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A CHARACTERIZATION OF GENERAL Z.P.I.-RINGS II

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A commutative ring R is a general Z.P.I.-ring if each ideal of R can be represented as a finite product of prime ideals. If R is not a general Z.P.I.-ring, it is still possible that each principal ideal of R can be represented as a finite product of prime ideals. In this paper, it is shown that if R is a commutative ring in which each ideal generated by two elements can be written as a finite product of prime ideals, then R must be a general Z.P.I.-ring.

Let R be a commutative ring. R is a *general Z.P.I.-ring* if each ideal of R can be represented as a finite product of prime ideals. In a previous paper, we proved that R is a general Z.P.I.-ring if each finitely-generated ideal of R can be represented as a finite product of prime ideals [4; Theorem 2.3]. If each ideal of R generated by n or fewer elements can be represented as a finite product of prime ideals, then we define R to be a *$\pi(n)$-ring*. Mori completely characterized the structure of $\pi(1)$-rings in a series of four papers [5, 6, 7, 8]. Using his characterization, it is not difficult to construct a $\pi(1)$-ring that is not a $\pi(n)$-ring for any $n > 1$. For this reason it is surprising that the main result of this paper is the following theorem.

THEOREM. *Let R be a commutative ring. Then the following conditions are equivalent*:

(a) *R is a general Z.P.I.-ring*;

(b) *for $n \geqq 2$, R is a $\pi(n)$-ring*;

(c) *R is a $\pi(2)$-ring.*

Throughout this paper, R denotes a commutative ring and n denotes an arbitrary positive integer.

2. **$\pi(n)$-rings without zero-divisors.** If D is an integral domain, we call a prime ideal P of D *minimal* if P is of height one. An integral domain D with identity is a *Krull domain* if there is a set of rank one discrete valuation rings $\{V_\alpha\}$ such that $D = \bigcap_\alpha V_\alpha$ and such that each nonzero element of D is a non-unit in only finitely many of the V_α.

EXAMPLE 2.1. An integral domain D with identity is a $\pi(1)$-ring if and only if D is a Krull domain in which each minimal prime ideal is invertible [4; Theorem 1.2]. If Z denotes the rational integers, then the polynomial ring in one indeterminate $Z[x]$ is a $\pi(1)$-ring and $Z[x]$ is not a $\pi(n)$-ring for any $n > 1$.

Henceforth we refer to $\pi(n)$-rings without zero-divisors as *$\pi(n)$-domains*.

LEMMA 2.2. *Let R be a $\pi(2)$-domain with identity. Then R is a Krull domain in which each prime ideal of height one is invertible. Moreover, the prime ideals of height one are pairwise comaximal.*

Proof. If R is a $\pi(2)$-domain, R is a $\pi(1)$-domain. It follows from [4; Theorem 1.2] that R is a Krull domain in which each minimal prime ideal is invertible. Let P_1 and Q be distinct minimal prime ideals of R. Let $a \in P_1 \backslash Q$. Then

$$(a) = \prod_{i=1}^{s} P_i^{e_i} ,$$

where, for each i, $e_i \geqq 1$, $P_i \neq Q$, and P_i is a minimal prime ideal. Let $b \in Q \backslash \bigcup_{i=1}^{s} P_i$. Then

$$(a, b) = \prod_{j=1}^{m} R_j ; (a, b^2) = \prod_{k=1}^{p} S_k ,$$

where for each j and k, R_j and S_k are prime ideals of R.

If $bt \in (a)$ for some $t \in R$, then $(bt) \subset \prod_{i=1}^{n} P_i^{e_i}$. If for each i, $1 \leqq i \leqq s$, we let v_i denote the valuation on R with respect to the minimal prime ideal P_i, then $v_i(bt) \geqq e_i$ while $v_i(b) = 0$. Hence $t \in P_i^{(e_i)}$, the e_ith symbolic power of P_i. Since for each i, P_i is invertible, it follows that $P_i^{(e_i)} = P_i^{e_i}$ [9; Lemma 21], and so $t \in P_i^{e_i}$. Because each P_i is invertible, we can use an induction argument on s to conclude that $t \in \prod_{i=1}^{s} P_i^{e_i} = (a)$.

If $\bar{R} = R/(a)$, and $\bar{b}$ is the image of b in $\bar{R}$, the above argument shows that $\bar{b}$ is a regular element of $\bar{R}$. In $\bar{R}$,

$$(\bar{b}) = \prod_{j=1}^{m} (R_j/(a))$$

$$(\bar{b}^2) = \prod_{k=1}^{p} (S_k/(a)) .$$

By [1; Theorem 1], the factorization of the ideal $(\bar{b}^2)$ is unique up to factors of $\bar{R}$. It follows that $p = 2m$, and that we can index the ideals S_k, $1 \leqq k \leqq p$, so that

$$R_j = S_{2j-1} = S_{2j} .$$

Hence $(a, b^2) = \prod_{k=1}^{p} S_k = \prod_{j=1}^{m} (R_j)^2 = (a, b)^2$. Thus

$$(a) \subset (a, b^2) = (a, b)^2 \subset (a^2, b) .$$

If $x \in (a)$, then $x = ra^2 + sb$, where $r, s \in R$. This implies that $sb \in (a)$, and, consequently, $s \in (a)$. We conclude that

$$(a) \subseteqq (a)(a, b) .$$

Since the reverse conclusion is always valid,

$$(a) = (a)(a, b) .$$

Because $a \neq 0$, it follows that

$$R = (a, b) \subseteqq (P_1, Q) \subseteqq R .$$

Hence the minimal prime ideals of R are comaximal. This completes the proof of the lemma.

