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CONCERNING BANACH SPACES WHOSE DUALS ARE ABSTRACT L-SPACES

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The purpose of this paper is to give general methods for constructing Banach spaces whose duals are linearly isometric to abstract L spaces. These methods are based on annihilating certain subspaces of the duals of certain Banach spaces and in the existence of "affine" maps from a compact Hausdorff space X to the space of regular Borel measures on X.

1. Introduction. There have been several papers concerned with the structure and classification of Banach spaces whose duals are linearly isometric to a space of the type $L_1(\mu)$ (see [25] and its references). General methods for the generation of such spaces have been developed in [6], [8], and [16]. The purpose of this paper is to present these theorems in their most general framework.

In §2 a general result is presented which indicates several ways in which Banach spaces whose duals are L-spaces can be constructed. Using this as a base, §3 begins with an application of the main result of section two, Theorem 2.2, to show the way in which certain affine symmetric maps generate these spaces. Finally it is shown how some of these maps from compact Hausdorff spaces to the corresponding space of regular Borel measures generate the Banach spaces whose duals are L-spaces.

All Banach spaces considered in this paper are over the real field. If X is a compact Hausdorff space then $C(X)$ denotes the Banach space of all real-valued continuous functions on X and $M(X)$ the Banach space dual of $C(X)$. If μ is a measure, $L_1(\mu)$ is the Banach space of all integrable real-valued functions (sometimes called an (abstract) L-space). The dual of a Banach space A is denoted by A^*. If A is a Banach space and $S\subset A$ and $\{a_x: x\in S\}$ is a set of numbers, then $\sum_{x\in S}(a_x)x$ denotes the limit (provided it exists) of the net of all sums $\sum_{x\in F}(a_x)x$ for finite sets $F\subset S$. The notation and terminology regarding Choquet simplexes, maximal measures, affine functions, etc. is that of [26].

2. Methods of generating Banach spaces whose duals are L-spaces. The purpose of this section is to prove a general result concerning the generation of Banach spaces whose duals are L-spaces. How to construct such spaces as subspaces of given Banach spaces having the same property is revealed in Theorem 2.2.

If A is a Banach space and B is a subset of A then $B^{\perp}=\{x^*\in A^*: x^*(x)=0$ for all $x\in B\}$. For $M\subset A^*$, ${}^{\perp}M=\{x\in A: x^*(x)=0$

for all $x^* \in M\}$. A *contractive* operator on A is a linear operator $T: A \to A$ satisfying $\|T\| \leqq 1$. Of course the term "projection" has its usual meaning.

A Banach space A whose isometric image in any other Banach space B is the range of a contractive projection on B is called a P_1 space. The relevant facts concerning P_1 spaces can be found in [9].

The following proposition serves a dual purpose. It provides motivation to the main theorem of this section as well as the key to the main result of §3.

PROPOSITION 2.1. *Let A and B be Banach spaces with $B \subset A$ and B^* an L-space. Then there is a contractive projection $P: A^* \to A^*$ whose kernel is $B^\perp$ and whose range is linearly isometric to B^* under the restriction map $x^* \to x^* | B$ for $x^* \in A^*$.*

Proof. Let $J: B^* \to B^{***}$, $K: A \to A^{**}$, $L: B \to B^{**}$ and $i: B \to A$ be the natural embeddings. Since B^* is an L-space, B^{**} is a P_1 space [see 9, 10, or 17]. Hence there is a contractive projection $Q: A^{**} \to A^{**}$ whose range is $i^{**}(B^{**})$. Set $T = [(i^{**})^{-1}QK]^*J$. Then $\|T\| \leqq 1$ and $Tx^*(ix) = x^*(x)$ for all $x^* \in B^*$ and $x \in B$. For,

$$\begin{aligned}&\langle [(i^{**})^{-1}QK]^*Jx^*, ix\rangle \\ &= \langle Jx^*, (i^{**})^{-1}QKix\rangle = \langle (i^{**})^{-1}QKix, x^*\rangle \\ &= \langle (i^{**})^{-1}Qi^{**}Lx, x^*\rangle = \langle Lx, x^*\rangle = x^*(x)\ .\end{aligned}$$

Hence T is a linear isometry of B^* into A^* and $P = Ti^*$ is the required projection.

