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ON PSEUDO-CONFORMAL MAPPINGS OF CIRCULAR DOMAINS

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In the present paper we investigate the condition whether the bounded domain B of C^2 is a pseudo-conformal image of a circular domain, say C. Under the assumption that this is the case and that the invariant $J_B(z_1, z_2; \bar{z}_1, \bar{z}_2)$ is not a constant, we characterize the center of a circular domain. This characterization is invariant with respect to pseudoconformal transformations. Assuming that B is a pseudoconformal image of a circular domain C and that there is in B one and only one point, say (t_1, t_2) which satisfies the conditions mentioned above, we determine the representative $R(B; t_1, t_2)$ of B. If B is a pseudo-conformal image of a circular domain C and (t_1, t_2) is the image in B of the center of C, then the representive $R(B; t_1, t_2)$ is a circular domain. The pair of functions v^{10}, v^{01} mapping B onto $R(B; t_1, t_2)$ can be written explicitly in terms of the kernel function of B.

A homeomorphism T of a domain, say B, of the z_1, z_2-space, $z_k = x_k + iy_k$, $k = 1, 2$, by a pair of holomorphic function

$$z_k^* = z_k^*(z_1, z_2) , \qquad (z_1, z_2) \in B , \tag{1}$$

of two complex variables is denoted a PCT (*pseudo-conformal transformation*).

A domain which admits the group

$$z_k^\dagger = z_k e^{i\varphi} , \qquad 0 \leqq \varphi \leqq 2\pi, z_k = x_k + iy_k , \tag{2}$$

of PCT's onto itself (automorphisms) is called a *circular domain*.

We assume that at every boundary point Q of C the Levi expression is negative (see (11), p. 11 and (16), p. 12, of [1]). (Hypothesis 1)

To decide whether a domain, say B, belongs to a given class of domains, for instance, whether B is a pseudo-conformal image of a circular domain C, is one of the interesting problems of the theory of PCT's. In the following we shall show that the theory of the kernel function permits us to answer this question in certain instances. In addition, if, $B = T(C)$, we shall determine the function pair mapping B onto the circular domain C.

REMARK. Concerning the application of the kernel function in the theory of conformal mapping of simply and multiply connected domains onto canonical domains and onto each other, see [3], Chapter VI, [5] and [6].

The first step in our approach (under the assumption that $B = \boldsymbol{T}(C)$, i.e., that B is a pseudo-conformal image of a circular domain C) is the determination of the image of the center O of C in B.

Using the considerations on p. 183 ff. of [3], we assume that the invariant (with respect to PCT's)

$$(3) \qquad J_B = J \equiv \frac{K}{T_{1\bar{1}}T_{2\bar{2}} - |T_{1\bar{2}}|^2}, \qquad T_{m\bar{n}} = \frac{\partial^2 \log K}{\partial z_m \partial \bar{z}_n},$$

is known and is *not constant*. (Hypothesis 2)

In accordance with the considerations on p. 183 of [3] J_B is invariant with respect to PCT's. Consequently,

$$(4) \qquad J_C(z, \bar{z}) = J_B(z^*, \bar{z}^*) .$$

$J_C(z, \bar{z})$ is a real analytic function of $z_1, z_2, \bar{z}_1, \bar{z}_2$ in C. From formula (33), p. 19, of [2] it follows that

$$(5) \qquad J_C(z, \bar{z}) > 0 , \qquad z \in C .$$

Since we assumed that at every boundary point of C Levi's expression, $L(\Phi)$, is negative, it follows that for every boundary point Q of C

$$(6) \qquad \lim_{z \to Q} J_C(z, \bar{z}) = \frac{2}{9\pi^2}$$

holds (see (1), p. 12, of [2]).[1] Since $J(z, \bar{z})$ is not constant in C, it must assume its maximum or minimum in C.

In Theorems 1 − 3 we shall discuss some properties of a connected set which includes the origin O where $J_C(z, \bar{z})$ has a (local) maximum or a minimum. These properties are preserved in PCT's and enable us to determine the image $\boldsymbol{T}(O)$ of the center O in $B = \boldsymbol{T}(C)$.

