C) *If* $\alpha < \omega$, $\lambda > \mu$ *then* $P3(\lambda, \mu, \alpha)$ *holds.*

(D) *If* $cf(\lambda) \leqq \mu < \lambda$, $(\forall \chi < \lambda) \neg P2(\chi, \mu, \alpha)$ *then not* $P2(\lambda, \mu, \alpha)$.

Proof. Immediate. We use (*D*) for (*B*).

Let us now prove the theorems.

DEFINITION 1.4. $\mathrm{Ded}(\mu)$ is the first cardinal λ such that there is no ordered set of cardinality λ with a dense subset of cardinality μ.

REMARK. Clearly $\mu^+ < \mathrm{Ded}(\mu) \leqq (2^\mu)^+$. By Mitchell [8] it is consistent with $ZF + AC$ that $\mathrm{Ded}(\aleph_1) < (2^{\aleph_1})^+$.

THEOREM 1.9. *If* $\mu < \lambda < \mathrm{Ded}(\mu)$ *then* $P3(\lambda, \mu, \omega)$ *does not hold.*

REMARK. Clearly Theorem 1.1 is an immediate conclusion of this theorem.

Proof. Let a tree mean a pair of a set and a well ordering of the set, which is not necessarily a total ordering. A branch of a tree is a maximal ordered subset. It can be easily shown that there is a tree $\langle A, <\rangle$ (A—the set, $<$—the ordering) such that $|A| = \mu$ and the tree has $\geqq \lambda$ branches. Let S_1 be the family of the branches of the tree and $S = A\ (-)\ S_1$. Clearly $|S| \geqq \lambda$, $|A| = \mu$ and S is a family of subsets of A. So it suffices to show that there is no $\bar{a} \in A^\omega$ which is strongly cut by S.

So suppose $\bar{a} \in A^\omega$ is strongly cut by S. By using Ramsey theorem ([11]) we know there is an infinite subsequence of $\bar{a}$, $\bar{b}$, such that exactly one of the following conditions is fulfilled

(1) for every $n < m < \omega$, $\bar{b}_n < \bar{b}_m$ (in the tree)

(2) for every $n < m < \omega$, $\bar{b}_n = \bar{b}_m$

(3) for every $n < m < \omega$, $\bar{b}_n > \bar{b}_m$

(4) for every $n < m < \omega$, $b_n b_m$ are incomparable, i.e., $b_n \neq b_m$, not $b_n > b_m$, and not $b_n < b_m$.

Now clearly also $\bar{b}$ is strongly cut by S. Hence (2) cannot be fulfilled. As $<$ is a well ordering (3) cannot be fulfilled. Now as $\bar{b}$ is strongly cut by S, there is a branch of $\langle A, <\rangle$ which contains two of the b_n's and so they are comparable, in contradiction to (4). So (1) is fulfilled. As $\bar{b}$ is strongly cut by S, there is $X \in S$ such that $\bar{b}_0 \in X$, $\bar{b}_1 \notin X$. But $A - X$ is a branch of the tree, $\bar{b}_1 \in A - X$, $\bar{b}_0 < \bar{b}_1$, hence $\bar{b}_1 \in A - X$, a contradiction.

THEOREM 1.2. *If* $\lambda > \sum_{0 \leqq \kappa < \chi} (\mu^\kappa + 2^{2^\kappa})$ *then* $P1(\lambda, \mu, \chi)$ *holds.*

Proof. Let S be a family of subsets of A, $|S| = \lambda$, $|A| = \mu$. We should prove there are $\bar{a}, \bar{b} \in A^\chi$ and $\bar{X} \in S^\chi$ such that, for every $\alpha, \beta < \chi$, $\bar{a}_\alpha \in \bar{X}_\beta \Leftrightarrow \bar{b}_\alpha \in \bar{X}_\beta$ iff $\beta < \alpha$.

Let us define, for every $T \subset S$, an equivalence relation E_T on A: $aE_T b$ holds if and only if for every $X \in T$, $a \in X \Leftrightarrow b \in X$. Clearly E_T is an equivalence relation, and the number of equivalence classes is $\leqq 2^{|T|}$.

Let us also define that $T \subset S$ *fixes* $X \in S$ if for every $a, b \in A$, $aE_T b$ implies $a \in X \Leftrightarrow b \in X$. Clearly the number of $X \in S$ which are fixed by T cannot be more than the number of subsets of the set of the E_T-equivalence classes. Hence $|\{X\colon X \in S, X \text{ is fixed by } T\}| \leqq 2^{2^{|T|}}$.

Let us now define by induction the families S_κ for $0 \leqq \kappa < \chi$ such that:

(1) $S_\kappa \subset S$, $|S_\kappa| \leqq \mu^\kappa$

(2) $\kappa_1 < \kappa_2$ implies $S_{\kappa_1} \subset S_{\kappa_2}$

(3) if $B, C \subset A$, $|B| \leqq \kappa$, $|C| \leqq \kappa$, and there is $X \in S$ such that $B \subset X$, $C \cap X = 0$, *then* there is $Y \in S_\kappa$ such that $B \subset Y$, $C \cap Y = 0$.