For motivational purposes, let K be a compact Choquet simplex and X any closed subset of K which contains the extreme points, EK, of K. Setting $A = \{f | X: f \in A(K)\}$, where $A(K)$ is the space of continuous affine functions on K, it follows from the Bauer maximum principle [4] that $f \to f | X$ is an order preserving linear isometric isomorphism of $A(K)$ onto $A \subset C(X)$. Since each maximal measure on K vanishes on $K \backslash X$, the maximal measures have support contained in X. For each $x \in X$ let μ_x be the maximal measure representing x $\left(\text{i.e., } f(x) = \int f d\mu_x \text{ for all } f \in A\right)$. The linear span, N, of the maximal probability measures on X is a band in $M(X)$ i.e. $M(X) = N \oplus N^p$ where $N^p = \{\sigma \in \mu(X): |\mu| \wedge |\sigma| = 0$ for all $\mu \in N\}$ and $(N^p)^p = N$, and so is the closed linear span, M, of the point masses ξ_x with $x \in Y = X \backslash EK$. For $(\sum_{x \in Y} a_x \xi_x) \in M$, $T(\sum_{x \in Y} a_x \xi_x) = \sum_{x \in Y} a_x \mu_x$ defines a contractive linear operator from M to N. It is well-known that $A = {}^\perp\{T\nu - \nu: \nu \in M\} = \left\{f \in C(X): f(x) = \int f d\mu_x \text{ for all } x \in X\right\}$ and $A^\perp$ is the weak*-closure of $\{T\nu - \nu: \nu \in M\}$. Note that $A^\perp \cap N = \{0\}$.

For, suppose $0 \neq \nu \in A^{\perp} \cap N$. Then since $1 \in A$, $||\nu^{+}|| = ||\nu^{-}||$ and we can assume that $||\nu^{+}|| = ||\nu^{-}|| = 1$. Since $\nu^{+}(h) = \nu^{-}(h)$ for all $h \in A$, ν^{+} and ν^{-} represent the same point of K and since ν^{+}, ν^{-} are both maximal measures and K is a simplex, $\nu^{+} = \nu^{-}$ (see [26]). From this it follows immediately that $N \cap A^{\perp} = \{0\}$.

The main results of this section, Theorem 2.2 below, demonstrates that by placing the preceding remarks in a more general setting, one is able to generate conjugate L-spaces in a nice way. The notation is as follows: A is a Banach space whose dual is an L-space, $P: A^{*} \rightarrow A^{*}$ is a contractive projection, N is the range of P, M is the kernel of P, $Q = I - P$, $T: M \rightarrow N$ is a bounded linear operator, M_0 is the weak*-closure of $\{Tx^{*} - x^{*}: x^{*} \in M\}$ and $B = \{x \in A: Tx^{*}(x) = x^{*}(x)$ for $x^{*} \in M\}$.

THEOREM 2.2. *Preserve the notation above. Then each of the conditions below ensure that B^{*} is an L-space linearly isometric to N.*

(1) *$M_0 \cap N = \{0\}$ and $||P + TQ|| \leqq 1$.*

(2) *M is weak*-closed, T, is weak*-continuous, $||T|| < 1$, and $||P + TQ|| \leqq 1$.*

(3) *N is a band, P is the band projection, $M_0 \cap N = \{0\}$ and T is contractive.*

(4) *N is a band, P is the band projection, M is weak*-closed, T is weak*-continuous, and $||T|| < 1$.*

Proof. The range of any contractive projection on an L-space is also linearly isometric to an L-space (see [17] or [30]).

Hence it is only required to show that B^{*} in each case is linearly isometric to N. The proofs of this fact in statements (1)—(4) are given in the correspondingly numbered paragraphs below.

(1) Let $S: N \rightarrow B^{*}$ be the restriction map. The $||Sx^{*}|| \leqq ||x^{*}||$ for all $x^{*} \in N$ and $Sx^{*} = 0$ implies that $x^{*} \in B^{\perp}$. Since $M_0 \cap N = \{0\}$, S is one-to-one. For $x^{*} \in B^{*}$ let $y^{*} \in A^{*}$ with $y^{*} | B = x^{*}$ and $||y^{*}|| = ||x^{*}||$. Then $z^{*} = (P + TQ)y^{*}$ belongs to N and

$$\begin{aligned} z^{*}(x) &= (P + TQ)y^{*}(x) \\ &= Py^{*}(x) + TQy^{*}(x) \\ &= y^{*}(x) \end{aligned}$$

for $x \in B$. Hence $Sz^{*} = x^{*}$. For $x \in A$, $|z^{*}(x)| \leqq ||P + TQ||\ |y^{*}(x)| \leqq ||y^{*}||\ ||x||$ and S is a linear isometry.

