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TRANSVERSAL MATROIDS AND HALL'S THEOREM

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Transversal matroids, not necessarily having finite character, are investigated. It is demonstrated that if $\mathfrak{A}(I) = (A_i: i \in I)$ is an arbitrary family of subsets of an arbitrary set E whose transversal matroid has at least one basis and has no coloops, then $\mathfrak{A}(I)$ has a transversal; in fact, each basis is a transversal of $\mathfrak{A}(I)$ but of no proper subfamily of $\mathfrak{A}(I)$. P. Hall's theorem on the existence of a transversal for a finite family, and indeed an extension of it, can be obtained from this result.

Some necessary conditions for a matroid to be a transversal matroid are derived. One of these is that a transversal matroid of rank r can have at most $\binom{r}{k}$ k-flats having no coloops $(1 \leqq k \leqq r)$.

1. Matroids. Let E be a set. A *matroid* [14, 15, 16] on E is a nonempty collection $\underline{M}$ of subsets of E such that

(i) $$A \in \underline{M},\ A' \subseteqq A \text{ imply } A' \in \underline{M}\,.$$

(ii) $$A_1, A_2 \in \underline{M},\ |A_1| < |A_2| < \infty \text{ imply there exists } x \in A_2 \backslash A_1 \text{ such that } A_1 \cup x^\dagger \in \underline{M}\,.$$

The members of M are called *independent sets*; those subsets of E not in $\underline{M}$ are *dependent sets*. The matroid $\underline{M}$ on E is said to have *finite character* provided

(iii) $$A \in \underline{M},\ A' \in \underline{M} \text{ for all finite sets } A' \subseteqq A \text{ imply } A \in \underline{M}\,.$$

If E is a finite set, a matroid on E is always a finite character matroid, and a matroid on E is the collection of independent sets of a *combinatorial pregeometry* [4] on E. Finite character matroids arise from many mathematical situations including graphs, vector spaces, geometry, and so on. For details the reader is referred to Crapo and Rota [4]. Matroids not necessarily having finite character also arise in important ways, and we shall be concerned with a certain class of such matroids.

Let $\underline{M}$ be a matroid on E. A *basis* of $\underline{M}$ is a maximal, with respect to set-theoretic inclusion, member of $\underline{M}$. Bases need not exist as is easily seen by taking E to be an uncountable set and $\underline{M}$ to be all finite or countably infinite subsets of E. However, if E is finite,

† The set $\{x\}$ is usually denoted by x.

bases surely exist; if $\underline{M}$ has finite character, then Zorn's lemma in conjunction with the finite character property (iii) guarantees the existence of bases and indeed that every independent set is contained in a basis. It is well-known [13, 2] that in a finite character matroid $\underline{M}$ all bases have the same cardinal number called the *rank* of $\underline{M}$. A *circuit* is a set $C \subseteqq E$ which is a minimal dependent set. If the matroid has finite character, it follows from (iii) that circuits are finite sets.

If $\underline{M}$ is a matroid on E and $A \subseteqq E$, then we define $\underline{M}_A$ by

$$\underline{M}_A = \{F: F \subseteqq A, F \in \underline{M}\} .$$

It is clear that $\underline{M}_A$ is a matroid on A, called the *restriction* of $\underline{M}$ to A. If $(E_i: i \in I)$ is a family of pairwise disjoint sets and $\underline{M}_i$ is a matroid on $E_i(i \in I)$, then a matroid $\underline{M}$ on $E = \bigcup_{i \in I} E_i$ can be defined by

$$\underline{M} = \left\{\bigcup_{i \in I} A_i: A_i \in \underline{M}_i(i \in I)\right\} .$$

The matroid $\underline{M}$ is called the *direct sum* of $\underline{M}_i(i \in I)$ and is denoted by $\bigoplus_{i \in I} \underline{M}_i$. If $|I| < \infty$ and $\underline{M}_i$ has finite rank $r_i(i \in I)$, then $\bigoplus_{i \in I} \underline{M}_i$ has finite rank $\sum_{i \in I} r_i$.

If $\underline{M}$ is a matroid on E, then $\underline{M}$ is *connected* or *nonseparable* provided it is impossible to partition E into nonempty sets E_1, E_2 in such a way that $\underline{M} = \underline{M}_{E_1} \oplus \underline{M}_{E_2}$. The element x of E is a *loop* of $\underline{M}$ if $\{x\} \notin \underline{M}$; thus loops can be part of no independent sets. The element x is called a *coloop* or *isthmus* provided $A \cup x \in \underline{M}$ whenever $A \in \underline{M}$; thus coloops are part of every basis. If x is either a loop or coloop, then $\underline{M} = \underline{M}_{\{x\}} \oplus \underline{M}_{E \setminus x}$ so that $\underline{M}$ cannot be connected. If X is a set of coloops of $\underline{M}$, then $\underline{M} = \underline{M}_X \oplus \underline{M}_{E \setminus X}$ where $\underline{M}_X$ is the *free matroid* or *Boolean algebra* on X. In what is to follow, matroids having no coloops play an important role; such matroids will be called coloop-free. For these matroids it is impossible to 'split off' a Boolean algebra.