Clearly we can define the S_κ. We shall now prove that

($*$) there is $Y \in S$ such that for any T, $T \subset S_\kappa$, $0 \leqq \kappa < \chi$, $|T| \leqq \kappa$, Y is not fixed by T.

Suppose ($*$) does not hold and we shall get a contradiction. So

$$S = \bigcup_{0 \leqq \kappa < \chi} \bigcup_{\substack{T \subset S_\kappa \\ |T| \leqq \kappa}} \{X: X \in S, X \text{ is fixed by } T\} .$$

We have proved that $|\{X: X \in S, X \text{ is fixed by } T\}| \leqq 2^{2^{|T|}}$, and by its contruction $|S_\kappa| \leqq \mu^\kappa$. Hence

$$\begin{aligned} \lambda = |S| &\leqq \sum_{0 \leqq \kappa < \chi} \sum_{\substack{T \subset S_\kappa \\ |T| \leqq \kappa}} 2^{2^{|T|}} \\ &\leqq \sum_{0 \leqq \kappa < \chi} |S_\kappa|^\kappa \times 2^{2^\kappa} = \sum_{0 \leqq \kappa < \chi} (|S_\kappa|^\kappa + 2^{2^\kappa}) \\ &\leqq \sum_{0 \leqq \kappa < \chi} (\mu^\kappa + 2^{2^\kappa}) < \lambda \end{aligned}$$

a contradiction. So ($*$) holds.

Now we shall define by induction a_k, b_k, X_k for $k < \chi$ such that:

(A) $a_k \in A$, $b_k \in A$, and $X_k \in S_{|k|+1}$

(B) if $l \leqq k$ then $a_l \in X_k$, $a_l \in Y$, $b_l \notin X_k$, and $b_l \notin Y$

(C) if $l < k$, then $a_k \in X_l$ if and only if $b_k \in X_l$.

Suppose a_l, b_l and X_l has been defined for every $l < k$. Let $1 + |k| = \kappa$, and $T = \{X_l: l < k\}$. Clearly $T \subset S_\kappa$, $|T| \leqq \kappa$. Hence, by the definition of Y, it is not fixed by T. So there are $a_k, b_k \in A$ such that: $a_k \in Y$, $b_k \notin Y$ and $a_k E_T b_k$, i.e., for every $l < k$, $a_k \in X_l$ if and only if $b_k \in X_l$. Clearly $\{a_l: l \leqq k\} \subset Y$, $\{c_l: l \leqq k\} \cap Y = 0$, $|\{a_l: l \leqq k\}| \leqq \kappa$, $|\{b_l: l \leqq k\}| \leqq \kappa$; hence by the definition of S_κ there is $X_k \in S_\kappa$ such that

$$\{a_l: l \leqq k\} \subset X_k, \{b_l: l \leqq k\} \cap X_k = 0 .$$

Clearly $\langle a_k: k < \chi\rangle$, $\langle b_k: k < \chi\rangle$, and $\langle X_k: k < \chi\rangle$ are the required sequences, and so Theorem 1.2 is proved.

THEOREM 1.3. *If $\chi^0 = \sum_{0 \leqq \kappa < \chi} 2^\kappa$, $\lambda > \mu^{\chi^0}$, then $P2(\lambda, \mu, \chi)$ holds.*

Proof. As the proof is very similar to the proof of Theorem 2, we shall only sketch it.

Suppose S is a family of subsets of A, $|S| = \lambda$, $|A| = \mu$. It is easy to find $S_1 \subset S$, $|S_1| \leqq \mu^{\chi^0}$ such that:

(1) if $B \subset A$, $|B| \leqq 2^\kappa$, $0 \leqq \kappa < \chi$, and $T \subset S_1$, $|T| \leqq \kappa$ and $Y \in S$ then there is $X \in S_1$ such that: (A) $X \cap B = Y \cap B$ (B) if C is an E_T-equivalence class then $C \subset X \Leftrightarrow C \subset Y$ and $C \cap X = 0 \Leftrightarrow C \cap Y = 0$.

(2) if X_l^k, $k < \alpha_l < \chi$, $l < \chi^0$, Y_l^k, $k < \beta_l < \chi$, $l < \chi^0$ and Z_l, $l < \chi^0$ are sets from S_1, and there is $X \in S$ such that: for every $l < \chi^0$

$$X \cap \bigcap_{k<\alpha_l} X_l^k \cap \bigcap_{k<\beta_l} (A - Y_l^k) = Z_l \cap \bigcap_{k<\alpha_l} X_l^k \cap \bigcap_{k<\beta_l} (A - Y_l^k)$$

then there is $X \in S_1$, which satisfies this condition.

Now we can repeat a construction similar to that which appears in the proof of Theorem 1.

As Theorem 1.4 is trivial, it remains to prove only

THEOREM 1.5. (A) *If* $\lambda > \mu$ *then* $P2(\lambda, \mu, \omega + 1)$ *holds.*

(B) *If* $\lambda > \mu = \sum_{0 \leqq \kappa < \chi} \mu^\kappa$, $\alpha \leqq \chi$ *and* $P2(\lambda, \mu, \alpha)$ *holds then* $P2(\lambda, \mu, \alpha + 1)$ *holds. Hence for every* n, *if in addition* $\alpha < \chi$, $P2(\lambda, \mu, \alpha + n)$ *holds.* (By 1.8D we can assume $cf(\lambda) > \mu$).

