PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 2, 1972

INTEGRATED ORTHONORMAL SERIES

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Throughout this paper the author defines

$$F_\alpha(t) = \sum_{m=1}^{\infty} |\Phi_m(t)|^\alpha = \sum_{m=1}^{\infty} \left| \int_a^t \varphi_m(x) dx \right|^\alpha$$

where $0 < \alpha \leqq 2, a \leqq t \leqq b$, and $\{\varphi_m\}$ is a sequence in $L^1[a, b]$, usually orthonormal. In this paper, $F_\alpha(t)$ is studied for the Haar, Walsh, trigonometric, and general orthonormal sequences. For instance, it is proved that for the Haar system $F_\alpha(t)$ satisfies a Lipschitz condition of order $\alpha/2$ in $[0, 1]$ and that this result is best possible for any complete orthonormal sequence. An application is also given regarding the absolute convergence of Walsh series.

Previously, Bosanquet and Kestelman essentially proved [3, p. 91]

THEOREM A. *Let $\{\varphi_m\}$ be orthonormal. Then the Fourier coefficients of every absolutely continuous function are absolutely convergent if and only if $F_1(t) \in L^\infty[a, b]$.*

Also, applying Parseval's equality to the characteristic function of $[a, t]$, we obtain

THEOREM B. *Let $\{\varphi_m\}$ be orthonormal. Then $\{\varphi_m\}$ is complete in $L^2[a, b]$ if and only if $F_2(t) = t - a, a \leqq t \leqq b$.*

For certain systems, such as the Haar system, the following extension of Theorem A is possible.

THEOREM 1. *Assume $\{\varphi_m\}$ is orthonormal, $\Phi_m(t)$ has constant sign on $[a, b]$ for each $m = 1, 2, \cdots$, and $\Sigma |\Phi_m(b)| < \infty$. Then the Fourier coefficients of every absolutely continuous function $f(t)$, such that $f'(t) \in L^p$, are absolutely convergent if and only if $F_1(t) \in L^q$, $1 \leqq p \leqq \infty$, $p^{-1} + q^{-1} = 1$.*

Proof. Necessity. Integrating by parts we obtain

$$\int_a^b f'(t) \sum_{m=1}^{\infty} |\Phi_m(t)| dt$$

exists for every $f' \in L^p$. Hence, $F_1(t) \in L^q$ [7, p. 166].

Sufficiency. By Hölder's inequality

$$\sum_{m=1}^{N}\left|\int_a^b f'(t)\Phi_m(t)dt\right| \leqq \int_a^b |f'(t)|\sum_1^N \Phi_m(t)\,|dt \leqq \|f'\|_p\|F_1\|_q .$$

If an orthonormal sequence $\{\varphi_m\}$ is not complete we still obtain $F_2(t)$ continuous since the "completed" series converges to a continuous function and hence (i.e. by Dini's theorem) the convergence must be uniform. In fact, we have

THEOREM 2. *If $\{\varphi_m\}$ is orthonormal, then $F_2(t) \in \mathrm{Lip}\,(1/2)$.*

Proof. Let $x, y \in [a, b]$. Using Bessel's inequality, we obtain

$$\begin{aligned}|F_2(x) - F_2(y)| &= \left|\sum_{m=1}^{\infty}[\Phi_m(x)]^2 - [\Phi_m(y)]^2\right| \\ &\leqq \sum_{m=1}^{\infty}|\Phi_m(x) - \Phi_m(y)|\{|\Phi_m(x)| + |\Phi_m(y)|\} \\ &\leqq \left\{\sum_{m=1}^{\infty}[\Phi_m(x) - \Phi_m(y)]^2\sum_{m=1}^{\infty}[\Phi_m(x)]^2\right\}^{1/2} \\ &\quad + \left\{\sum_{m=1}^{\infty}[\Phi_m(x) - \Phi_m(y)]^2\sum_{m=1}^{\infty}[\Phi_m(y)]^2\right\}^{1/2} \\ &\leqq 2|b - a|^{1/2}|x - y|^{1/2} .\end{aligned}$$

REMARK 1. This result is best possible in the following sense: For every $\varepsilon > 0$ if we set $\varphi_1(x) = (1 - x)^{(\varepsilon-1)/2}$, $0 \leqq x < 1$, then $\varphi_1 \in L^2[0, 1]$ but $[\Phi_1(t)]^2 \notin \mathrm{Lip}\,(1/2 + \varepsilon)$.

REMARK 2. It would be interesting to know if $F_2(t)$ is absolutely continuous and if $F_2'(t) \in L^2$ for any orthonormal sequence $\{\varphi_m\}$.