Finally, we introduce the notion of a flat. A set $F \subseteqq E$ is a *flat* of the matroid $\underline{M}$ on E or is *closed* provided $A \subseteqq F$, $A \in \underline{M}$ and $x \in E \setminus F$ imply that $A \cup x \in \underline{M}$. If $\underline{M}_F$ has finite rank, then this means that enlarging F in any way increases the rank or equivalently that given $x \in E \setminus F$ there is no circuit C with $x \in C \subseteqq F \cup x$. If the rank of $\underline{M}_F$ is $k < \infty$, then F is called a k-flat. Observe that each coloop is a 1-flat (but not conversely) and that the set of all loops is the only 0-flat. In case $\underline{M}$ has finite rank, the collection of flats form a geometric lattice [4] with respect to set-theoretic inclusion.

2. Transversal matroids. An important class of matroids, discovered by Edmonds and Fulkerson [6], are those known as transversal matroids. These are defined as follows. Let $\mathfrak{A} = \mathfrak{A}(I) = (A_i: i \in I)$ be a family of subsets of E. A set $T \subseteqq E$ is a *transversal* of $\mathfrak{A}$ provided there is a bijection $\theta: T \rightarrow I$ such that $x \in A_{\theta(x)} (x \in T)$. If θ is only an injection, then T is a *partial transversal*; in this case T is a transversal of the subfamily $\mathfrak{A}(K) = (A_i: i \in K)$ where $K = \theta(T)$. If $\underline{M}(\mathfrak{A})$ denotes the collection of all partial transversals of $\mathfrak{A}$, then $\underline{M}(\mathfrak{A})$ is a matroid on E [6, 11]. If each element of E is a member of only finitely many sets of the family $\mathfrak{A}$, then $\underline{M}(\mathfrak{A})$ is a finite-character matroid. A matroid $\underline{M}$ on E is a *transversal matroid* provided there is a family $\mathfrak{A}$ of subsets of E such that $\underline{M} = \underline{M}(\mathfrak{A})$.

The bases of the transversal matroid $\underline{M}(\mathfrak{A})$, if there are any, are the maximal partial transversals of $\mathfrak{A}$. These need not however be transversals of $\mathfrak{A}$. However the following theorem is proved in Brualdi and Scrimger [1], although not stated in this form.

THEOREM 2.1. *Let $\mathfrak{A}(I) = (A_i: i \in I)$ be a family of subsets of E. Let B be a basis of $\underline{M}(\mathfrak{A})$ and let $\mathfrak{A}' = \mathfrak{A}(K) = (A_i: i \in K)$ be any subfamily of $\mathfrak{A}$ of which B is a transversal. Then $\underline{M}(\mathfrak{A}) = \underline{M}(\mathfrak{A}')$ and every basis of $\underline{M}(\mathfrak{A})$ is a transversal of $\mathfrak{A}'$.*

One of the results of this paper is that in Theorem 2.1 K can only be I if $A_i \neq \phi (i \in I)$ and the matroid $\underline{M}(\mathfrak{A})$ has no coloops. Before getting to this, it is convenient to place our discussion in a graph-theoretic setting, for some of our proofs are graph-theoretic in nature.

A *bipartite graph* may be regarded as a triple (X, Δ, Y) where X and Y are disjoint sets and $\Delta \subseteqq X \times Y$. The members of Δ are the *edges* of the graph, which we regard as undirected. Let $A \subseteqq X$, $B \subseteqq Y$. Then A and B are *linked* in the bipartite graph (or A is linked to B or B is linked to A) provided there is a bijection $\theta: A \rightarrow B$ with $\Delta' = \{(x, \theta(x)): x \in A\} \subseteqq \Delta$. The bijection θ is called a *linking of A to B* and the members of Δ' are the *edges of the linking*. If $\mathfrak{A}(I) = (A_i: i \in I)$ is a family of subsets of a set E, then a bipartite graph (E, Δ, I) can be associated where $(e, i) \in \Delta$ if and only if $e \in A_i (e \in E, i \in I)$. The partial transversals of $\mathfrak{A}$ are precisely those subsets of E which are linked to some subset of I. It is then clear that a matroid $\underline{M}$ on a set X is a transversal matroid provided there is a bipartite graph (X, Δ, Y) such that $\underline{M}$ consists of those subsets of X which are linked to at least one subset of Y. Such a bipartite graph is said to *induce* $\underline{M}$ on X. The bipartite graph likewise induces a matroid $\underline{M}'$ on Y. For $x \in X$, $A \subseteqq X$, we set $\Delta(x) = \{y \in Y: (x, y) \in \Delta\}$ and $\Delta(A) = \bigcup_{x \in A} \Delta(x)$. If $y \in Y, B \subseteqq Y$, then $\Delta(y)$ and $\Delta(B)$ are defined analogously.

