PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 1, 1972

STRONG HEREDITY IN RADICAL CLASSES

R. L. TANGEMAN

In a recent paper, W. G. Leavitt has called a radical class $\mathscr{P}$ in a universal class $\mathscr{W}$ of not necessarily associative rings strongly hereditary if $\mathscr{P}(I) = I \cap \mathscr{P}(R)$ for all ideals I of any ring $R \in \mathscr{W}$. In this paper, strongly hereditary radicals are investigated and a new construction is provided for the minimal strongly hereditary radical containing a given class in $\mathscr{W}$. Nonassociative versions of some results of E. P. Armendariz on semisimple classes are proved, including a characterization of semisimple classes corresponding to strongly hereditary radicals.

Unless otherwise indicated, $\mathscr{W}$ is assumed to be a universal class of not necessarily associative rings. If $\mathscr{P}$ is any radical class in $\mathscr{W}$, we denote the class of $\mathscr{P}$-semisimple rings in $\mathscr{W}$ by $\mathscr{S}\mathscr{P}$. We use the notation $I \leqq R$ to denote that I is an ideal of R. For any class $\mathscr{M}$ we denote by $\mathscr{H}\mathscr{M}$ and $\mathscr{I}\mathscr{M}$, respectively, the homomorphic closure and ideal closure of $\mathscr{M}$.

For any radical class $\mathscr{P} \subseteqq \mathscr{W}$, Leavitt in [7] has defined $\mathscr{G}\mathscr{P} = \{J' \mid J \leqq I \leqq R, J \in \mathscr{P}$, and J' is the ideal of R generated by $J\}$. Radical classes $\mathscr{P}$ for which $\mathscr{P} = \mathscr{G}\mathscr{P}$ are said to satisfy property (a). Theorem 1 of [7] states that a hereditary radical class $\mathscr{P}$ is strongly hereditary if and only if $\mathscr{P}$ satisfies property (a). In [8], it is shown that any subclass $\mathscr{M}$ of $\mathscr{W}$ is contained in a unique minimal radical class satisfying property (a).

Some preliminary results are required.

LEMMA 1.1. [2]. *Let $\mathscr{P}$ be any radical class in $\mathscr{W}$. Then $\mathscr{S}\mathscr{P}$ is hereditary if and only if for each $R \in \mathscr{W}$ with $I \leqq R$ we have* $\mathscr{P}(I) \subseteqq (R)$.

LEMMA 1.2. *Let $\mathscr{P}$ be any radical class. Then $\mathscr{P}$ is strongly hereditary if and only if both $\mathscr{P}$ and $\mathscr{S}\mathscr{P}$ are hereditary.*

Proof. If $\mathscr{P}$ is strongly hereditary, $\mathscr{P}(I) = I \cap \mathscr{P}(R)$ for each $I \leqq R$, so $\mathscr{P}$ and $\mathscr{S}\mathscr{P}$ are hereditary. Suppose $\mathscr{P}$ and $\mathscr{S}\mathscr{P}$ are hereditary and let $I \leqq R$. By Lemma 1.1 $\mathscr{P}(I) \subseteqq I \cap \mathscr{P}(R)$. Also since $\mathscr{P}(R) \in \mathscr{P}$ and $\mathscr{P}$ is hereditary, $I \cap \mathscr{P}(R) \in \mathscr{P}$. Since $I \cap \mathscr{P}(R) \leqq I$, we have $I \cap \mathscr{P}(R) \subseteqq \mathscr{P}(I)$.

LEMMA 1.3. *Let $\mathscr{P}$ be a radical class satisfying property* (a). *Then $\mathscr{S}\mathscr{P}$ is hereditary. If $\mathscr{P}$ is hereditary, $\mathscr{P}$ satisfies property*

(a) *if and only if* $\mathscr{SP}$ *is hereditary.*

Proof. Suppose $\mathscr{P}$ has property (a). Let $J \leqq I \leqq R$ where $J \in \mathscr{P}$. Then J', the ideal of R generated by J, belongs to $\mathscr{P}$, so $J' \subseteq \mathscr{P}(R)$. Thus $J \subseteq \mathscr{P}(R)$ and $\mathscr{P}(R)$ contains all the $\mathscr{P}$-ideals of I, so $\mathscr{P}(I) \subseteq \mathscr{P}(R)$. By Lemma 1.1, this means $\mathscr{SP}$ is hereditary. If $\mathscr{P}$ and $\mathscr{SP}$ are hereditary, then $\mathscr{P}$ is strongly hereditary by Lemma 1.2, so $\mathscr{P}$ satisfies property (a) by Theorem 1 of [7].