THEOREM 3. *For any complete orthonormal system $\{\varphi_m\}$, $F_\alpha(t) \notin \mathrm{Lip}\,(\alpha/2 + \varepsilon)$ for any $\varepsilon > 0$.*

Proof. Let $t \in [a, b]$. By Parseval's equality

$$[F_\alpha(t)]^{1/\alpha} \geqq [F_2(t)]^{1/2} = (t - a)^{1/2},\ 0 < \alpha \leqq 2 ,$$

since for any nonnegative sequence $\{a_m\}$, $[\Sigma a_m^\alpha]^{1/\alpha}$ is a non-increasing function of α for $\alpha > 0$.

We will now determine which Lipschitz class $F_\alpha(t)$ belongs to for the Haar, Walsh, and trigonometric systems.

DEFINITION. If $0 < \alpha \leqq 1$, set

$$N_\alpha(f) = \sup|f(x) - f(y)|\,|x - y|^{-\alpha} \quad \text{for} \quad x \neq y \quad \text{and} \quad x, y \in [a, b] .$$

LEMMA 1. *Let $\alpha > 0$ and $0 < \alpha - \beta \leqq 1$. If*

$$\textstyle\sum_{m=1}^{n} N_\alpha(f_m) = O(n^\beta)$$

and

$$\textstyle\sum_{m=n}^{\infty} ||f_m||_\infty = O(n^{\beta-\alpha}) ,$$

then

$$f(t) = \textstyle\sum_{m=1}^{\infty} f_m(t) \in \mathrm{Lip}\,(\alpha - \beta) .$$

Proof. Let $2^{-n-1} < h \leqq 2^{-n}$. Then

$$|f(t+h) - f(t)| \leqq \sum_{m=1}^{\infty} |f_m(t+h) - f_m(t)| = \sum_{m=1}^{2^n} + \sum_{m=2^n+1}^{\infty} = P + Q .$$

$$P = O\Big[h^\alpha \sum_{m=1}^{2^n} N_\alpha(f_m)\Big] = O(h^{\alpha-\beta}) ,$$

$$Q = O\Big[\sum_{m=2^n+1}^{\infty} ||f_m||_\infty\Big] = O(h^{\alpha-\beta}) .$$

LEMMA 2. (a) *If* $\sum_{m=2^n+1}^{2^{n+1}} |a_m| m^\alpha = O(2^{n\beta})$, *then*

$$\sum_{m=n}^{\infty} |a_m| = O(n^{\beta-\alpha}),\ \beta - \alpha < 0 .$$

(b) *If* $\displaystyle\sum_{m=2^n+1}^{2^{n+1}} |a_m| = O(2^{n\beta})$, *then* $\displaystyle\sum_{m=1}^{n} |a_m| m^\alpha = O(n^{\alpha+\beta}),\ \alpha + \beta > 0$.

Proof. Straightforward.

LEMMA 3. *Let* $0 < \gamma \leqq 1$ *and suppose* $f \in \mathrm{Lip}\,\gamma$.
(a) *If* $0 < \alpha \leqq 1$, $|f|^\alpha \in \mathrm{Lip}\,(\alpha\gamma)$.
(b) *If* $\alpha > 1$, $|f|^\alpha \in \mathrm{Lip}\,\gamma$.

Proof. We may assume $f(t) \geqq 0$ because

$$||f(t+h)| - |f(t)|| \leqq |f(t+h) - f(t)| .$$

Part (a). Since $|x+y|^\alpha \leqq |x|^\alpha + |y|^\alpha$, $0 < \alpha \leqq 1$, we obtain

$$|f^\alpha(t+h) - f^\alpha(t)| \leqq |f(t+h) - f(t)|^\alpha = O(h^{\alpha\gamma}) .$$

Part (b). Since $|x^\alpha - y^\alpha| \leqq ||\alpha t^{\alpha-1}||_\infty |x - y|$, $\alpha \geqq 1$, it follows that

$$|f^\alpha(t+h) - f^\alpha(t)| \leqq ||\alpha f^{\alpha-1}(t)||_\infty |f(t+h) - f(t)| = O(h^\gamma) .$$

THEOREM 4. *Let* $0 < \gamma \leqq 1$ *and assume* $f \in \mathrm{Lip}\,\gamma$ *and is of period* $b - a$.
(a) *If* $0 < \alpha \leqq 1$, $0 < \alpha\gamma - \delta \leqq 1$, *and*

$$\sum_{m=1}^{n} |a_m| m^{\alpha\gamma} = O(n^\delta) ,$$

then

$$f_\alpha(t) = \sum_{m=1}^{\infty} a_m |f(mt)|^\alpha \in \mathrm{Lip}\,(\alpha\gamma - \delta) \,.$$