(C) *If* $\lambda > \mu^{\aleph_0}$, *then* $P2(\lambda, \mu, \omega + n)$.

REMARK. (1) Clearly (A) cannot be improved by [5] $P2(\aleph_1, \aleph_0, \omega + 2)$ does not hold.

(2) Part of the proof is a generalization of a proof of A. Máté which appeared in [5].

Proof. As the proof of (B) is obvious from the proof of A, we shall prove A only. (C follow from B).

So let S be a family of subsets of A, $|S| = \lambda$, $|A| = \mu$.

First, there is $a^0 \in A$ such that $S_1 = \{X: X \in S, a^0 \in X\}$ is of cardinality $> \mu$. Otherwise

$$\begin{aligned} \lambda = |S| &= \left| \bigcup_{a \in A} \{X: X \in S, a \in X\} \cup \{0\} \right| \\ &\leqq \sum_{a \in A} |\{X: X \in S, a \in X\}| + 1 = \mu \cdot \mu + 1 = \mu < \lambda \end{aligned}$$

a contradiction. Similarly there is $a^1 \in A$ such that $S_2 = \{X: X \in S_1, a^1 \notin X\}$ is of cardinality $> \mu$. Now at first we assume

($*$) there is $A^1 \subset A$, and $S^1 \subset \{Y \cap A^1: Y \in S_2\}$ such that $|S^1| > \mu$; and for every $X \in S^1$,

$$|\{Y \cap X: Y \in S^1\}| \leqq \mu .$$

Then it can be easily seen that if $X_1, \cdots, X_n \in S^1$, $X = X_1 \cup \cdots \cup X_n$ then

$$|\{Y \cap X: Y \in S^1\}| \leqq \mu .$$

So we can easily find $S^2 \subset S^1$, $|S^2| \leqq \mu$ such that: if $X_1, \cdots, X_n \in S^2$, $X \in S^1$ and $X \subset X_1 \cup \cdots \cup X_n$ then $X \in S^2$; and if $a_0, \cdots, a_n \in A$, $X \in S^1$, then there is $Y \in S^2$ such that $\{a_0, \cdots, a_n\} \cap X = \{a_0, \cdots, a_n\} \cap Y$. Now let $Y^0 \in S^1$, $Y^0 \notin S^2$. (Y^0 exists as $|S^1| > \mu \geqq |S^2|$). Now we shall define by induction on n, a_n, X_n such that: $a_n \in Y^0$, $X_n \in S^2$, and

$a_n \notin X_0, a_n \notin X_1, \cdots, a_n \notin X_n; a_0, \cdots, a_{n-1} \in X_n$. Suppose a_n, X_n has been defined for every $n < m < \omega$. As $Y^0 \notin S^2$, $Y^0 \not\subset X_0 \cup \cdots \cup X^{m-1}$, hence there is $a_m \in Y^0$, $a_m \notin X_0 \cup \cdots \cup X^{m-1}$. Also there is $X_m \in S^2$ such that $\{a_0, \cdots, a_m\} \cap X_m = \{a_0, \cdots, a_m\} \cap Y^0$.

Now clearly if we define $a_\omega = a^1$, clearly $\langle a_\alpha \mid \alpha < \omega + 1\rangle \in A^{\omega+1}$ and is strongly cut by S; so the conclusion of theorem holds.

Similarly the conclusion of the theorem holds if

(**) there is $A^1 \subset A$ and $S^1 \subset \{Y \cap A^1: Y \in S_2\}$ such that $|S^1| > \mu$, and for every $X \in S^1$

$$|\{Y \cap (A^1 - X): Y \in S^1\}| \leqq \mu .$$

Hence we can assume (*) and (**) do not hold. So there is $X^0 \in S_2$ such that $S_3 = \{Y \cap X^0: Y \in S_2\}$ is of cardinality $> \mu$. (Otherwise, taking $A^1 = A$, $S^1 = S_2$, (*) holds.) Similarly there is $X^1 \in S_3$ such that $S_4 = \{Y \cap (X^0 - X^1): Y \in S_3\}$ is of cardinality $> \mu$ (otherwise taking $A^1 = X^0$, $S^1 = S_3$, (**) holds). Now $|S_4| > \mu \geqq |X^0 - X^1|$, and S_4 is a family of subsets of $X^0 - X^1$. Hence there is $\bar{a} \in (X^0 - X^1)^\omega$ which is strongly cut by S_4 or by $(X^0 - X^1)(-)\, S_4$. Taking as $\bar{a}_\omega$, a^0 or a^1 (accordingly), we get a sequence from $A^{\omega+1}$ which is strongly cut by S or $A\,(-)\,S$. So we prove Thorem 1.5A.

Naturally the question arises on the finite case. More exactly

DEFINITION 1.5. For natural numbers m, n let $f(m, n)$ be the first ordinal α such that $P3(\alpha, m, n)$ holds.