The semisimple class $\mathscr{SP}$ may be hereditary even when $\mathscr{P}$ does not satisfy property (a). To see this, let R be the ring generated over $GF(2)$ by the nonassociative symbols $\{x, y, z\}$, subject to the relations $x^2 = xy = yx = xz = x$, $yz = zy = zx = y$, $z^2 = z$, $y^2 = 0$. Then the only proper nonzero ideal of R is $I = \{0, x, y, x + y\}$ and the only proper ideal of I is $J = \{0, x\}$. R/I and J are isomorphic simple idempotent rings, and I/J is simple and nilpotent. Let $\mathscr{W} = \{0, R, J, I, I/J\}$ and $\mathscr{P} = \{0, R, J\}$. Then $\mathscr{P} \neq \mathscr{GP}$ but $\mathscr{SP} = \{0, I/J\}$ is hereditary.

2. Radical classes. In Theorem 2 of [2], it is shown that if $\mathscr{P}$ is a radical class in an alternative class $\mathscr{W}$, and if $R \in \mathscr{W}$ with $I \leqq R$, then $\mathscr{P}(I) \leqq R$. The following theorem shows that radicals satisfying property (a) in an arbitrary universal class have the same property.

THEOREM 2.1. *Let* $\mathscr{P}$ *be any radical class in* $\mathscr{W}$. *The* $\mathscr{P}$ *satisfies property* (a) *if and only if for each* $R \in \mathscr{W}$, $I \leqq R$ *implies* $\mathscr{P}(I) \leqq R$.

Proof. Suppose $I \leqq R$ implies $\mathscr{P}(I) \leqq R$. Then $\mathscr{P}(I) \subseteq \mathscr{P}(R)$ so $\mathscr{SP}$ is hereditary by Lemma 1.1. If $\mathscr{P}$ does not have property (a), we have some $J \leqq I \leqq R$ with $J \in \mathscr{P}$ and $J' \notin \mathscr{P}$, where J' is the ideal of R generated by J. This means $\mathscr{P}(J') \neq J'$. Since $\mathscr{SP}$ is hereditary and $J \leqq J'$, we have $\mathscr{P}(J) = J \subseteq \mathscr{P}(J')$. Since $\mathscr{P}(J') \leqq R$, we have J contained in an ideal of R properly smaller than J', contradicting the definition of J'.

Conversely, suppose $\mathscr{P}$ has property (a) and let $I \leqq R$. Then $\mathscr{P}(I) \leqq I \leqq R$, so $\mathscr{P}(I)' \in \mathscr{P}$ and from $\mathscr{P}(I)' \leqq I$ if follows that $\mathscr{P}(I)' \subseteq \mathscr{P}(I)$. Thus $\mathscr{P}(I) = \mathscr{P}(I') \leqq R$.

In [3] (Lemma 5) a result is proved which may be restated for our purposes as

LEMMA 2.2. *If* $\mathscr{P}$ *is a radical class in an alternative class, then* $\mathscr{P}$ *satisfies property* (a).

This lemma, when applied with Theorem 2.1, shows that in alternative classes $I \leqq R$ implies $\mathscr{P}(I) \leqq R$, thus providing another proof of Theorem 2 of [2].

We next note that property (a) may be satisfied by possibly nonradical classes of rings. For an example of this, let $\mathscr{M}$ be the class of nilpotent rings in the universal class of all associative rings. Then if $J \leqq I \leqq R$ with $J \in \mathscr{M}$, we have $(J')^3 \subseteqq J$ so that $J' \in \mathscr{M}$. The following lemma shows that such classes, if also homomorphically closed, are only one step removed from being radical.

LEMMA 2.3. *If $\mathscr{M}$ is homomorphically closed and satisfies property* (a), *then $\mathscr{L}\mathscr{M} = \mathscr{M}_2$ in the lower radical construction.* (*For details of this construction see* [5]).

Proof. Let $R \in \mathscr{M}_3$ and let R_1 be an arbitrary homomorphic image of R. Then R_1 has an ideal $I \in \mathscr{M}_2$ and I has a nonzero ideal $J \in \mathscr{M}$. Since $\mathscr{M}$ satisfies property (a), J', the ideal of R_1 generated by J, is in $\mathscr{M}$. Thus each image of R has a nonzero ideal in $\mathscr{M}$, which means $\mathscr{M}_3 \subseteqq \mathscr{M}_2$ so $\mathscr{L}\mathscr{M} = \mathscr{M}_2$.