(b) *If* $\alpha > 1, 0 < \gamma - \delta \leqq 1,$ *and*

$$\sum_{m=1}^{n} |a_m| m^\gamma = O(n^\delta) \,,$$

then

$$f_\alpha(t) = \sum_{m=1}^{\infty} a_m |f(mt)|^\alpha \in \mathrm{Lip}\,(\gamma - \delta) \,.$$

Proof. *Part* (a). By hypothesis and Lemma 3 (a)

$$\sum_{m=1}^{n} N_{\alpha\gamma}[a_m |f(mt)|^\alpha] = O\Big(\sum_{1}^{n} |a_m| m^{\alpha\gamma}\Big) = O(n^\gamma) \,.$$

Also, by Lemma 2 (a), if $0 < \alpha\gamma - \delta$, then

$$\sum_{m=n}^{\infty} || a_m |f(mt)|^\alpha ||_\infty = O\Big(\sum_{n}^{\infty} |a_m|\Big) = O(n^{\delta - \alpha\gamma})$$

and so our result follows by Lemma 1.

Part (b). By hypothesis and Lemma 3 (b)

$$\sum_{m=1}^{n} N_\gamma[a_m |f(mt)|^\alpha] = O\Big(\sum_{1}^{n} |a_m| m^\gamma\Big) = O(n^\delta) \,.$$

Also, by Lemma 2 (a), if $0 < \gamma - \delta$, then

$$\sum_{m=n}^{\infty} || a_m |f(mt)|^\alpha || = O\Big(\sum_{n}^{\infty} |a_m|\Big) = O(n^{\delta - \gamma}) \,,$$

and so our result again follows from Lemma 1.

THEOREM 5. *Let* $0 < \alpha \leqq 2$ *and assume* $\varphi \in L^\infty[a, b]$, $\varphi_m(x) = \varphi(mx)$, *and* $\Phi_1(t)$ *is of period* $b - a$. *If*

$$\sum_{m=1}^{n} |b_m| = O(n^\beta), 0 < \alpha - \beta < 1 \,,$$

then

$$G_\alpha(t) = \sum_{m=1}^{\infty} b_m |\Phi_m(t)|^\alpha \in \mathrm{Lip}\,(\alpha - \beta) \,.$$

Proof. $\Phi_m(t) = m^{-1}\Phi_1(mt)$ and so

$$G_\alpha(t) = \sum_{m=1}^{\infty} b_m m^{-\alpha} |\Phi_1(mt)|^\alpha .$$

Now let $\gamma = 1$ and $a_m = b_m m^{-\alpha}$ in Theorem 4. Then, if $0 < \alpha \leqq 1$, our result follows by Theorem 4 (a) with $\delta = \beta$.

If $\alpha > 1$ and $\alpha - \beta < 1$, then by Lemma 2 (b)

$$\sum_{m=1}^{n} |a_m| m^1 = \sum_{1}^{n} |b_m| m^{1-\alpha} = O(n^{\beta-\alpha+1}) .$$

Thus, utilizing Theorem 4 (b) with $\delta = \beta - \alpha + 1$, we obtain

$$G_\alpha(t) \in \mathrm{Lip}\,[1 - (\beta - \alpha + 1)] = \mathrm{Lip}\,(\alpha - \beta) .$$

COROLLARY 1. (a) $\sum_{m=1}^{\infty} \left| \int_0^t \sin mx\, dx \right|^\alpha \in \mathrm{Lip}\,(\alpha - 1)$, $1 < \alpha < 2$, *on* $[0, 2\pi]$.

(b) *If* $1 < \alpha < 2$ *and* $\{w_m(x)\}$ *and* $\{r_m(x)\} = \{r_1(2^{m-1}x)\}$ *denote the Walsh and Rademacher functions* (*defined in* [1]), *then*

$$\sum_{m=0}^{\infty} \left| \int_0^t w_m(x) dx \right|^\alpha = t^\alpha + \sum_{m=1}^{\infty} 2^{m-1} \left| \int_0^t r_m(x) dx \right|^\alpha \in \mathrm{Lip}\,(\alpha - 1) \text{ on } [0, 1] ,$$

since $\left| \int_0^t w_m(x) dx \right| = \left| \int_0^t r_k(x) dx \right|$ *for* $2^{k-1} \leqq m < 2^k$, $k = 1, 2, \cdots$, *as can be easily seen directly.*

(c) *If* $0 < \alpha < 2$ *and* $\{h_m\}$ *denotes the Haar system* (*defined in* [1]), *then*

$$\sum_{m=0}^{\infty} \left| \int_0^t h_m(x) dx \right|^\alpha = t^\alpha + \sum_{m=1}^{\infty} 2^{(m-1)\alpha/2} \left| \int_0^t r_m(x) dx \right|^\alpha \in \mathrm{Lip}\,(\alpha/2) \text{ on } [0, 1] ,$$

since $\sum_{m=2^{k-1}}^{2^k-1} \left| \int_0^t h_m(x) dx \right| = 2^{(k-1)\alpha/2} \left| \int_0^t r_k(x) dx \right|$ *for* $k = 1, 2, \cdots$.