The result is $f(m, n) = 1 + \sum_{k=0}^{n-1} \binom{m}{k}$. The proof follows from a little more complex result, of Perles and Shelah.

Another natural generalization is the relation $P4(\lambda, \mu, \chi)$ which is

DEFINITION 1.5. $P4(\lambda, \mu, \chi)$ holds if whenever $|S| = \lambda$, $|A| = \mu$, and S is a family of subsets of A, there exists $B \subset A$, $|B| = \chi$, such that for every $C \subset B$ there is $X \in S$ such that $X \cap B = C$.

Clearly $P4(\lambda, \mu, \chi)$ implies $P3(\lambda, \mu, \chi)$ and $P3(\lambda, \mu, \alpha)$ for every $\alpha < \chi^+$. The only result known to me is that if $\lambda \geqq \mathrm{Ded}(\mu)$, λ is regular and χ is finite, then $P_4(\lambda, \mu, \chi)$ holds. (see Shelah [15]). Perles and I prove that if μ and χ are finite $P4(\lambda, \mu, \chi)$ holds if and only if $\lambda > \sum_{k=0}^{\chi-1} \binom{\mu}{k}$. Later and independently Sauer [19] proved it.

2. **On stable models and theories.** In this section we shall apply a combinatorial theorem from § 1 to get results in the theory of models.

Let L be a first-order language; $L_{\lambda,\omega}$ will be its extension by permitting conjunctions on sets of $< \lambda$ formulas, provided that in the conjunction, only finitely many variables appear free. $L_{\infty,\omega}$ will be the *class* of formulas $\bigcup_\lambda L_{\lambda,\omega}$. T will denote a *set* of sentences from $L_{\infty,\omega}$. Δ will denote a *set* of formulas $\varphi(\bar{x})$ from $L_{\infty,\omega}$ (more exactly, Δ is a set of pairs $\langle\varphi, \bar{x}\rangle$ where $\varphi \in L_{\infty,\omega}$, $\bar{x}$ is a finite sequence of variables, and every free variable of φ appears in $\bar{x}$). Δ is closed if it is closed under negation, finite conjunction (hence all connective), adding dummy variables and changing the order of the variables. $\bar{\Delta}$ is the closure of Δ. M, N shall denote models (L-models, if not said otherwise). $|M|$ is the set of elements of M. If $A \subset |M|$, p is a (Δ, m)-type over A iff p is a set whose elements are of the form $\varphi(\bar{x}, \bar{a})$ where $\bar{x} = \langle x_0, \cdots, x_{m-1}\rangle$, $\varphi(\bar{x}, \bar{y}) \in \Delta$ and $\bar{a} \in A$ (or more exactly $\bar{a}_0, \bar{a}_1, \cdots \in A$).

For $\bar{c} \in |M|$, the Δ-type $\bar{c}$ realizes over A, $p(\bar{c}, A, M, \Delta)$ is

$$\{\varphi(\bar{x}, \bar{a}): \bar{a} \in A,\ \varphi(\bar{x}, \bar{y}) \in \Delta,\ M \models \varphi[\bar{c}, \bar{a}]\}\ .$$

Let

$$S^m(A, M, \Delta) = \{p(\bar{c}, A, M, \Delta): \bar{c} \in |M|^m\}\ .$$

The model M is called (Δ, λ)-*stable* if $|A| \leqq \lambda$ implies $|S^1(A, M, \Delta)| \leqq \lambda$; otherwise M is (λ, Δ)-unstable.

Let $\lambda \in Od_\Delta(M)$ if there is $n < \omega$, and sequences $\bar{a}^l \in |M|^n$, $l < \lambda$; and a formula $\varphi(\bar{x}, \bar{y}) \in \Delta$ such that $M \models \varphi[\bar{a}^k, \bar{a}^l]$ if and only if $k < l$ for every $k, l < \lambda$.

THEOREM 2.1. *Suppose M is (Δ, κ)-unstable, $\Delta = \bar{\Delta}$, $\kappa = \sum_{0 \leqq \mu < \lambda} (\kappa^\mu + 2^{2^\mu})$ and $\kappa = \kappa^{|\Delta|}$. Then $\lambda \in Od^\Delta(M)$.*

Proof. Let $\Delta = \{\varphi_k(x, \bar{y}^k): k < |\Delta|\}$, $\Delta_k = \{\varphi_k(x, \bar{y}^k)\}$. As M is (Δ, κ)-unstable, there is $A \subset |M|$, $|A| \leqq \kappa$ such that $|S^1(A, M, \Delta)| > \kappa$. If for every $k < |\Delta|$, $|S^1(A, M, \Delta_k)| \leqq \kappa$ then

$$\kappa < |S^1(A, M, \Delta)| \leqq \left|\prod_{k<|\Delta|} S^1(A, M, \Delta_k)\right| = \prod_{k<|\Delta|} |S^1(A, M, \Delta_k)| \leqq \kappa^{|\Delta|} = \kappa$$

a contradiction. Hence there is $k < \kappa$ such that $|S^1(A, M, \Delta_k)| > \kappa$. Let $\varphi = \varphi_k$. Now clearly $S^1(A, M, \Delta_k)$ is a set of subsets of

$$\Phi = \{\varphi_k(x, \bar{a}): \bar{a} \in A,\ \bar{a} \text{ is of the length of } \bar{y}^k\}\ .$$