Using Lemma 2.3, we next prove that property (a) is preserved by passing to the lower radical.

THEOREM 2.4. *If $\mathscr{M}$ is homomorphically closed and satisfies property* (a), *then $\mathscr{L}\mathscr{M}$ satisfies property* (a).

Proof. Let $\mathscr{P} = \mathscr{L}\mathscr{M} = \mathscr{M}_2$, and suppose $\mathscr{P}$ does not have property (a). Then by Theorem 2.1 there is some R with $I \leqq R$ but $\mathscr{P}(I)$ not an ideal of R. Since the union of a chain of $\mathscr{P}$-ideals of R is again a $\mathscr{P}$-ideal of R (see [1]), we may select by Zorn's Lemma a $\mathscr{P}$-ideal F of R which is maximal among those $\mathscr{P}$-ideals of R which are contained in $\mathscr{P}(I)$ (F may of course be zero). We claim $\mathscr{P}(I)/F = \mathscr{P}(I/F)$. Clearly $\mathscr{P}(I)/F \subseteqq \mathscr{P}(I/F)$. Let K be the ideal of I such that $K/F = \mathscr{P}(I/F)$. Then $\mathscr{P}(I) \subseteqq K$ and $(K/F)/(\mathscr{P}(I)/F) \cong K/\mathscr{P}(I) \in \mathscr{P}$, forcing $K \in \mathscr{P}$, which means $K \subseteqq \mathscr{P}(I)$ so $\mathscr{P}(I)/F = \mathscr{P}(I/F)$. Now if R/F has a nonzero $\mathscr{P}$-ideal $P/F \subseteqq \mathscr{P}(I)/F$, we would have $P \in \mathscr{P}$ violating the maximality of F.

Thus by passing if necessary to a homomorphic image, we may assume $\mathscr{P}(I) \subseteqq I \leqq R$, and $\mathscr{P}(I)$ contains no nonzero $\mathscr{P}$-ideal of R. Since $\mathscr{P}(I) \in \mathscr{M}_2$, $\mathscr{P}(I)$ has a nonzero ideal $J_1 \in \mathscr{M}$, and the ideal J of I generated by J_1 is also in $\mathscr{M}$. Thus we have $J_1 \subseteqq J \subseteqq \mathscr{P}(I) \subseteqq I \subseteqq R$ where $J \in \mathscr{M}$ and $J \leqq I$. Also J', the ideal of R generated by J, is contained in I, and $J' \in \mathscr{M} \subseteqq \mathscr{P}$. Thus $J' \subseteqq \mathscr{P}(I)$, contradicting the assumption that $\mathscr{P}(I)$ contains no nonzero $\mathscr{P}$-ideals of R.

The example following the proof of Lemma 1.3 may be used to show that in the nonassociative case the requirement in Theorem 2.4

that $\mathscr{M}$ be homomorphically closed cannot be dropped. Let $\mathscr{W}$ be as in the example and $\mathscr{M} = \{R\}$. Then $\mathscr{M}$ has property (a) but $\mathscr{H}\mathscr{M} = \mathscr{P}$ does not.

It is shown in [8] that an arbitrary class is contained in a unique minimal radical class satisfying property (a) and in a unique minimal strongly hereditary radical class. The next few results provided countable construction which are at most one (Kuroš) step from these classes.

THEOREM 2.5. *Let $\mathscr{M} \subseteqq \mathscr{W}$ with $\mathscr{W}$ any universal class. There exists a unique minimal class in $\mathscr{W}$ containing $\mathscr{M}$ which is homomorphically closed and has property* (a).

Proof. Define $\mathscr{M}_1 = \mathscr{M}$ and $\mathscr{M}_{n+1} = \mathscr{G}\mathscr{H}\mathscr{M}_n$ for each $n \geqq 1$. Then set $\mathscr{M}^* = \cup \mathscr{M}_n$, the union being taken over all positive integers n. $\mathscr{M}^*$ is easily seen to be homomorphically closed. Also $\mathscr{M}^*$ satisfies property (a), for if $J \leqq I \leqq R$ with $J \in \mathscr{M}^*$, then $J \in \mathscr{M}_n$ for some n so that $J' \in \mathscr{M}_{n+1} \subseteqq \mathscr{M}^*$.

If $\mathscr{A}$ is any homomorphically closed class containing $\mathscr{M}$ and satisfying property (a), an easy induction shows $\mathscr{M}_n \subseteqq \mathscr{A}$ for each n so that $\mathscr{M}^* \subseteqq \mathscr{A}$.