3. Induced matroids. If we phrase the first part of Theorem 2.1 in terms of bipartite graphs, it becomes: *if* (X, Δ, Y) *is a bipartite graph inducing the matroid* $\underline{M}$ *on* X, *if* B *is a basis of* $\underline{M}$ *and* Z *is any subset of* Y *to which* B *is linked, then the bipartite graph* (X, Δ', Z) *where* $\Delta' = \Delta \cap \{X \times Z\}$ *already induces* $\underline{M}$ *on* X.

The main result of the section deals with a closer analysis of the above situation. Before stating it we record a lemma.

LEMMA 3.1. *Let* (X, Δ, Y) *be a bipartite graph inducing the matroid* $\underline{M}$ *on* X. *Assume that* $\underline{M}$ *is coloop-free. Then if* B *is a basis of* $\underline{M}$ *and* $z \in B$, *there exists* $x \in X \backslash B$ *such that* $\{B \backslash z\} \cup x$ *is a basis of* $\underline{M}$.

The result is true for any coloop-free finite-character matroid. The matroid $\underline{M}$ in the lemma above is not necessarily a finite-character one, but is assumed to be a transversal matroid.

Proof. Since $\underline{M}$ is coloop-free, $B \neq X$. Assume $\{B \backslash z\} \cup x$ is not a basis of $\underline{M}$ for all $x \in X \backslash B$. Let $A \in \underline{M}$ with $z \notin A$. We will show that $A \cup z \in \underline{M}$, so that z is a coloop of $\underline{M}$, a contradiction.

If $A \subseteqq B$, then $A \cup z \in \underline{M}$. Thus we may assume $A \backslash B \neq \phi$. Let Δ_1 be the edges of a linking of B to a set $Z_1 \subset Y$ and Δ_2 the edges of a linking of A to a set $Z_2 \subset Y$. (We could assume from the result mentioned above that $Z_2 \subseteqq Z_1$.) Each $x \in A \backslash B$ determines a path P_x beginning at x whose edges alternate in Δ_2 and Δ_1. Let Δ_i^x denote the set of edges of Δ_i on this path $(i = 1, 2)$. If P_x is either an infinite path or terminates at an element of $Z_2 \backslash Z_1$, then $\{\Delta_1 \backslash \Delta_1^x\} \cup \Delta_2^x$ is the set of edges of a linking of $B \cup x$ to a subset of Y. This contradicts the basis property of B. The only other alternative is that P_x terminates at an element w_x of $B \backslash A$. If $w_x = z$, then $\{\Delta_1 \backslash \Delta_1^x\} \cup \Delta_2^x$ is the set of edges of a linking of $\{B \backslash z\} \cup x$ to a subset of Y. It follows as in [1], that $\{B \backslash z\} \cup x$ is a basis of $\underline{M}$. Since we are assuming this is not the case $w_x \neq z$. Since this is true for all $x \in A \backslash B$, the path Q_z determined by z whose edges alternate in Δ_1 and Δ_2 must either be infinite or terminate in $Z_1 \backslash Z_2$. For, if Q_z terminates at some $x \in A \backslash B$, the only other alternative, we would have that P_x terminates at z and thus $w_x = z$. Thus following the above convention, $\{\Delta_2 \backslash \Delta_2^z\} \cup \Delta_1^z$ is the set of edges of a linking of $A \cup \{z\}$ to a subset of Y, so that $A \cup \{z\} \in \underline{M}$. Since this is true for all $A \in \underline{M}$, it follows that z is a coloop. This completes the proof of the lemma.

We now state and prove the main result.

THEOREM 3.2. *Let (X, Δ, Y) be a bipartite graph inducing the matroid $\underline{M}$ on X. Assume that $\underline{M}$ is coloop-free. Then if B is any basis of $\underline{M}$ and Z is any subset of Y to which B is linked, then $\Delta(X) = Z$.*

We point out once more that since we are not assuming $\underline{M}$ has finite character, the matroid $\underline{M}$ may not have any bases, in which case the theorem says nothing.

Proof. There is no loss in generality in assuming that $\Delta(X) = Y$, for those elements $y \in Y$ with $\Delta(y) = \phi$ play no role whatsoever. The conclusion of the theorem is then that $Z = Y$.