THEOREM 1. *If $J(z_1, z_2)$ has a maximum, minimum or minimax at an isolated point P of $B = \boldsymbol{T}(C)$, then*

$$(7) \qquad P = \boldsymbol{T}(O) .$$

Proof. Let $\widetilde{C} = C \cap [y_2 = 0]$. The function $J(z_1, x_2)$ is defined in $\widetilde{C}$. If $J(z_1, x_2)$ assumes a value, say c_0, at a point $z_1^0, x_2^0, (z_1^0, x_2^0) \neq O$, then $J(z_1, z_2) = c_0$ along a line

$$(8) \qquad o^1(z_1^0, x_2^0) = [z_1 = z_1^0 e^{i\varphi}, z_2 = x_2^0 e^{i\varphi}, 0 \leqq \varphi \leqq 2\pi] ,$$

the orbit of (z_1^0, x_2^0). If and only if $z_1^0 = 0, x_2^0 = 0$ (i.e., if (z_1^0, x_2^0) is the origin O), o^1 degenerates to a point. In accordance with (4), if $J_B(z^*, \bar{z}^*)$

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* We assume here that we approach Q in the sense A^I in a way described in [2], p. 10.

has a minimum or a maximum at an isolated point P, $P \in B$, then (7) holds.

THEOREM 2. *It is impossible that $J(z, \bar{z})$ has a maximum or a minimum along a (one-dimensional) connected set including O.*

(We assume here that the set does not include a segment of an orbit.)

Proof. Suppose that

$$p = \bigcup_{\alpha=0}^{a} P(\alpha) , \qquad a > 0, P(0) = (0, 0) = O \tag{9}$$

is a (one-dimensional) connected set consisting of points $P(s)$, $0 \leqq s \leqq a$, where $J_C(z, \bar{z})$ assumes a minimum or a maximum. Then to every point $P_\nu(\alpha)$, $0 < \alpha \leqq a$, corresponds the orbit $o^1(P_\nu(\alpha))$, along which $J_C(z, \bar{z})$ assumes a constant value. Thus $J_C(z, \bar{z})$ assumes the same value on

$$\bigcup_{\alpha=0}^{a} o^1(P_\nu(\alpha)) , \qquad 0 \leqq \alpha \leqq a . \tag{10}$$

Each $o^1(P_\nu(\alpha))$, α constant, $\alpha > 0$, is one dimensional. Two different orbits $o^1(P_\nu(\alpha_1))$ and $o^1(P_\nu(\alpha_2))$, $\alpha_1 \neq \alpha_2$, are disjoint and therefore (10) is a two-dimensional set. $T[\bigcup_{\alpha=0}^{a} o^1\overline{(P_\nu(\alpha))}]$ is also two dimensional since T is a homeomorphism.

In the following we shall consider two cases where $J(z, \bar{z})$ equals a maximum or a minimum in a three-dimensional segment s^3.

THEOREM 3a. *Suppose that $J_C(z, \bar{z})$ = maximum (or minimum) in a (three-dimensional) set s^3, $P \in s^3$. We assume that s^3 is connected in C, $s^3 - P$ is a sum of two (or in general n, $n < \infty$) disconnected sets. Then P is the center O of C.*

If $B = T(C)$, i.e., if B is a pseudo-conformal image of C, then $T(s^3)$ has the property indicated above and if $T(s^3) - P^*$ is a sum of n, $n > 1$, disconnected parts, then $P^* = T(O)$.

Proof. If two parts, say s_1^3 and s_2^3, $s_k^3 \in C$, are connected at one point, say (z_1^0, z_2^0), $|z_1^0|^2 + |z_2^0|^2 > 0$, then they are connected along a line segment

$$o^1(z_1^0, z_2^0) = [z_1^0 e^{i\varphi}, z_2^0 e^{i\varphi}, 0 \leqq \varphi \leqq 2\pi] . \tag{11}$$

If we delete one point, say $Q = (z_1^0 e^{i\varphi_1}, z_2^0 e^{i\varphi_1})$, from (11), then s_1^3 and s_2^3 will still be connected along

$$[z_1^0 e^{i\varphi}, z_2^0 e^{i\varphi}, 0 \leqq \varphi \leqq 2\pi] - (z_1^0 e^{i\varphi_1}, z_2^0 e^{i\varphi_1}) . \tag{12}$$

Thus by deleting the point Q, s_1^3 and s_2^2 can become disconnected only if $Q = O = (0, 0)$, in which case the orbit (11) degenerates to a point.

THEOREM 3b. *Suppose that* $J(z, \bar{z}) =$ *maximum* (*or minimum*), $z \in C$, *is a connected set* s^3, $P \in s^3$, $s^3 = s_1^3 \cup s_2^3 \cup P$, *and that one can define sufficiently small neighborhoods* $N(s_1^3)$ *and* $N(s_2^3)$ *such that* $N(s_1^3) \cup N(s_2^3) \cup P$ *is connected and* $N(s_1^3) \cup N(s_2^3)$ *is not connected. Then* $P = O$.