Clearly $|\Phi| \leqq \kappa$. Hence by Theorem 1.2, there are $p_l \in S^1(A, M, \Delta_k)$ $\bar{a}^l, \bar{b}^l \in |A|$ for $l < \lambda$ such that $\varphi(x, \bar{a}^l) \in p_j \Leftrightarrow \varphi(x, \bar{b}^l) \in p_j$ if and only if $j < l$. Let $p_l = p(\bar{c}^l, A, M, \Delta_k)$, and $\bar{d}^l = \bar{a}^l \frown \bar{b}^l \frown \bar{c}^l$ (the juxtaposition of the three sequences). Clearly $M \models \varphi[\bar{c}^j, \bar{a}^l] \equiv \varphi[\bar{c}^j, \bar{b}^l]$ if and only if $j < l$. As $\Delta = \bar{\Delta}$, we can easily find $\psi(\bar{x}, \bar{y}) \in \Delta$ such that for $k, l < \lambda$; $M \models \psi[\bar{d}^k, \bar{d}^l]$ if and only if $k < l$. Hence $\lambda \in Od_\Delta(M)$.

DEFINITION 2.1. Let $A, C \subset |M|$. C is Δ-indiscernible over A in M if for every n, and every n different elements $c_0, \cdots, c_{n-1}$ of C, and every additional n different elements $c^0, \cdots, c^{n-1}$ of C

$$p(\langle c_0, \cdots, c_{n-1}\rangle, A, M, \Delta) = p(\langle c^0, \cdots, c^{n-1}\rangle, A, M, \Delta) .$$

THEOREM 2.2. *Suppose M is $(\bar{\Delta}, \lambda)$-stable, $\lambda \notin Od_{\bar{\Delta}}(M)$, $A \subset |M|$, $C \subset |M|$, $|A| \leqq \lambda < |C|$, and the cofinality of λ is greater than $|\Delta|$. Then there exists $C_1 \subset C$, $|C_1| > \lambda$ such that C_1 is Δ-indiscernible in M over A.*

REMARK. Taking a Souslin tree, we can see that the condition $\lambda \notin Od_{\bar{\Delta}}(M)$ is necessary. (More exactly, this is consistent with $ZF + AC$.) Instead $cf(\lambda) > |\Delta|$ we can demand $\exists \mu < \lambda$, $\mu \notin Od_{\bar{\Delta}}(M)$.

Morley in [9] Theorem 4.6 proved a similar theorem for models of a complete, first-order, countable, totally transcendental theory. In [12] this was generalized to models of stable theories, and in [13], Theorem 3.1 to models with stable finite diagram. Another generalization is Theorem 5.9A of Shelah [15]. Theorem 2.2, in fact, implies all these theorems. (For 5.9A [15] we should note that if Δ is finite, then there is a finite Δ_1, $\Delta \subset \Delta_1 \subset \bar{\Delta}$, such that for any M, λ; M is (Δ_1, λ)-stable if and only if it is $(\bar{\Delta}, \lambda)$-stable.)

Proof. As the proof is very similar to the proof of Theorem 3.1 [13], we omit it.

DEFINITION 2.2. T is (Δ, λ)-stable if every model of T is (Δ, λ)-stable. T is Δ-stable, if for at least one λ it is (Δ, λ)-stable, T is (Δ, λ)-unstable [Δ-unstable] if it is not (Δ, λ)-stable [Δ-stable]. Let $\lambda \in Od_\Delta(T)$ if for at least one model M of T, $\lambda \in Od_\Delta(M)$. T is stable if it is Δ-stable for every Δ; otherwise-unstable.

REMARK. If T has no model of cardinality $> \lambda$, then it is (Δ, λ)-stable, and hence stable.

THEOREM 2.3. *Suppose $T, \Delta \subset L_{\lambda^+,\omega}$, $|T| \leqq \lambda$, $|L| \leqq \lambda$, T is (Δ, κ)-unstable, $\kappa^{\mu(\lambda)} = \kappa$. Then T is Δ-unstable.*

REMARK. (1) $\mu(\lambda)$ is the first cardinality such that if a sentence of a language $L^1_{\lambda^+,\omega}$ has a model of cardinality $\mu(\lambda)$, it has models in any cardinalty $\geqq \lambda$.

(2) We can demand only: $T, \Delta \subset L_{\lambda^+,\omega}$, $|T| + |\Delta| \leqq \lambda$, and for every $\mu < \mu(\lambda)$ there is $\kappa = \kappa^\mu$ such that T is (Δ, κ)-unstable.

(3) We can demand only $T, \Delta \subset L_{\lambda^+,\omega}$, $|T| \leqq \lambda$, $|L| < \mu(\lambda)$, $\kappa =$

$\sum_{\mu<\mu(\lambda)} \kappa^{\mu}$ and T is (Δ, κ)-unstable.