Let B_1 be a basis of $\underline{M}$ with B_1 linked to $Z_1 \subseteqq Y$. If $x \in X \backslash B_1$, then the maximality of B_1 implies that $\Delta(x) \subseteqq Z_1 \subseteqq \Delta(B_1)$. Suppose $Y \backslash Z_1 \neq \phi$. Then there exists a $w \in B_1$ such that $\Delta(w) \cap \{Y \backslash Z_1\} \neq \phi$. By Lemma 2.1 there exists $x \in X \backslash B_1$ such that $B_2 = \{B_1 \backslash w\} \cup x$ is a basis of $\underline{M}$. By Theorem 2.1 B_2 is linked to Z_1. Since $w \notin B_2$, this means that $B_2 \cup w$ is linked to $Z_1 \cup z$ where $z \in \Delta(w) \cap \{Y \backslash Z_1\}$. Hence $B_2 \cup w \in \underline{M}$, and this contradicts the fact that B_2 is a basis of $\underline{M}$. The $Y = Z_1$ and the theorem is proved.

In case the matroid $\underline{M}$ has finite rank, the above theorem takes on an appealing form. It is proved by Mason in [9, 10]. It can also be proved using a canonical decomposition of bipartite graphs which was derived by Dulmage and Mendelsohn [5] as an extension of a result of Ore [12].

COROLLARY 3.3. *Let (X, Δ, Y) be a bipartite graph inducing the matroid M of finite rank r on X. If M is coloop-free, then $|\Delta(X)| = r$.*

In this case $|B| = |Z| = r$. Effectively what the corollary says is that a coloop-free transversal matroid $\underline{M}$ on a set X with rank r can only be induced by bipartite graphs (X, Δ, Y) where $|Y| = r$. Observe that the corollary applies in case $\underline{M}$ is connected with $|X| > 1$. As an example it applies to any matroid of the form $\underline{M} = \mathscr{P}_r(X) = \{A \subseteqq X: |A| \leqq r\}$, $1 \leqq r \leqq |X|$. This matroid is easily seen to be a connected, transversal matroid. If $r = |X| - 1$, it is just a circuit.

THEOREM 3.4. *Let (X, Δ, Y) be a bipartite graph inducing the matroid $\underline{M}$ on X. Let $A \subseteqq X$ and suppose $\underline{M}_A$ is coloop-free. If B' is any basis of $\underline{M}_A$, there is a unique subset of Y, namely $\Delta(A)$, to which B' is linked. In particular, if $\underline{M}_A$ is coloop-free of rank t, then $|\Delta(A)| = t$.*

Proof. It is clear that (A, Δ', Y) induces $\underline{M}_A$ on A where $\Delta' = \Delta \cap \{A \times Y\}$. Let B' be a basis of $\underline{M}_A$ and let Z' be *any* subset of Y to which B' is linked. Applying Theorem 3.2 to this new bipartite graph, we conclude that $\Delta'(A) = Z'$. Since $\Delta(A) = \Delta'(A)$, this established the theorem.

THEOREM 3.5. *Let the bipartite graph (X, Δ, Y) induce the matroid $\underline{M}$ on X and the matroid $\underline{M}'$ on Y. If $\underline{M}$ has a basis and is coloop-free, then $\underline{M}'$ is the free matroid on Y.*

Proof. Let B be a basis of $\underline{M}$. Since $\underline{M}$ has no coloops, it follows from Theorem 3.2 that $\Delta(X) \in \underline{M}'$. Thus if $y \in \Delta(X)$, y is a coloop of $\underline{M}'$, while if $y \in Y \backslash \Delta(X)$, y is a loop of $\underline{M}'$. The conclusion is now obvious.

A particular case of Theorem 3.5 asserts that if one of $\underline{M}$ and $\underline{M}'$ bas a basis (e.g. if one has finite character), not both of $\underline{M}$ and $\underline{M}'$ can be coloop-free and, in particular, not both can be connected.

4. Applications to transversal theory. Let $\mathfrak{A} = \mathfrak{A}(I) = (A_i: i \in I)$ be a family of subsets of a set E. For $K \subseteqq I$, let $A(K) = \bigcup_{i \in K} A_i$. If $|I| < \infty$, so that $\mathfrak{A}$ is a finite family, then the well-known theorem of P. Hall [7] asserts that $\mathfrak{A}$ has a transversal if and only if $|A(K)| \geqq |K|$ for all $K \subseteqq I$. If $|I| = \infty$ but each A_i is a finite set $(i \in I)$, the extension due to M. Hall Jr. [8] of this result asserts that $\mathfrak{A}$ has a transversal if and only if $|A(K)| \geqq |K|$ for all finite sets $K \subseteqq I$. We offer the following theorem.