(2) The conditions T is weak*-continuous and $||T|| < 1$ imply that $\{Tx^{*} - x^{*}: x^{*} \in M\}$ is weak*-closed when M is. For if $||Tx^{*} - x^{*}|| \leqq 1$, then $||x^{*}|| \leqq ||Tx^{*} - x^{*}|| + ||Tx^{*}||$ and so $||x^{*}|| \leqq$

$(\|Tx^* - x^*\|)/(1 - \|T\|)$. By the usual compactness argument the unit sphere of $\{Tx^* - x^*: x^* \in M\}$ is weak*-compact and thus $\{Tx^* - x^*: x^* \in M\}$ is weak*-closed. Clearly, $N \cap \{Tx^* - x^*: x^* \in M\} = \{0\}$ and (2) now follows from (1).

(3) The only difference in the proof of (3) and (1) is to notice that because P is the band projection, $\|x^*\| = \|Px^*\| + \|Qx^*\|$ for every $x^* \in A^*$. Hence for $y^* = (P + TQ)x^*$,

$$\begin{aligned}\|y^*\| &\leqq \|Px\| + \|TQx^*\| \\ &\leqq \|Px^*\| + \|T\| \ \|Qx^*\| \\ &\leqq \|Px^*\| + \|Q^*\| \\ &= \|Px^* + Qx^*\| \\ &= \|x^*\| .\end{aligned}$$

It now follows that the restriction map in (1) is an isometry.

(4) Combine the remarks of (2) and (3) above.

Contractive projections (in fact band projections) are plentiful in L-spaces and by Theorem 2.2, many of these projections in dual L-spaces generate Banach spaces whose duals are L-spaces.

To get a less abstract idea of situations analogous to the requirements of Theorem 2.2, some important special cases of the above theorem are summarized below.

1. If $T = 0$ and M weak*-closed (or (weak-*closure of $M) \cap N = \{0\}$), then $B = {}^{\perp}M$.

2. If N has finite codimension, then M and M_0 are automatically weak*-closed. So, T can be any linear operator satisfying $\|P + TQ\| \leqq 1$, or in case P is the band projection onto a band N, just $\|T\| \leqq 1$.

3. If $P: A \to A$ is a contractive projection, then $P^*: A^* \to A^*$ is a contractive projection and kernel $P^* = (\text{Range } P)^{\perp}$ is already weak*-closed.

4. Let X be a compact Hausdorff space and $F \subset X$ a nonempty closed subset. Then $A = \{f \in C(X): f(F) = 0\}$ is a closed sub-lattice of $C(X)$ and $A^{\perp} = \{\mu \in C(X)^*: \mu(X \backslash F) = 0\}$ is a weak*-closed band in $C(X)^*$. Taking N as the complementary band to $A^{\perp}$ and P the band projection provides a nice example of (4).

3. Generation of general Banach spaces whose duals are L-spaces. At this point it is natural to consider a rather simple application of Theorem 2.2. which allows the generation of Banach spaces whose duals are L-spaces. This application is presented in Theorem 3.2 below. First; however, additional terminology is need.

DEFINITION 3.1. Let R be an L-space with unit sphere S. A convex set $F \subset S$ is called a *biface* if its linear span is a band in R (see [14] for a discussion of bifaces).

Examples of such sets are plentiful in L-spaces. Specifically the norm closed absolutely convex hull of any face of a compact Choquet simplex is such a set.

Within the notation of the definition above if E is any subset of the extreme points of S (assuming that S has extreme points), then $E = \{\sum_{e \in E}(a_e)e\colon \sum_{e \in E} |a_e| \leqq 1\}$ is a biface in L.

An additional example of a biface is given by the set of all regular Borel measures on a compact Hausdorff space X whose total variation is 0 on some Borel subset B. If $X \backslash B$ is closed this biface is weak*-closed and hence compact.