Proof. Here we use Ehrefeucht-Mostowski models (see [2]) and the method of Morley [10]. All the results we use appeared in Chang [1]. As T is (Δ, κ)-unstable, T has a model M and $A \subset |M|$ such that $|S^1(A, M, \Delta)| > \kappa \geqq |A|$. It is well known that $\chi < \mu(\lambda)$ implies $2^{\chi} < \mu(\lambda)$; hence $\chi < \mu(\lambda)$ implies $2^{2^{\chi}} < \mu(\lambda)$. So $\kappa = \sum_{\chi<\mu(\lambda)} (\kappa^{\chi} + 2^{2^{\chi}})$. As $|\Delta| \leqq |L_{\lambda^+,\omega}| < \mu(\lambda)$, exactly as in the proof of Theorem 2.1, this implies that there are sequences $\bar{a}^k$, $\bar{b}^k$, $k < \mu(\lambda)$ from A and $c_k \in |M|$, $k < \mu(\lambda)$ and a formula $\varphi(x, \bar{y}) \in \Delta$ such that:

$$\text{for every} \quad k, l < \mu(\lambda),\ M \models \varphi[c_l, \bar{a}^k] \equiv \varphi[c_l, \bar{b}^k] \quad \text{if and only if } l < k\ .$$

Now we add to M the one place relation $P^M = \{c_k : k < \mu(\chi)\}$, and the functions F_1^M, F_2^M defined by $F_1^M(\bar{a}^k) = c_k$, $F_2^M(\bar{b}^k) = c^k$, and otherwise $F_1^M(\bar{a}) \notin P^M$, $F_2^M \notin P^M$.

Now using Morley's method we get (in fact we need an improvement of Chang [1]):

(*) for every ordered set I, there is a model M_I of T, in which there are c_s, $\bar{a}_s$, $\bar{b}_s$ for every $s \in I$ such that: for every $s, t \in I$

$$M_I \models \varphi[c_t, \bar{a}_s] \equiv [c_t, \bar{b}_s] \text{ if and only if } t < s\ .$$

Let χ be any cardinality, and we shall prove T is (Δ, χ)-unstable. We can find easily an ordered set I, $|I| > \chi$, with a dense subset J, $|J| \leqq \chi$ (If $\chi_1 = \inf\{\chi_1 : 2^{\chi_1} > \chi\}$, then I can be the set of sequences of ones and zeroes of length χ_1, ordered lexicographically.) Let $M = M_I$, and let $A = \bigcup\{\mathrm{Rang}\,\bar{a}_s \cup \mathrm{Rang}\,\bar{b}_s : s \in J\}$. Clearly $|A| \leqq \aleph_0 + |J| \leqq \chi$. On the other hand we shall show that $t_1 \neq t_2$, $t_1, t_2 \in I$ implies $p(c_{t_1}, A, M, \Delta) \neq p(c_{t_2}, A, M, \Delta)$. Hence $|S^1(A, M, \Delta)| > \chi$, so T is (Δ, χ)-unstable.

Suppose $t_1 \neq t_2$, $t_1, t_2 \in I$. Without loss of generality suppose $t_1 < t_2$. As J is a dense subset of I, there is $s \in J$, $t_1 < s < t_2$. By the definition of M_I,

$$M \models \varphi[c_{t_1}, \bar{a}_s] \equiv [c_{t_1}, \bar{b}_s]$$
$$M \models \neg(\varphi[c_{t_2}, \bar{a}_3] \equiv \varphi[c_{t_2}, \bar{b}_s])\ .$$

Hence

$$\varphi(x, \bar{a}_s) \in p(c_{t_1}, A, M, \Delta) \text{ if and only if } \varphi(x, \bar{b}_s) \in p(c_{t_1}, A, M, \Delta)$$

and

$$\varphi(x, \bar{a}_s) \in p(c_{t_2}, A, M, \Delta) \text{ if and only if } \varphi(x, \bar{b}_s) \notin p(c_{t_2}, A, M, \Delta)\ .$$

So $p(c_{t_1}, A, M, \Delta) \neq p(c_{t_2}, A, M, \Delta)$, and as noted before this implies T

is (Δ, χ)-unstable, for every χ.

Similarly we can prove

THEOREM 2.4. (1) *If* $T, \Delta \subset L_{\lambda^+,\omega}$; $|T| + |\Delta| \leqq \lambda$, *and for every* $\kappa < \mu(\lambda)$, $\kappa \in Od_\Delta(T)$, *then every* $\kappa \in Od_\Delta(T)$.

(2) *If every* $\kappa \in Od_\Delta(T)$, *then* T *is* $\bar{\Delta}$-*unstable.*

REMARK. In 2.4.2 we use the following fact: if M is $(\bar{\Delta}, \lambda)$-stable, $A \subset |M|$, $|A| \leqq \lambda$, $m < \omega$ then $|S^m(A, M, \Delta)| \leqq \lambda$.