Proof. Suppose that the point $Q = (z_1^0, z_2^0) \neq O$ belongs to s^3. Then the orbit (11) must also belong to s^3. Let $P_k \in N(s_k^3)$, $k = 1,2$. According to the assumption of the the theorem, one can connect P_1 and P_2 by a line segment passing the point $Q = (z_1^0, z_2^0)$. Since

$$|z_1^0|^2 + |z_2^0|^2 > 0 \, , \tag{13}$$

one can also connect P_1 and P_2 by a segment passing by the point $Q^* = (z_1^0 e^{i\varphi_1}, z_2^0 e^{i\varphi_1})$, $0 < \varphi_1 < 2\pi$. $N(s_1^3) - P$ and $N(s_2^3) - P$ become disconnected only if the assumption (13) does not hold, i.e., if $P = O = (0, 0)$.

REMARK. It is interesting to give an example of a set s^3 and to describe a construction of $N(s^3)$ possessing the properties indicated in Theorem 3b. Suppose that s_1^3 lies in $x_2 > 0$, s_2^3 in $x_2 < 0$, and that they are connected by a one-dimensional set which lies in s^3 and which includes O (it lies in $x_2 = 0$). Let $Q \neq O$, $Q \in s^3$. To construct the desired neighborhood, we draw around every point $Q \in s^3 - O$ a hypersphere $H(Q, \rho)$ with the center at Q and of radius $\rho > 0$, $\rho = \rho(Q) < d(Q)$, where $d(Q)$ denotes the distance between Q and $x_2 = 0$. Then $N(s_1^3) = [\cup H(Q, 1/2d(Q)), Q \in s_1^3]$. Obviously $N(s_1^3)$ has no points lying in $x_2 = 0$. Naturally, instead $x_2 = 0$ one can use another hypersurface possessing the necessary property.

THEOREM 4. *Suppose that* J *assumes its maximum* (*or minimum*) *on a two-dimensional connected set* s^2, $O \in s^2$. *We assume that* $s^2 - O$ *is a sum of* n *disconnected segments*, $1 < n < \infty$. *Then* O *is the center of* C.

Proof. The proof proceeds as the proof of Theorem 3a. To every point P, $P \neq O$, of $s^2 \cap (y_2 = 0)$ corresponds the orbit (8), i.e.,

$$s^2 = \bigcup_{\beta=0}^{\alpha} o^1(P_\nu(\alpha)) \, , \qquad 0 \leqq \alpha < a \, .$$

Suppose that P_1 and P_2 are two points of s^2 which lie in different orbits $o^1(P_k)$, $k = 1.2$. If the line segment connecting P_1 and P_2 passes through O, the segments s_k^2, $P_k \in s_k^2 - O$, $s_1^2 \cup s_2^2 = s^2 - O$ are disconnected. We note that if we delete a point Q, $Q \neq O$, then s_1^2 and s_2^2 can be connected by a segment passing through another point of the orbit $o^1(Q)$. Only for $Q = O$ this orbit shrinks to a point.

REMARK. It is not necessary to consider the case where the domain s (where $J_B(z, \bar{z}) =$ maximum (or minimum)) is four dimensional. If this holds, then

$$J_B(z, \bar{z}) = \text{const}\ , \qquad (z_1, z_2) \in B\ , \tag{14}$$

is in contradiction with Hypothesis 2.

Obviously there exist situations for which our procedures do not enable us to determine the location of $\boldsymbol{T}(O)$ in B. For example, suppose that s^2 is a segment in $C \cap [x_2 = 0,\ y_2 = 0]$.

In some of these cases we can use in addition to J a second invariant (with respect to PCT's) $J_2(z, \bar{z})$ which is linearly independent of $J(z, \bar{z})$. Concerning conditions for such domains B, see [4]. We assume that the intersection

$$[J(z, \bar{z}) = \text{const} = c_1] \cap [J_2(z, \bar{z}) = \text{const} = c_2] \tag{15}$$

either includes an isolated point or a closed Jordan curve.

THEOREM 5a. *Suppose that the set* (15) *in B consists of disconnected components and one of these components is an isolated point, say Q. Then $Q = \boldsymbol{T}(O)$.*

Proof. Suppose that (15) in C is a point $(z_1^0, z_2^0) \neq O$. Since C admits the group (2) of PCT's onto itself, the orbit

$$o^1(z_1^0, z_2^0) = [z_k = z_k^0 e^{i\varphi},\ 0 \leqq \varphi \leqq 2\pi,\ k = 1, 2] \tag{16}$$

must belong to (15). (16) is a closed Jordan curve and its image $\boldsymbol{T}(o^1(z_1^0, z_2^0))$ is also a closed Jordan curve. It degenerates to a point only if $(z_1^0, z_2^0) = O$.