THEOREM 4.1. *Let $\mathfrak{A}(I) = (A_i: i \in I)$ be a family of nonempty subsets of a set E. If the matroid $\underline{M}(\mathfrak{A})$ has a basis and is coloop-free, then the family $\mathfrak{A}(I)$ has a transversal.*

Proof. Assume, without loss in generality, that $I \cap E = \phi$, and consider the bipartite graph (E, Δ, I) where $\Delta = \{(e, i): e \in A_i, i \in I\}$. This bipartite graph induces the matroid $\underline{M}(\mathfrak{A})$ on E. If B is any basis of $\underline{M}(\mathfrak{A})$ and J is any subset of I to which B is linked, then from Theorem 3.2 we conclude that $\Delta(E) = J$. On the other hand since $A_i \neq \phi (i \in I)$, $\Delta(E) = I$. Hence $J = I$ and B is linked to I in the bipartite graph. But this means that B is a transversal of the family $\mathfrak{A}(I)$.

If in the theorem each element of E is a member of only finitely many A's, then $\underline{M}(\mathfrak{A})$ is a finite character matroid and hence has bases.

COROLLARY 4.2. *If the matroid of a family $\mathfrak{A}(I)$ of nonempty* *subsets of a set E with $|E| > 1$ has a basis and is connected, then the family has a transversal.*

A connected matroid on a set with more than one element cannot have any coloops.

A more detailed analysis produces the following theorem which contains P. Hall's theorem as a special case, but not necessarily M. Hall's theorem. (On the other hand, M. Hall's theorem follows easily from P. Hall's theorem through a simple application of Rado's selection principle or other theorems dependent on the axiom of choice.)

To say that a matroid $\underline{M}$ has only a finite number of coloops is equivalent to saying that an infinite Boolean algebra can not be "split off" from $\underline{M}$.

THEOREM 4.3. *Let $\mathfrak{A}(I) = (A_i\colon i \in I)$ be a family of subsets of a set E. Assume the matroid $\underline{M}(\mathfrak{A})$ has a basis and only a finite number of coloops. Then $\mathfrak{A}(I)$ has a transversal if and only if*

$$(1) \qquad |A(K)| \geqq |K| \quad (K \text{ finite, } K \subseteqq I).$$

Proof. Let (E, Δ, I) be the bipartite graph associated as before with the family $\mathfrak{A}(I)$. Let F be the set of coloops of $\underline{M} = \underline{M}(\mathfrak{A})$ and $E' = E\backslash F$. By assumption F is a finite set. The matroid $\underline{M}_{E'}$ has a basis, since $\underline{M}$ has a basis: if B is a basis of $\underline{M}$, then $B\backslash F$ is a basis of $\underline{M}_{E'}$. Moreover the matroid $\underline{M}_{E'}$ has no coloops. For, if x were a coloop of $\underline{M}_{E'}$ and $A \in \underline{M}$, then $\{A \cap E'\} \cup x \in \underline{M}$ and thus $A \cup x \subseteqq \{A \cap E'\} \cup x \cup F \in \underline{M}$. Thus x is a coloop of $\underline{M}$ with $x \notin F$, and this contradicts the choice of F. Let B' be a basis of $\underline{M}_{E'}$. By Theorem 3.4 B' is linked in the bipartite graph to $J = \Delta(E')$ and J is the only subset of I having this property. Since $J = \Delta(E')$, it follows that $\Delta(I\backslash J) \subseteqq F$. Thus $\mathfrak{A}(I)$ has a transversal if and only if the subfamily $\mathfrak{A}(I\backslash J)$, which is a family of subsets of the *finite* set F, has a transversal.

Suppose now condition (1) is satisfied for all finite $K \subseteqq I$ and thus for all finite $K \subseteqq I\backslash J$. Since F is a finite set and $|A(K)| \leqq |F|$ for all $K \subseteqq I\backslash J$, the set $I\backslash J$ must be finite. Since $B' \cup F \in \underline{M}$ and B' is linked only to J, it follows that F is linked to a subset of $I\backslash J$, so that $|I\backslash J| \geqq |F|$. But then

$$|F| \geqq |A(I\backslash J)| \geqq |I\backslash J| \geqq |F| ,$$

so that $|I\backslash J| = F$. Hence F is linked to $I\backslash J$. This means that $\mathfrak{A}(I)$ has a transversal, namely $B' \cup F$. Since condition (1) is obviously necessary for $\mathfrak{A}(I)$ to have a transversal, the proof of the theorem is complete.

The proof of the theorem indicates how to find a single set $\bar{K} \subseteqq I$ such that $\mathfrak{A}(I)$ has a transversal if and only if $|A(\bar{K})| \geqq |\bar{K}|$. For taking $\bar{K} = I \backslash J$, it was demonstrated in the proof that if $|A(\bar{K})| \geqq |\bar{K}|$, then $\mathfrak{A}(I)$ has a transversal, while if $|A(\bar{K}) < |\bar{K}|$ this would mean that $|F| < |\bar{K}|$ so that $\mathfrak{A}(I)$ could not have a transversal.

As a corollary to Theorem 4.3 we obtain P. Hall's theorem [7].