THEOREM 2.5. *Suppose* $T \subset L_{\lambda^+,\omega}$, $|T| \leqq \lambda$, $|L| \leqq \lambda$, *and* T *is unstable. Then there exists* $\Delta_1 \subset L_{\lambda^+,\omega}$, $|\Delta_1| \leqq \lambda$ *such that* T *is* Δ_1-*unstable.*

Proof. As in the proof of Theorem 2.3, we depend on the method of Morley [10], Chang [1]. So let T be Δ-unstable. Without loss of generality, let $\Delta = \bar{\Delta}$ and $\Delta \subset L_{\kappa^+,\omega}$. From Theorem 2.1 it follows that every $\mu \in Od_\Delta(T)$ [as T is $(\Delta, 2^{2^{(\mu+\kappa+|\Delta|+|L|)}})$-unstable]. Let $\lambda^1 = \mu(\lambda + |T| + \kappa + |\Delta| + |L|)$. So T has a model M such that $\lambda^1 \in Od_\Delta(M)$. We expand now M to M^1 in the following way:

(1) For every subformula $\varphi(\bar{x})$ of a formula from $T \cup \Delta$ (including the formulas form Δ themselves) we add to M the relation $R_\varphi^{M^1} = \{\bar{a}: M \models \varphi[\bar{a}]\}$.

(2) M^1 has Skolem function for every first-order formula in its language.

Let $L^1 = L(M^1)$ be the first-order language associated with M^1. Clearly $|L(M^1)| \leqq |L| + |T| + |\Delta| + \kappa + \lambda$. As $\lambda^1 \in Od_\Delta(M)$, there are $\bar{a}^k$, $k < \lambda^1$ from M^1 and there is $\varphi_0(\bar{x}, \bar{y}) \in \Delta$ such that $M^1 \models \varphi_0[\bar{a}^k, \bar{a}^l]$ if and only if $k < l$. For simplicity we shall assume the sequences $\bar{a}^k$ are of length one, and $\bar{a}^k = \langle a_k \rangle$.

Hence there is a model N and $a_s \in |N|$ for $s \in I$, which satisfy the following properties:

(1) the first-order language associated with N is L^1.

(2) N, M^1 are elementarily equivalent.

(3) N is a model of T, and for every subformula $\varphi(\bar{x})$ of a formula from $T \cup \Delta$, $N \models (\forall \bar{x})[\varphi(\bar{x}) \equiv R_\varphi(\bar{x})]$.

(4) I is an ordered set isomorphic to the rationals (s, t will denote elements of I).

(5) for each $s, t \in I$; $N \models \varphi_0[a_s, a_t]$ if and only if $s < t$.

(6) for each $c \in N$, there are $s_1 < \cdots < s_n (\in I)$ and a term B of L^1 such that

$$N \models c = B[a_{s_1}, \cdots, a_{s_n}] .$$

(7) for every $\varphi(x_1, \cdots, x_n) \in L^1$, $s_1 < \cdots < s_n$, and $t_1 < \cdots < t_n$

the following holds:

$$N \models \varphi[a_{t_1}, \cdots, a_{tn}] \text{ if and only if } N \models \varphi[a_{s_1}, \cdots a_{s_n}] .$$

As I is dense, by [7], [17], this holds also for every $\varphi \in L^1_{\infty,\omega}$.

Let $\bar{x}^0 = \langle x_0, x_1\rangle$, $\bar{x}^1 = \langle x_2, x_3\rangle$.

Let $\{\varphi_{k,n}(\bar{x}^0, \bar{x}^1, y_0, \cdots y_{n-1}): n < \omega, k < |L|\}$ be the list of the atomic formulas of L. Let

$$\Phi_n(\bar{x}^0, \bar{x}^1, y_0, \cdots, y_{n-1}, z_0, \cdots, z_{n-1}) =$$
$$= \bigwedge_{k<|L|} (\varphi_{k,n}(\bar{x}^0, \bar{x}^1, y_0, \cdots, y_{n-1}) \equiv \varphi_{k\,n}(\bar{x}^0, \bar{x}^1, z_0, \cdots, z_{n-1}))$$
$$\Phi(\bar{x}^0, \bar{x}^1) =$$
$$= (\exists y_0 \forall z_0 \exists z_1 \forall y_1, \exists y_2 \forall z_2 \exists z_3 \forall y_3, \cdots, \exists y_{2m} \forall z_{2m} \exists z_{2m+1} \forall y_{2m+1}, \cdots)_{m<\omega}$$
$$[\neg \bigwedge_{n<\omega} \Phi_n(\bar{x}^0, \bar{x}^1, y_0, \cdots, y_{n-1}, z_0, \cdots, z_{n-1})] .$$

By Shelah [14], for every L-model M_1, and $\bar{a}, \bar{b} \in |M_1|^2$, $M_1 \models \Phi[\bar{a}, \bar{b}]$ if and only if $\bar{a}$ and $\bar{b}$ realizes different $L_{\infty,\omega}$-types (i.e., there is $\varphi(\bar{x}^0) \in L_{\infty,\omega}$ such that

$$M_1 \models \varphi[\bar{a}],\ M_1 \models \neg\varphi[\bar{b}]) .$$

REMARK. The definition of the satisfaction of $\Phi[\bar{a}, \bar{b}]$ is self-evident. Discussion about languages with such expressions can be found in Keisler [6].