COROLLARY [Leavitt]. *For $\mathscr{M} \subseteqq \mathscr{W}$ with $\mathscr{W}$ any universal class there is a unique minimal radical class in $\mathscr{W}$ containing $\mathscr{M}$ which satisfies property* (a).

Proof. This is immediate from Theorems 2.4 and 2.5.

Note that by Lemma 2.2 the radical $\mathscr{L}\mathscr{M}^*$ coincides with the lower radical $\mathscr{L}\mathscr{M}$ in alternative classes, and thus for such classes the above construction may be regarded as an alternate lower radical construction.

THEOREM 2.6. *Let $\mathscr{W}$ be a universal class $\mathscr{M} \subseteqq \mathscr{W}$. There is a unique minimal class $\mathscr{M}' \supseteqq \mathscr{M}$ which is homomorphically closed, hereditary, and satisfies property* (a).

Proof. Define $\mathscr{M}_1 = \mathscr{M}$ and for $n \geqq 1$ let $\mathscr{M}_{n+1} = \mathscr{G}\mathscr{I}\mathscr{H}(\mathscr{M}_n)$. Now set $\mathscr{M}' = \cup \mathscr{M}_n$, the union being taken over all positive integers.

As in the proof of Theorem 2.5, $\mathscr{M}'$ is homomorphically closed, hereditary, and has property (a). Also as before (induction) $\mathscr{M}' \subseteqq \mathscr{A}$ where $\mathscr{A}$ is any homomorphically closed hereditary class with property (a) containing $\mathscr{M}$.

COROLLARY 2.7. *If* $\mathscr{M}$ *is any class,* $\mathscr{L}\mathscr{M}'$ *is the unique minimal strongly hereditary radical class containing* $\mathscr{M}$.

Proof. Since $\mathscr{M} \subseteqq \mathscr{M}'$, $\mathscr{M} \subseteqq \mathscr{L}\mathscr{M}'$. Since $\mathscr{M}'$ is homomorphically closed, hereditary, and satisfies property (a), $\mathscr{L}\mathscr{M}'$ has the same properties by Theorem 2.4 together with Theorem 2 of [6]. Now let $\mathscr{M} \subseteqq \mathscr{P}$ where $\mathscr{P}$ is a strongly hereditary radical class. Then $\mathscr{P}$ is homomorphically closed and hereditary, hence satisfies property (a) by Lemmas 1.2 and 1.3. Hence by Theorem 2.6 $\mathscr{M}' \subseteqq \mathscr{P}$ and therefore $\mathscr{L}\mathscr{M}' \subseteqq \mathscr{P}$.

3. Semisimple classes. Using Theorem 2.1, nonassociative versions of certain theorems concerning semisimple classes can be given. In [4], semisimple classes of associative rings are characterized as those classes $\mathscr{Q}$ satisfying the following four properties:

(1) $\mathscr{Q}$ is hereditary

(2) $\mathscr{Q}$ is closed under subdirect sums

(3) $\mathscr{Q}$ is extension closed

(4) If $I \leqq R$ and $0 \neq I/B \in \mathscr{Q}$ for some ideal B of I, there is an ideal A of R with $A \subseteqq I$ and $0 \neq I/A \in \mathscr{Q}$.

For possibly nonassociative classes, we have

THEOREM 3.1. $\mathscr{Q}$ *is a semisimple class for a radical class* $\mathscr{P}$ *satisfying property* (a) *if and only if* $\mathscr{Q}$ *satisfies properties* (1), (2), (3), *and* (4).

Proof. If $\mathscr{Q} = \mathscr{S}\mathscr{P}$ where $\mathscr{P}$ has property (a), then the proof of (1), (2), (3), and (4) go through as in the associative case (see [4]) using Theorem 2.1 and Lemma 1.3. Conversely, suppose $\mathscr{Q}$ satisfies (1), (2), (3), and (4). Then again as in the associative case $\mathscr{Q}$ is semisimple for some radical $\mathscr{P}$. Suppose $\mathscr{P}$ does not satisfy property (a). Then by Theorem 2.1 there is some R with an ideal I for which $\mathscr{P}(I)$ is not an ideal of R. Let T be the ideal of R generated by $P(I)$, then $\mathscr{P}(I) \leqq T \leqq I \leqq R$. Then by (1) and two applications of Lemma 1.1, $\mathscr{P}(\mathscr{P}(I)) = \mathscr{P}(I) \subseteqq \mathscr{P}(T) \subseteqq \mathscr{P}(I)$ so $\mathscr{P}(I) = \mathscr{P}(T) \neq T \leqq R$, and T is the ideal of R generated by $\mathscr{P}(T)$. Also $T/\mathscr{P}(T) \in \mathscr{Q}$ so by (4) there is an ideal K of R with $K \subseteqq T$ so that T/K is nonzero in $\mathscr{Q}$. Thus $K \geqq \mathscr{P}(T)$ so K is an ideal of R containing $\mathscr{P}(T)$, and K is proper in T, a contradiction which proves the theorem.