COROLLARY 4.4. (P. Hall). *Let $\mathfrak{A} = (A_i: 1 \leqq i \leqq n)$ be a family of subsets of E. Then $\mathfrak{A}$ has a transversal if and only if*

$$|A(K)| \geqq |K| \ (K \subseteqq \{1, \cdots, n\}) .$$

In this case the matroid $\underline{M}(\mathfrak{A})$ has finite rank, so that it has a basis and can only have a finite number of coloops.

We also remark here that Theorem 4.3 applies to any family $\mathfrak{A}(I)$ of subsets of E such that each element of E is a member of only finitely many A's and the matroid $\underline{M}(\mathfrak{A})$ has only a finite number of coloops.

If in Theorem 4.3 the matroid $\underline{M}(\mathfrak{A})$ has an infinite number of coloops, then condition (1) is no longer sufficient for $\mathfrak{A}$ to have a transversal. This is already seen from M. Hall's much quoted example [8] where $I = E = \{1, 2, \cdots\}$ and $A_1 = E, A_i = \{i + 1\}$ $(1 \geqq 2)$. In this case $E \in \underline{M}(\mathfrak{A})$ so that each element of E is a coloop. Condition (1) is satisfied but there is no transversal.

5. **Transversal matroids.** In general it is difficult to decide whether a given matroid is a transversal matroid. A characterization of finite-character transversal matroids in terms of a rank inequality on unions of circuits is given by Mason [9, 10], but it is difficult to check. The following result is contained implicity in [1].

THEOREM 5.1. *Let $\underline{M}$ be a transversal matroid on a set E. Let B_1 and B_2 be bases of $\underline{M}$. Then there exists a bijection $\sigma: B_1 \rightarrow B_2$ such that both $\{B_1 \backslash x\} \cup \sigma(x)$ and $\{B_2 \backslash \sigma(x)\} \cup x$ are bases for all $x \in B_1$.*

For finite-character matroids a σ satisfying the exchange property in this theorem can always be defined, as is proved in [2], but σ need not be a bijection or injection. Indeed the example given in [2] for which it is impossible to define a bijective σ amounts to the cycle matroid of the complete graph on 4 nodes, K_4. (The *cycle matroid* of a graph is the matroid on its edge set such that a set of edges is independent if and only if it does not contain the edges of a polygon; thus the circuits are the edge sets of polygons.) Thus Theorem

5.1 furnishes a necessary, but not sufficient, condition for a matroid to be a transversal matroid. We shall use the results of § 3 to obtain other necessary conditions.

THEOREM 5.2. *Let $\underline{M}$ be a transversal matroid on a set E with finite rank r. Let k be any integer with $1 \leqq k \leqq r$. Then $\underline{M}$ has at most $\binom{r}{k}$ coloop-free k-flats.*

Proof. Let (E, Δ, Y) be a bipartite graph which induces $\underline{M}$ on E. By Corollary 2.2 we may assume $|Y| = r$. Let $1 \leqq k \leqq r$ and let $(F_j: j \in J)$ be the family of distinct coloop-free k-flats of $\underline{M}$, indexed by J. We need to show that $|J| \leqq \binom{r}{k}$. By Theorem 3.4, since $\underline{M}_{F_j}$ is coloop-free with rank k, $|\Delta(F_j)| = k$ $(j \in J)$. Suppose $|J| > \binom{r}{k}$. It would then follow, since $|Y| = r$, that $\Delta(F_{j_1}) = \Delta(F_{j_2})$ for some $j_1, j_2 \in J$ with $j_1 \neq j_2$. This would mean that $|\Delta(F_{j_1} \cup F_{j_2})| = k$ and thus that $\underline{M}_{F_{j1} \cup F_{j2}}$ has rank k. But since F_{j_1}, F_{j_2} are distinct k-flats, $\underline{M}_{F_{j1} \cup F_{j2}}$ has rank greater than k. This is a contradiction and the theorem is proved.

As an example consider once again the rank 3 cycle matroid $\underline{M}$ on the set of edges $E = \{1, 2, \cdots, 6\}$ of the complete 4-graph K_4. (Figure 1) The set $\{1, 2, 5\}, \{2, 3, 6\}, \{3, 4, 5\}, \{1, 4, 6\}$ are all coloop-free 2-flats of $\underline{M}$. Since $4 > \binom{3}{2}$, it follows by Theorem 5.2 that $\underline{M}$ is not a transversal matroid.

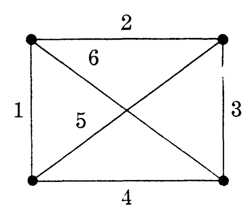

FIGURE 1

Theorem 5.2 can also be used to demonstrate that matroids of infinite rank are not transversal. This is so because if $\underline{M}$ is a transversal matroid on E and $A \subseteqq E$, then $\underline{M}_A$ is also a transversal matroid. Thus if A is chosen so that $\underline{M}_A$ has finite rank, we can use Theorem 5.2 on $\underline{M}_A$.