THEOREM 3.2. *Let A be a Banach space whose dual is an L-space and F a weak*-closed biface in $S(A^*)$, the unit sphere of A^*. Let M be the span of F and N be the complementary band to M. If $\alpha\colon F \to N \cap S$ is an affine symmetric weak*-continuous function, then for $B = \{x \in A\colon \alpha(x^*)(x) = x^*(x)$ for all $x^* \in F\}$, B^* is linearly isometric to N.*

Proof. Since $F = M \cap S(A^*)$ and F is weak*-closed, M is weak*-closed (see [11]). Because α is affine and symmetric, it has a quique weak*-continuous linear extension $T\colon M \to N$ with $\|T\| \leqq 1$. Theorem 2.2 now completes the proof.

An interesting special case of the above is the following situation. Let $x_1^*, \cdots, x_n^*$ be positive extreme points of A^* and $F = \{\sum_{i=1}^n a_i x_i^*\colon \sum_{i=1}^n |a_i| \leqq 1\}$. Letting $y_1^*, \cdots, y_n^*$ be any elements of A^* such that $|y_i^*| \wedge |x_j^*| = 0$ for $i, j = 1, \cdots, n$, define $\alpha(\sum_{i=1}^n a_i x_i^*) = \sum_{i=1}^n a_i y_i^*$ to obtain a weak*-continuous affine symmetric map from F to the complementary band determined by F. Then $B = \{x \in A\colon y_i^*(x) = x_i^*(x)$ for $i = 1, 2, \cdots, n\}$ is a Banach space whose dual is an L-space.

In particular, if X is a compact Hausdorff space $x_1, \cdots, x_n$ are in X, $\mu_1, \cdots, \mu_n$ are regular Borel measures with $\|\mu_i\| \leqq 1$ and $|\mu_i|(\{x_1, \cdots, x_n\}) = 0$ for $i = 1, 2, \cdots, n$ then $A = \{f \in C(X)\colon f(x_i) = \int f d\mu_i; i = 1, 2, \cdots, n\}$ is a Banach space whose dual is an L-space. This result was obtained by the first author in [6], later refined in [8] and later by Gleit in a different setting in [15].

In view of the preceding theorem, one is naturally lead to ask under what conditions maps from compact Hausdorff spaces to sets of measures induce Banach spaces whose duals are L-spaces. A. Gleit [15] has obtained some partial results. It is possible to use Theorems 2.2 and 3.2 to provide a complete answer.

To motivate subsequent results, return to the setting immediately following Proposition 2.1. That is, let K be a compact Choquet simplex, X any compact set containing EK and μ_x be the unique maximal measure with resultant $x \in K$. For $x \in X$ let $\rho(x) = \mu_x$. From [26, p. 71], $\rho: X \to M(X)$ is Borel measurable (i.e. $x \to \int f d\rho(x)$ is Borel measurable for each $f \in C(X)$). It is also true [27] that $f \to f_\rho$ where $f_\rho(x) = \int f d\rho(x)$ for all Borel measurable functions is a positive projection when restricted to the space $H = \{f + g_\rho: f, g \in C(X)\}$. Equivalently, $(f_\rho)_\rho = f_\rho$ for each continuous f on X. Denoting this projection by P, one has $\|P\| = 1$ when H is given the uniform norm.

Effros has shown that the continuous functions on $\overline{EK}$ which satisfy $f(x) = \int f du_x$ for $x \in \overline{EK}$ are precisely those having a unique continuous affine extension to all K[13]. Obviously the statement remains valid if $\overline{EK}$ is replaced by X. So $A_\rho = \{f: f_\rho = f, f \in C(X)\}$ is linearly order isometric to $A(K)$.

Observe that if $\int f d\mu = \int f d\nu$ for all $f \in A_\rho$ where μ, ν are probability measures on X, then μ and ν have the same resultant in K. Appealing to [26, p. 63] it follows that the function defined by $f_\rho(x) = \int f d\mu_x$ where $f \in C(X)$ is affine and $\int f_\rho d\mu = f_\rho(x)$ for any probability measure μ on K having x as a resultant. Thus $\int f_\rho d\mu = \int f_\rho d\nu$ for all $f \in C(X)$.

Thus $A(K)$ is generated by a natural Borel measurable map ρ defined on a compact set X with $EK \subset X \subset K$. This map satisfies the following three conditions.

(i) f_ρ is Borel measurable for each $f \in C(X)$,

(ii) $\|\rho(x)\| \leqq 1$ for all $x \in X$,

(iii) if $\mu, \nu \in M(X)$ and $\int f d\mu = \int f d\mu$ for all $f \in A$, then $\int f_\rho du = \int f_\rho d\nu$ for all $f \in C(X)$.