In [4], an ideal I of a ring R is said to be large in R if I has nonzero intersection with every nonzero ideal of R. It is proved there that a radical class $\mathscr{P}$ in an associative universal class is hereditary if and only if $\mathscr{S}\mathscr{P}$ satisfies property (λ): If $I \leqq R$ with $I \in \mathscr{S}\mathscr{P}$ and I large in R, then $R \in \mathscr{S}\mathscr{P}$. The same proof given there proves the following theorem, which is valid in an arbitrary universal class.

THEOREM 3.2. *Let $\mathscr{P}$ be a radical class satisfying property* (a). *Then $\mathscr{P}$ is hereditary if and only if $\mathscr{S}\mathscr{P}$ satisfies property* (λ).

Theorem 3.1 and 3.2 may be combined to give the following charracterization of semisimple classes for strongly hereditary radicals:

THEOREM 3.3. *$\mathscr{Q}$ is a semisimple class for a strongly hereditary radical if and only if $\mathscr{Q}$ satisfies properties* (1), (2), (3), (4), *and* (λ).

Proof. Suppose $\mathscr{Q} = \mathscr{S}\mathscr{P}$ where $\mathscr{P}$ is strongly hereditary. Then $\mathscr{P}$ is hereditary and has property (a) so $\mathscr{Q}$ satisfies (1), (2), (3), (4), and (λ). Conversely, if $\mathscr{Q}$ satisfies (1), (2), (3), (4), and (λ), then $\mathscr{Q} = \mathscr{S}\mathscr{P}$ for a radical $\mathscr{P}$ satisfying property (a) by Theorem 3.1 and $\mathscr{P}$ is hereditary by Theorem 3.2. Thus $\mathscr{P}$ is strongly hereditary by Lemma 1.2.

The following proposition and its corollary show that certain semisimple classes of associative rings satisfy property (a).

PROPOSITION 3.4. *If $\mathscr{M}$ is a class of associative rings which is hereditary, extension closed, and contains all nilpotent associative rings, then $\mathscr{M}$ satisfies property* (a).

Proof. Suppose $J \leqq I \leqq R$ where $J \in \mathscr{M}$, and let J' be the ideal of R generated by J. Then $(J')^3 \subseteqq J$ so $(J')^3 \in \mathscr{M}$. Also $J'/(J')^3 \in \mathscr{M}$, so $J' \in \mathscr{M}$ since $\mathscr{M}$ is extension closed.

COROLLARY 3.5. *If $\mathscr{P}$ is a radical in an associative universal class such that $R^2 = R$ for all $R \in \mathscr{P}$, then $\mathscr{S}\mathscr{P}$ has property* (a).

Proof. $\mathscr{S}\mathscr{P}$ is easily verified to satisfy the hypotheses of Proposition 3.4.

REFERENCES

1. S. A. Amitsur, *Radicals in rings and bicategories*, Amer. J. Math., **76** (1954), 100-125.

2. T. Anderson, N. Divinsky, and A. Sulinski, *Hereditary radicals in associative and alternative rings*, Canad. J. Math., **17** (1965), 594-603.

3. ———, *Lower Radical Properties for associative and alternative rings*, J. London Math. Soc., **41** (1966), 417-424.

4. E. P. Armendariz, *Closure properties in radical theory*, Pacific J. Math., **26** (1968), 1-7.

5. N. Divinsky, *Rings and radicals*, University of Toronto Press, Tronto, 1965.

6. A. E. Hoffman and W. G. Leavitt, *Properties inherited by the lower radical*, Portugal. Math., **27** (1968), 63-66.
7. W. G. Leavitt, *Strongly hereditary radicals*, Proc. Amer. Soc., **21** (1969), 703-705.
8. ———, *Radical and semisimple classes with specified properties*, Proc. Amer. Math. Soc., **24** (1970), 680-687.

Received April 12, 1971.

ARKANSAS STATE UNIVERSITY