The conditions on the number of k-flats as given in Theorem 5.2 are not sufficient to guarantee that a matroid is a transversal matroid. To obtain an example, consider the rank 4 cycle matroid $\underline{M}$ on the set of edges $E = \{1, 2, \cdots, 7\}$ of the graph of Figure 2. Then $\underline{M}$ has two coloop-free 2-flats, namely $\{3, 4, 7\}$ and $\{4, 5, 6\}$ and three coloop-free 3-flats, namely $\{1, 2, 3, 6\}, \{1, 2, 5, 7\}, \{3, 4, 5, 6, 7\}$. Hence the

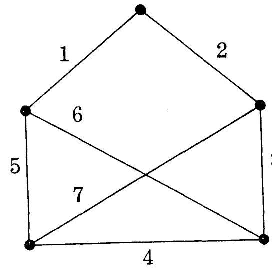

FIGURE 2

conditions on the numbers of coloop-free k-flats are satisfied ($k = 1$ and $k = r$ are always trivially satisfied). But $\underline{M}$ is not a transversal matroid as can be seen from Theorem 5.1. For $B_1 = \{1, 2, 4, 6\}$ and $B_2 = \{1, 3, 5, 7\}$ are both bases with $\{B_1\backslash 4\} \cup y$ and $\{B_2\backslash y\} \cup 4$ both bases only for $y = 7$ of B_2, and $\{B_1\backslash 2\} \cup z$ and $\{B_2\backslash z\} \cup 2$ are both bases only for $z = 7$ of B_2. Thus the bijection of Theorem 5.1 cannot exist, so that $\underline{M}$ is not a transversal matroid.

Theorem 5.2 is interesting because it gives a bound on the number of coloop-free k-flats of a transversal matroid on E which does not depend on the size of E but on the rank of the matroid. The total number of k-flats cannot be bounded in terms of r. For if E is a set with $|E| \geqq r$ and $\underline{M} = \mathscr{P}_r(E) = \{A \subseteqq E: |A| \leqq r\}$, then $\underline{M}$ is a transversal matroid of rank r and every subset of E of k elements, $1 \leqq k \leqq r - 1$, is a k-flat. Hence $\underline{M}$ has $\binom{|E|}{k}$ k-flats $(1 \leqq k \leqq r - 1)$, all of which have coloops.

Before getting to another necessary condition for a matroid to be transversal, we require a definition. Let $\underline{M}$ be a matroid on a set E. We say that $\underline{M}$ has the *direct sum property* provided:

Whenever $(E_k: k \in K)$ is a family of pairwise disjoint subsets of E such that $\underline{M}_{E_k}$ is a coloop-free matroid with basis on E_k $(k \in K)$, then

$$\underline{M}_{E_k \cup E_l} = \underline{M}_{E_k} \oplus \underline{M}_{E_l} \qquad (k, l \in K, k \neq l)$$

imply

$$\underline{M}_{\cup(E_j \cdot j \in J)} = \bigoplus (\underline{M}_{E_j}: j \in J) .$$

A matroid need not have the direct sum property as the cycle matroid $\underline{M}$ on the set of edges $E = \{1, 2, \cdots, 9\}$ of the graph of Figure 3 shows. If we take $E_1 = \{1, 2, 3\}$, $E_2 = \{4, 5, 6\}$, $E_3 = \{7, 8, 9\}$, then $\underline{M}_{E_i}$ is coloop-free $(1 \leqq i \leqq 3)$ and $\underline{M}_{E_i \cup E_j} = \underline{M}_{E_i} \oplus \underline{M}_{E_j}$ $(1 \leqq i \neq j \leqq 3)$. But $\underline{M}_{E_1 \cup E_2 \cup E_3} \neq \underline{M}_{E_1} \oplus \underline{M}_{E_2} \oplus \underline{M}_{E_j}$, for $\{2, 6, 9\}$ is a circuit whose intersections with E_i are independent $(1 \leqq i \leqq 3)$.

We do, however, have the following theorem.