THEOREM 5b. *Suppose that* (15) *includes a closed Jordan curve, say p^1. If we draw around every point $R \in p^1$ a (invariant) hypersphere $\bar{\sigma}^3(R, \rho)$ of radius ρ, then for ρ sufficiently small all hyperspheres $\bar{\sigma}^3(R, \rho)$, $R \in p^1$, have no common point. If ρ increases, there exists a smallest ρ, say $\rho = \rho_0$, such that all $\bar{\sigma}^3(R, \rho_0)$ have a common point, say Q. Then*

$$Q = \boldsymbol{T}(O)\ . \tag{17}$$

Proof. Since the construction described in Theorem 5b is invariant with respect to PCT, we can consider it either in C or in $B = \boldsymbol{T}(C)$. We shall consider it in C. By PCT (2)p^1 goes onto itself. Therefore the invariant distance, say ρ_0, of every point $R \in p^1$ from O is the same. For $\rho < \rho_0$ all hyperspheres $\bar{\sigma}^3(R, \rho)$ have no common point. Suppose that there exists a point, say S, $S \neq O$, such that the invariant distance between p^1 and S is $\rho_1 < \rho_0$. Then p^1 would be simultaneously an orbit around two different points, O and S. But the orbits around the two different points cannot coincide. If $\rho = \rho_0$, where ρ_0 is the invariant distance of p^1 from O, all (closed) hyperspheres $\bar{\sigma}^3(R, \rho_0)$ will have the point O in common. Thus (17) holds.

Suppose that the domain B (in the z_1, z_2-space) is a pseudoconformal image of a circular domain C, i.e., $B = \boldsymbol{T}(C)$. The previous considerations in most cases enable us to determine in B the image $t = \boldsymbol{T}(O)$, $t = (t_1, t_2)$ of the center O of C. In the following we shall indicate, using the above result, how we can determine the pair $v^{10}(z, t)$, $v^{01}(z, t)$, $z = (z_1, z_2)$, $t = (t_1, t_2)$ of analytic functions which transform B onto a circular domain.

We shall use, without proof, the following:

LEMMA. *A circular domain, say C, is transformed by a linear* PCT *again onto a circular domain.*

A mapping pair $w_k(z_1, z_2)$ is said to be normalized at $t = (t_1, t_2)$ if

$$w_k(t_1, t_2) = t_k, \; \frac{\partial w_k(z_1, z_2)}{\partial z_n}\bigg|_{z_p = t_p} = \delta_{kn} , \tag{21}$$

$\delta_{kn} = 0$ for $k \neq n$, $\delta_{kn} = 1$ for $k = n$, $k = 1, 2$, $p = 1, 2$.

In (50), (51), pp. 188, 189 of [3] the pair $V_t = (v^{10}(z, t), v^{01}(z, t))$ normalized at t is given (in terms of the kernel function $K_B(z, t)$) which maps B onto the representative $R(B, t)$ of B, see Theorem, p. 186 of [3].

THEOREM 6. *Suppose that B is a pseudo-conformal image of a circular domain C, i.e., $B = T(C)$, where T is* a *PCT. Let $t^* = (t_1^*, t_2^*)$ be the image in B of the center O of C. Then the representative $R(B, t^*)$, $t^* = \boldsymbol{T}(O)$, is a circular domain. Here $R(B, t^*)$, $t^* = \boldsymbol{T}(O)$, is the domain which we obtain from B using the* PCT

$$z_1 = v^{10}(z, t) \; , \qquad z_2 = v^{01}(z, t) \; . \tag{22}$$

(v^{10}, v^{01}) *is the pair of functions introduced in* (50), (51), pp. 188–189 of [3].

Proof. According to our considerations $R(B, t)$ and C are both representatives of B with respect to the same point $t = \boldsymbol{T}(O)$. According to [3], p. 190, two representatives of B with respect to the same point can be transformed into each other by a *linear* PCT, say by

$$v^{*10} = \alpha_{1\bar{1}} v^{10} + \alpha_{1\bar{2}} v^{01} ,$$
$$(23) \qquad \alpha_{\nu\bar{\mu}} \text{ constants} ,$$
$$v^{*01} = \alpha_{2\bar{1}} v^{10} + \alpha_{2\bar{2}} v^{01} .$$

In accordance with Lemma 1, $R(B, t)$ is also a circular domain since C is a circular domain and (23) is a linear PCT.

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