REMARK 3. For the Haar system $F_1(t)$ has no finite derivative anywhere [5, p. 279].

THEOREM 6. *Let* $0 < ||\varphi||_1 < \infty$, $\varphi_m(x) = \varphi(mx)$, *and assume* $\Phi_1(t)$ *is of period* $b - a$.

(a) $\sum |a_m| m^{-\alpha} < \infty$ *if and only if* $\sum |a_m|\, |\Phi_m(t)|^\alpha \in L^1[a, b]$.

(b) *If* $\sum |a_m| m^{-\alpha} = \infty$, *then* $\sum |a_m|\, |\Phi_m(t)|^\alpha = \infty$ *almost everywhere.*

Proof. Part (a). Since $\Phi_m(t) = m^{-1}\Phi_1(mt)$, we obtain

$$\int_a^b |\Phi_m(t)|^\alpha dt = m^{-\alpha} \int_a^b |\Phi_1(mt)|^\alpha dt = m^{-\alpha} \int_a^b |\Phi_1(t)|^\alpha dt .$$

Part (b). Applying Fejer's Lemma [7, p. 49], we obtain for every set E of positive measure

$$\lim \int_E |\Phi(mt)|^\alpha dt = \frac{\mu(E)}{b-a}\int_a^b |\Phi_1(t)|^\alpha dt > 0 \quad \text{as} \quad m \to \infty ,$$

and so by a theorem of Orlicz [1, p. 327]

$$\sum |a_m| m^{-\alpha} |\Phi_1(mt)|^\alpha = \sum |a_m| |\Phi_m(t)|^\alpha = \infty$$

almost everywhere.

COROLLARY 2. *There exists an absolutely continuous function whose Walsh-Fourier series is absolutely divergent.*

Proof. For the Walsh system $F_1(t) \notin L^\infty$ by Theorem 6 and so the result follows from Theorem A.

It now seems appropriate to prove

THEOREM 7. *Let*

$$\omega^2(\delta, f) = \sup_{0<h\leqq\delta} \left\{\int_0^1 [f(x+h) - f(x)]^2 dx\right\}^{1/2} .$$

If $\sum 2^{n/2}\omega^2(2^{-n}, f) < \infty$, *then the Walsh-Fourier series of* f *converges absolutely.*

Proof. Let $\{c_n\}$ denote the Walsh-Fourier coefficients of f and let $x \dotplus y = \sum_{n=1}^\infty |x_n - y_n| 2^{-n}$ where $x = \sum x_n 2^{-n}$ and $y = \sum y_n 2^{-n}$ are the binary expansions of x and y (where for dyadic rationals we choose the finite expansion). N. Fine proved [4, p. 395]

$$\sum_{k=2^{n-1}}^{2^n-1} c_k^2 \leqq \int_0^1 [f(x \dotplus 2^{-n}) - f(x)]^2 dx .$$

Also, by definition of $\dotplus$, we obtain

$$\begin{aligned}\int_0^1 &[f(x \dotplus 2^{-n}) - f(x)]^2 dx \\ &= \int_{E_0} [f(x + 2^{-n}) - f(x)]^2 dx + \int_{E_1} [f(x - 2^{-n}) - f(x)]^2 dx \\ &= 2\int_{E_0} [f(x + 2^{-n}) - f(x)]^2 dx\end{aligned}$$

where $E_p = \{x \in [0, 1]: x_n = p\}$ for $p = 0, 1$. Hence,

$$\sum_{2^{n-1}}^{2^n-1} c_k^2 \leqq 2[\omega^2(2^{-n}, f)]^2 ,$$

and so by Schwarz's inequality

$$\sum_{k=2^{n-1}}^{2^n-1} |c_k| \leqq \left(\sum_{2^{n-1}}^{2^n-1} c_k^2\right)^{1/2} \left(\sum_{2^{n-1}}^{2^n-1} 1\right)^{1/2} \leqq \omega^2(2^{-n}, f) 2^{n/2} .$$

REMARK 4. Previously N. Fine [4, p. 394] and N. Vilenkin [6, p. 32] proved that if $f \in \mathrm{Lip}\,\alpha$, $\alpha > 1/2$, then the Walsh-Fourier series of f converges absolutely. By Theorem 7 it follows that all of the sufficiency theorems on absolute convergence for trigonometric series [2, p. 154–161] in terms of modulus of continuity carry over completely for the Walsh system.

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Received February 25, 1971 and in revised form April 24, 1972. Supported by NSF grant GP-14135.

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