As will be demonstrated any map $\rho: X \to M(X)$ having these three properties generates a Banach space whose dual is an L-space. The essential ingredient of condition (i) is that f_ρ is universally integrable (i.e. integrable for each $\mu \in M(X)$).

DEFINITION 3.2. A function $\rho: X \to M(X)$ is said to be *affine* if

(i) f_ρ is universally integrable for all $f \in C(X)$,

(ii) $\|\rho(x)\| \leqq 1$ for all $x \in X$

(iii) if $\mu, \nu \in M(X)$ and $\int f d\mu = \int f d\mu$ for all $f \in C(X)$ such that

$f = f_\rho$, then $\int f_\rho d\mu = \int f_\rho d\nu$ for all $f \in C(X)$.

If p is an affine map on a compact Hausdorff space X, then for $\mu \in M(X)$, $P\mu$ denotes the unique element of $M(X)$ such that $\int fdP\mu = \int f_\rho d\mu$ for all $f \in C(X)$, which exists since the map $f \to \int f_\rho d\mu$ is a continuous linear functional on $C(X)$.

LEMMA 3.4. *Preserve the notation above. Then*

(1) *P is a contractive projection,*

(2) *the range of P is the set $R = \{\mu: \mu \in M(X), \int (f - f_\rho)d\mu = 0$ for all $f \in C(X)\}$.*

(3) *the kernel of P is weak*-closed.*

Proof. (1) that P is linear is trivial. Because $\|\rho(x)\| \leqq 1$, $\|f_\rho\| \leqq \|f\|$ and hence $\|P\| \leqq 1$. Since for $f = f_\rho \in C(X)$ one has $\int fdP_\mu = \int f_\rho d\mu = \int fd\mu$, it follows from (iii) that $\int fdP[P\mu] = \int f_\rho dP\mu = \int fdP\mu$ for all $f \in C(X)$ and, thus, $P[P\mu] = P\mu$.

(2) This follows immediately from the definition of P and (1).

(3) Let $\{\mu_i\}$ be a net element in the kernel of P which is weak*-convergent to μ. Then $\int fd\mu_i \to \int fd\mu$ for $f \in C(X)$ and hence $\int fd\mu = 0$ when $f = f_\rho$. Using (iii) it follows that $\int f_\rho d\mu = 0$ for all $f \in C(X)$ and thus $P\mu = 0$.

The following theorem is an immediate consequence of the above lemma and Theorem 2.2.

THEOREM 3.5. *Let ρ be an affine map from X to $M(X)$. Then the Banach space*

$$A\rho = \{f: f = f_\rho \in C(X)\}$$

is a Banach space whose dual is the L-space $\left\{\mu: \int (f - f_\rho)d\mu = 0 \text{ for } f \in C(X)\right\}$.

Proof. It suffices to show that $A_\rho^\perp$ is the kernel of the contractive projection P defined through ρ. But trivially, by (iii) of the definition of affine map $\int fd\mu = 0$ when $f \in A$ implies $\int f_\rho d\mu = 0$ for $f \in C(X)$. Hence $P\mu = 0$. As the converse is just as trivial, $A_\rho^\perp$ is the kernel of P. Theorem 22 now completes the proof.

The significance of Theorem 3.5 is enhanced by the fact that every Banach space whose dual is an L-space arises in this fashion. The proof of this is based upon recent works of Lazar [23] and Effros [14].

Let A be a Banach space whose dual is an L-space and let $S(A^*)$ denote the unit ball in A^*. Then $S(A^*)$ is an absolutely convex weak*-compact set and so the general Choquet theory may be used to obtain the maximal measures on $S(A^*)$. Such measures are supported by the weak*-closure, X of the set of extreme points of $S(A^*)$.

Define the homeomorphism $\sigma: S(A^*) \rightarrow S(A^*)$ by $\sigma(x) = -x$. Then σ induces natural order preserving isometries $f \rightarrow \sigma f$ and $\mu \rightarrow \sigma\mu$ on $C(S(A^*))$ and $M(S(A^*))$ respectively. These are defined by the formulas

$$\sigma f(x) = f(\sigma x)$$

and

$$\sigma\mu(C) = \mu(\sigma C) .$$

For $\mu \in M(S(A^*))$ let odd $(\mu) = 1/2(\mu - \sigma\mu)$. From [23] it follows that odd $(\mu) =$ odd (ν) for any pair of maximal measures on $S(A^*)$ having the same resultant in $S(A^*)$. Since each maximal measure on $S(A^*)$ vanishes off X, they can be considered as maximal measures on X. For $x \in X$ let μ be any maximal measure which represents x and let $\rho(x) =$ odd μ. Then $\rho: X \rightarrow M(X)$ and $\|\rho(x)\| \leqq 1$ for all $x \in X$.