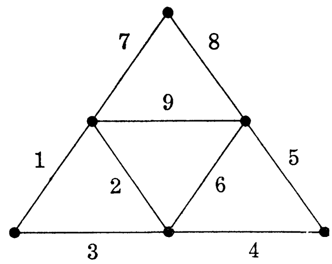

FIGURE 3

THEOREM 5.3. *Let $\underline{M}$ be a transversal matroid on E. Then $\underline{M}$ has the direct sum property.*

Proof. Let (E, Δ, Y) be a bipartite graph which induces $\underline{M}$ on E. Let $(E_k: k \in K)$ be a family of pairwise disjoint subsets of E such that $\underline{M}_{E_k}$ is a coloop-free matroid with basis on E_k $(k \in K)$ and $\underline{M}_{E_k \cup E_l} = \underline{M}_{E_k} \oplus \underline{M}_{E_l}$ $(k, l \in K, k \neq l)$. Since $\underline{M}_{E_k}$ has a basis B_k, it follows from Theorem 4.2 that B_k is linked to $\Delta(E_k)$ and to no other subset of $Y(k \in K)$. Since $\underline{M}_{E_k \cup E_l} = \underline{M}_{E_k} \oplus \underline{M}_{E_l}$ $(k \neq l)$, $B_k \cup B_l$ is a basis of $\underline{M}_{E_k \cup E_l}$ where $B_k \cap B_l = \phi$. Since $\underline{M}_{E_k} \oplus \underline{M}_{E_l}$ is coloop-free, $B_k \cup B_l$ is linked to $\Delta(E_k \cup E_l) = \Delta(E_k) \cup \Delta(E_l)$. Since B_k, resp. B_l, is only linked to $\Delta(E_k)$, resp. $\Delta(E_l)$, it follows that $\Delta(E_k) \cap \Delta(E_l) = \phi$. Since this is true for all $k, l \in K$ with $k \neq l$, the result follows.

Since the matroid of the graph of Figure 3 does not have the direct sum property, it follows it is not a transversal matroid.

The direct sum property does not characterize transversal matroids among all matroids. The cycle matroid of the graph of Figure 2 has the direct sum property but is not transversal as we have already seen. In fact the direct sum property holds trivially, for if $F \subseteqq E$ with $\underline{M}_F$ coloop-free $|F| \leqq 3$. Since $|E| = 7$ in this case, the direct sum property is valid.

To conclude we wish to mention one further consequence of the results of § 3. For this we need another definition which, to keep things simple we make only for finite character matroids. Let $\underline{M}$ be a finite character matroid on E, and let $F \subseteqq E$ with B a basis of $\underline{M}_{E \backslash F}$. Let

$$\underline{M}_{\otimes F} = \{A: A \subseteqq F, A \cup B \in \underline{M}\} .$$

Then it is well-known [14, 3] that $\underline{M}_{\otimes F}$ is independent of the choice of basis B and that $\underline{M}_{\otimes F}$ is a finite-character matroid on F, called the *contraction of* $\underline{M}$ to F. The contraction of a transversal matroid need not be a transversal matroid. An example which contains 2-element circuits (thus not a combinatorial geometry [4]) is given in [9]. The cycle matroid $\underline{M}$ on the set of edges of the graph of Figure

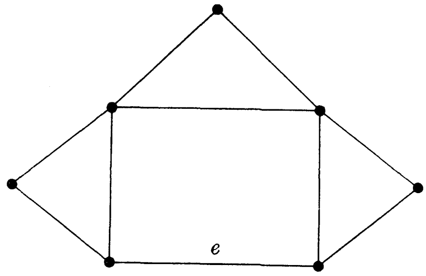

FIGURE 4

4 is a transversal matroid, as is not difficult to see. If we take $F = E \backslash e$, then $\underline{M}_{\otimes F}$ is isomorphic to the matroid of the graph of Figure 3 and hence is not a transversal matroid. It is therefore of interest to determine when the contraction of a transversal matroid is also a transversal matroid. We offer the following theorem.

THEOREM 5.4. *Let $\underline{M}$ be a finite-character transversal matroid on a set E. Let $F \subseteqq E$ and suppose $\underline{M}_{E\backslash F}$ is coloop-free. Then $\underline{M}_{\otimes F}$ is a (finite-character) transversal matroid.*

Proof. Let the bipartite graph (E, Δ, Y) induce the matroid $\underline{M}$ on E. Since $M_{E\backslash F}$ has no coloops, it follows from Theorem 3.4 that if B is a basis of $\underline{M}_{E\backslash F}$ then B is linked only to the subset $\Delta(E/F)$ of Y. Let the bipartite graph (F, Δ', Z) be defined by $Z = Y \backslash \Delta(E \backslash F)$ and $\Delta' = \Delta \cap \{F \times Z\}$. Let $A \subseteqq F$. Then $A \in \underline{M}_{\otimes F}$ if and only if $A \cup B \in M$; $A \cup B \in M$ if and only if $A \cup B$ is linked in (E, Δ, Y); $A \cup B$ is linked in (E, Δ, Y) if and only if A is linked in (F, Δ', Z). Hence the bipartite graph (F, Δ', Z) induces $\underline{M}_{\otimes F}$ on F so that $\underline{M}_{\otimes F}$ is a transversal matroid.

There is no difficulty in obtaining examples where $\underline{M}_{E\backslash F}$ has coloops but $\underline{M}_{\otimes F}$ is a transversal matroid. In fact the matroid of a graph which is a triangle already furnishes an example.

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