Hence we can find functions $F_1, \cdots, F_n, \cdots$ whose domains and ranges are $|N|$, each with a finite number of places such that:

(*) if N_1 is a submodel of a reduct of N, whose associated first order language include L, and $|N_1|$ is closed under the functions $\{F_n: n < \omega\}$ then for every $\bar{a}, \bar{b} \in |N_1|^2$, $N \models \Phi[\bar{a}, \bar{b}]$ implies $N_1 \models \Phi[\bar{a}, \bar{b}]$.

Now as in the downward Lowenheim-Skolem theorem, we can find a model N_1 such that:

(A) $|N_1| \subset |N|$, $\{a_s: s \in I\} \subset |N_1|$, $\|N_1\| \leqq \lambda$ and N_1 is a submodel of a reduct of N.

(B) $|N_1|$ is closed under $\{F_n: n < \omega\}$

(C) if $\bar{a} \in |N_1|$, $\varphi(x, \bar{y})$ is a subformula of $\psi \in T$, and $N \models (\exists x)\varphi(x, \bar{a})$, then for some $b \in |N_1|$, $N \models \varphi[b, \bar{a}]$. Hence N_1 is a model of T.

(D) if $s_1 < \cdots < s_n$, $t_1 < \cdots < t_n$, B is a term from L^1, and $B^N[a_{s_1}, \cdots, a_{s_n}] \in |N_1|$, then $B^N[a_{t_1}, \cdots, a_{t_n}] \in |N_1|$.

REMARK. Notice that by property (7) of N, if $B_1^N[a_s, \cdots, a_{s_n}] = B_2^N[a_{s_1}, \cdots, a_{s_n}]$ then $B_1^N[a_{t_1}, \cdots, a_{t_n}] = B_2^N[a_{t_1}, \cdots, a_{t_n}]$.

(E) The language of N_1, L^2, contains, L, is of cardinality λ, is contained in L^1, and for each $c \in |N_1|$ there is a term B from L^2 such that $c = B^N[a_{s\cdot}, \cdots, a_{s_n}]$ for some $s_1 < \cdots < s_n$.

It is easy to prove that N_1 satisfies properties (6) and (7) of N, with L^1 replaced by L^2. It is also clear, by (C), that N_1 is a model of T. Let $s < t$, we know that $N \models \varphi_0[a_s, a_t]$, but $N \models \neg \varphi_0[a_s, a_t]$. Hence $\langle a_s, a_t\rangle, \langle a_t, a_s\rangle$ do that satisfy the same $L_{\infty\,\omega}$-type in N. By $(*)$ and (B), $\langle a_s, a_t\rangle, \langle a_t, a_s\rangle$ also do not realize the same $L_{\infty\,\omega}$-type in N_1. As $\|N_1\| \leqq \lambda$, by Chang [1] it follows that $\langle a_s, a_t\rangle, \langle a_t, a_s\rangle$ do not realize the same $L_{\lambda^+,\omega}$-type in N_1. So there is a formula $\varphi_1(x, y) \in L_{\lambda^+\cdot\omega}$ such that $N_1 \models \varphi[a_s, a_t]$, $N_1 \models \neg\varphi[a_t, a_s]$. Let $\Delta_0 = \{\varphi_1(x, y)\}$, $\Delta_1 = \bar{\Delta}_0$. We shall prove that T is Δ_1-unstable, and so prove the theorem.

By Theorem 2.4.2 it suffices to prove that for every κ, $\kappa \in Od_{\Delta_1}(T)$. Let κ be any cardinal, and J a dense order set, $I \subset J$, and J contain a subset with order-type κ. We shall define now N_2 as an extension of N_1 such that:

(α) $\{a_s : s \in J\} \subset |N_2|$

(β) for every element c of N_2 there are $s_1 < \cdots s_n \in J$ and term $B \in L^2$ such that

$$c = B^{N2}[a_{s_1}, \cdots, a_{s_n}]$$

(γ) if $\varphi(x_1, \cdots, x_n)$ is an atomic formula, $s_1 < \cdots < s_n \in J$, $t_1 < \cdots < t_n \in J$ then

$$N_2 \models \varphi[a_{s_1}, \cdots, a_{s_n}] \text{ if and only if } N_2 \models \varphi[a_{t_1}, \cdots, a_{t_n}] .$$

It can be easily seen that N_2 exists. We can also show by induction on formulas of $L_{\lambda^+,\omega}$ that N_2 is an $L_{\lambda^+\,\omega}$-elementary extension of N_1. (See [7], [17].) Hence N_2 is a model of T. It is also clear that for every $s, t \in J$, $N_2 \models \varphi_1[a_s, a_t]$ if and only if $s < t$. By the definition of J and Δ_1 this implies $\kappa \in Od_{\Delta_1}(N_2)$ hence $\kappa \in Od_{\Delta_1}(T)$, and by 2.4.2, this implies T is Δ_1-unstable, where $|\Delta_1| \leqq \lambda$, $|\Delta_1| \subset L_{\lambda^+,\omega}$.

THEOREM 2.6. *If T is unstable, $T \subset L_{\lambda^+\omega}$, $\mu > \lambda + |T|$, then T has exactly 2^μ non-isomorphic models of cardinality μ.* (For most cases it suffices to demand $\mu \geqq \lambda + |T| + \aleph_1$.)

Proof. By Theorem 2.5, and Shelah [16].

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