It is shown in the sequel that ρ is an affine map. A positive measure μ on $S(A^*)$ is maximal if and only if $\int f d\mu = \int \bar{f} d\mu$ for all $f \in C(S(A^*))$ (see [26]). For such a measure it follows that $\int f d\sigma\mu = \int \bar{f} d\sigma\mu$ and $\int \sigma f d\mu = \int \overline{\sigma f} d\mu$ for all $f \in C(S(A^*))$. The main theorem of [23] then demonstrates that

$$\frac{1}{2}[\bar{f}(x) - \sigma\bar{f}(x)] = \int \frac{1}{2}[f - \sigma f] d\mu$$

for any maximal measure with resultant x and continuous convex function f on $S(A^*)$. Since

$$\frac{1}{2}\int [f - \sigma f] d\mu = \int f d \text{ odd } (\mu) ,$$

it is true that

$$(*) \qquad \frac{1}{2}[\bar{f}(x) - \sigma\bar{f}(x)] = \int f d[\text{odd } (\mu)] .$$

The preceding remarks provide a portion of the proof of the lemma below.

LEMMA 3.6. *Preserve the notation above. Then $\rho: X \to M(X)$ is affine.*

Proof. Condition (ii) of the definition is already verified. That condition (i) is true is an easy consequence of (*) above. Only the verification of (iii) is lacking.

Define $P: M(X) \to M(X)$ as follows: For $\nu \geqq 0$ in $M(X)$ let β be any maximal measure which dominates in the ordering of Choquet (see [26]) and set $P\nu = \text{odd}\ \beta$. For arbitrary $\nu = \nu^+ - \nu^- \in M(X)$ let $P\nu = P\nu^+ - P\nu^-$. Then P is linear, $\|P\| \leqq 1$ and because odd (odd β) = odd (β) for any $\beta \in M(X)$, P is a projection.

The kernel of the projection P is the set of measures μ in $M(X)$ which satisfy $\int f\rho d\mu = 0$ for all $f \in C(X)$. If $P = 0$ and f is continuous and convex, then

$$\int f_\rho du = \frac{1}{2}\int[\bar{f} - \sigma\bar{f}]d\mu = \frac{1}{2}\int[\bar{f} - \sigma\bar{f}]d\mu^+ - \frac{1}{2}\int[\bar{f} - \bar{f}]d\mu .$$

Now, let μ_1, μ_2 be maximal measures majorizing μ^+, μ^- respectively in Choquet's ordering. Since $\bar{f} - \bar{f}$ is affine and satisfies the bary centric formula [26, p. 100] one has

$$\begin{aligned} \int f_\rho d\mu &= \frac{1}{2}\int[\bar{f} - \sigma\bar{f}]d\mu_1 - \frac{1}{2}\int[\bar{f} - \sigma\bar{f}]d\mu_2 \\ &= \frac{1}{2}\int[f - \sigma f]d\mu_1 - \frac{1}{2}\int[f - \sigma f]d\mu_2 \\ &= \int fdP\mu^+ - \int fdP\mu^- = \int fdP\mu = 0 . \end{aligned}$$

It is now clear that $P\mu = 0$ implies $\int f_\rho d\mu = 0$ for all $f \in C(X)$. On the other hand the condition $\int f_\rho dP\mu = 0$ for all $f \in C(X)$ together with the fact that $\sigma f = -f$ if $f \in A \subset A(S(A^*))$ implies that $\int fd\mu = 0$ when $f \in A$. Let μ_1 and μ_2 be maximal measures whose resultants x_1 and x_2 are the same as those of μ^+ and μ^-. If $x_1 \neq x_2$ there is some $g \in A$ such that $g(x_1) \neq g(x_2)$. For g,

$$\int gd(\mu_1 - \mu_2) = g(x_1) - g(x_2) = \int gd(\mu^+ - \mu^-) \neq 0$$

is a contradiction and so $x_1 = x_2$. Thus odd (μ_1) = odd (μ_2) and $P\mu = 0$.