An integral domain with identity that is a general Z.P.I.-ring is called a *Dedekind domain*.

THEOREM 2.3. *Let R be an integral domain with identity. The following conditions are equivalent*:

(1) *R is a Dedekind domain,*

(2) *for $n \geqq 2$, R is a $\pi(n)$-domain;*

(3) *R is a $\pi(2)$-domain.*

Proof. $(1 \rightarrow 2)$ By definition of Dedekind domain.

$(2 \rightarrow 3)$ By definition of $\pi(n)$-ring.

$(3 \rightarrow 1)$ By Lemma 2.1, R is a Krull domain in which prime ideals of height one are invertible. To conclude that R is a Dedekind domain, it suffices to show that R is of Krull dimension one [3; Theorem 35.16]. Each non-unit of R is contained in some minimal prime ideal. Hence, if R has a unique minimal prime ideal P, P is also the unique maximal ideal of R, and R is of Krull dimension one. If R has more than one minimal prime ideal, then by Lemma 2.1, all these prime ideals are comaximal. If Q is any nonzero proper prime ideal of R, there is a minimal prime ideal P such that $P \subseteqq Q$ [3; Corollary 35.10]. If $P \neq Q$, there exists $b \in Q \backslash P$. $(b) = \prod_{i=1}^{t} S_i$, where for each i, S_i is a minimal prime ideal of R and $S_i \neq P$. Since $b \in Q$, for some $i, 1 \leqq i \leqq t$, $S_i \subset Q$. But this implies that $R = (P, S_i) \subseteqq Q$. Hence $Q = P$, and R is of Krull dimension one. This completes the proof of the theorem.

THEOREM 2.4. *Let R be a $\pi(2)$-domain without identity. Then R is a general Z.P.I.-ring.*

Proof. Each minimal prime ideal of R is a principal ideal [8;

Theorem 26]. If R contains a unique minimal prime ideal (p), then it must be the case that $R = (p)$ [8; Lemma II]. We assume that R contains two distinct minimal prime ideals, (p) and (q). Using the same argument we did in Lemma 2.2, we can show that

$$(p) = (p)(p, q) .$$

Since (p) is a regular ideal, it follows that R must have an identity [2; Corollary 5.2]. Therefore, since R has no identity, it must be the case that R is the only nonzero prime ideal of itself.

Let A be a nonzero ideal of R. Then there is a smallest positive integer n such that $R^n \subset A \subseteqq R^{n-1}$. Let $a \in A \backslash R^n$. Since $(a) = R^k$ for some $k < n$, it follows that $R^n \subset (a) = R^k \subseteqq A \subseteqq R^{n-1}$. Hence $A = R^{n-1}$. Because each ideal of R is a power of R it follows that R is a general Z.P.I.-ring [10; Theorem 2]. This completes the proof of this theorem.

3. Main result.

LEMMA 3.1. *Let R be a $\pi(2)$-ring with identity. If R is the direct sum of finitely many rings, $R = \sum_{i=1}^k R_i$, then each direct summand R_i is also a $\pi(2)$-ring.*

Proof. Let R_j be one of the direct summands of R, and let $A_j = (a_{1j}, a_{2j})$ be an ideal of R_j generated by two elements of R_j. Let e_i denote the identity of the direct summand R_i, $1 \leqq i \leqq k$. Then if A is the ideal of R generated by the two elements $(\sum_{i \neq j} e_i) + a_{1j}$ and $(\sum_{i \neq j} e_i) + a_{2j}$, then

$$A = \prod_{r=1}^{t} P_r$$

where for each r, $1 \leqq r \leqq t$, P_r is a prime ideal of R. Then $A_j = AR_j = (\prod_{r=1}^t P_r)R_j = \prod_{r=1}^t (P_r R_j)$. Since for each r, $P_r R_j$ is a prime ideal of R_j, A_j can be expressed as a finite product of prime ideals. Hence R_j is a $\pi(2)$-ring.

A principal ideal ring R with identity is called a *special primary ring* if R contains only one prime ideal $M \neq R$ and if $M^k = (0)$ for some positive integer k.

THEOREM 3.2. *Let R be a commutative ring. Then the following conditions are equivalent:*

(a) *R is a general Z.P.I.-ring;*

(b) *for $n \geqq 2$, R is a $\pi(n)$-ring;*

(c) *R is a $\pi(2)$-ring.*

Proof. It is clear that (a) implies (b) and that (b) implies (c). We now show that (c) implies (a). We consider three cases: (1) R is a commutative ring with identity; (2) R is a commutative ring without identity, but with zero divisors; (3) R is an integral domain without identity.

If R is a commutative ring with identity, then R is a direct sum of $\pi(1)$-domain with identity and special primary rings by [7; Hauptsatz]. Using [10; Theorem 2], we can conclude that R is a general Z.P.I.-ring if any summand R_i of R that is a domain is Dedekind. From Lemma 3.1 it follows that each summand of R is a $\pi(2)$-ring. Hence if the summand R_i is a domain, R_i is Dedekind by Theorem 2.3. Thus a $\pi(2)$-ring with identity is a general Z.P.I.-ring.

If R is a commutative ring without identity, but with zero-divisors, then $R = M$ or $R = M + K$, where K is a field and M is a ring without identity such that each ideal of M is a power of M [8; Hauptsatz 11]. R is a general Z.P.I.-ring by [10; Theorem 2].

The last case is settled by Theorem 2.4.

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