A consequences of the above is the fact that $\int f_\rho d(I-P)\mu = 0$ for all $\beta \in M(X)$. This is so because the range of $I - P$ is precisely the kernel of P. Suppose $\beta \geqq 0$ and $||\beta|| = 1$. If μ is a maximal measure which dominates β in Choquet's ordering and $f \in C(S(A^*))$, then since

$f_\rho = \frac{1}{2}(\bar{f} - \sigma\bar{f})$,

$$\int f_\rho dP\beta = \frac{1}{4}\Big[\int \bar{f} d\mu - \int \bar{f}\sigma d\mu - \int \sigma\bar{f} d\mu + \int \sigma f d\sigma\mu\Big]$$
$$= \frac{1}{2}\Big[\int f d\mu - \int f d\sigma\mu\Big] = \int f dP\beta \, .$$

From this it follows that $\int g dP\beta = \int g_\rho dP\beta$ for all $g \in C(X)$ and all $\beta \in M(X)$. Hence $\int f dP\beta = \int f_\rho d\beta$ for all $f \in C(X)$.

Arguing as in the case which showed that $\{\mu : P\mu = 0\} = \left\{\mu : \int f_\rho d\mu = 0\right\}$ it is possible to show that $\int f d\beta = 0$ for $f = f_\rho$ implies that $P\beta = 0$. Hence if $\mu, \nu \in M(X)$ and $\int f d\mu = \int f d\nu$ when $f = f_\rho$, $P(\mu - \nu) = 0$ so that $\int g dP\mu = \int g dP\nu$ for all $g \in C(X)$. Hence $\int g_\rho d\nu = \int g_\rho d\nu$ for all $g \in C(X)$. This completes the proof of the lemma.

Lemma 3.6 is fundamental in the proof of 3.7 below.

THEOREM 3.7. *Suppose A is a Banach space whose dual is an L-space. Then there is a compact Hausdorff space X and affine map $\rho: X \rightarrow M(X)$ such that*

$$A = \left\{f \in C(X) : f(x) = \int f d\rho(x)\right\} .$$

Proof. Let ρ and X be as in Lemma 3.6 and the remarks preceding it. By Lemma 3.6 and Theorem 3.5, $A_\rho = \{f : f = f_\rho \in C(X)\}$ is a Banach space whose dual is an L-space. Since $A^\perp = \{\mu : P\mu = 0\}$, $A\rho = A$ and the proof is complete.

These affine mappings can be used to obtain some known results in an efficient manner. For example, if $\{K_i\}_{i \in I}$ is a family of compact Choquet simplexes and $K = \prod_{i \in I} K_i$ is the product of the K_i's, let $\rho : K \rightarrow M(K)$ be defined by $\rho(\{X_i\}) = \prod_{i \in I} \mu_i$ where μ_i is the maximal measure representing x_i for each $i \in I$. Then ρ is an affine mapping and $A_\rho = A(L)$ where L is the product simplex as defined in [21].

Another example is the following. Suppose A is a Banach space whose dual is an L-space and that the extreme points X of $S(A^*)$

form a closed set. Let $\Sigma: X \to X$ be defined by $\Sigma(x^*) = -x^*$. Then Σ is an involutary homeomorphism. The ρ of Theorem 3.7 is given by $\rho(x^*) = 1/2[\xi_{x^*} - \xi_{\Sigma(x^*)}]$ and $A = \{f \in C(X): f(x^*) = -f(\Sigma x^*)\}$, i.e., $A = C_\Sigma(X)$. (This was proved in [25].)

We close with an application which seems to be new. Let A be a Banach space whose dual is an L-space and suppose that the weak*-closure X of the set E of extreme points of the unit sphere of A^* is $E \cup \{0\}$. The mapping defined by $\sigma x^* = -x^*$ is an involutary homeomorphism on X, i.e., $\sigma^2 =$ identity.

THEOREM 3.8. *$A = C_\sigma(X) = \{f \in C(X): f(x^*) = -f(\sigma x^*)$ for all $x^* \in X\}$.*

Proof. The mapping ρ is given by $\rho(x^*) = 1/2[\xi_{x^*} - \xi_{\sigma x^*}]$ for $x^* \in E$ and $\rho(0) = 0$. By Theorem 3.7,

$$A = A_\rho = \left\{f \in C(X): f(x^*) = \int f d\rho(x^*)\right\} = C_\sigma(X) .$$

This result is similar to the one for $C_\Sigma(X)$ above when the extreme points are closed.

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