PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 3, 1972

ON ABSOLUTE DE LA VALLÉE POUSSIN SUMMABILITY

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Gronwall proved that $(C, r) \subseteqq (V - P)$ for $r \geqq 0$, where (C, r) and $(V - P)$ denote Cesáro and de la Vallée Poussin summability. It is proved in this paper that $|C, r| \subseteqq |V - P|$ for $r \geqq 0$.

1. Introduction.

Let

$$V_n = \sum_{k=1}^{n} \frac{(n!)^2}{(n-k)!(n+k)!} a_k \quad (n \geqq 0) .$$

If $\lim_{n\to\infty} V_n = s$, we say that the series is summable $(V - P)$ to s. If

$$\sum_{n=1}^{\infty} \left| V_n - V_{n-1} \right| < \infty .$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|V - P|$.

Hyslop [2] proved that the $(V - P)$ method is equivalent to the $(A, 2)$ method defined by

$$\lim_{x\to 0+} \sum_{n=0}^{\infty} a_n e^{-n^2 x} = s$$

for all series $\sum_{n=0}^{\infty} a_n$ which satisfy the condition $a_n = 0(n^c)$, where c is any constant, and that the inclusion $(A, 2) \subseteqq (V - P)$ is false without restriction.

Kuttner [3] has shown that $(V - P) \subseteqq (A, 2)$ without restriction.

Gronwall [1] proved that $(C, r) \subseteqq (V - P)$ for $r \geqq 0$, where (C, r) denotes the Césaro summability of order r.

In this paper, we shall prove

THEOREM A. *$|C, r| \subseteqq |V - P|$ for $r \geqq 0$.*

2. Proof of Theorem A.

Since it is well-known that $|C, r|$ implies $|C, r'|$ for $-1 < r \leqq r'$, it is enough to consider the case r an integer. Now, writing

$$V_n = v_0 + v_1 + \cdots + v_n ,$$

we find that

$$(1) \qquad \begin{cases} v_0 = a_0 , \\ v_n = \displaystyle\sum_{k=1}^{n} \frac{((n-1)!)^2}{(n-k)!(n+k)!} k^2 a_k \quad (n \geqq 1) . \end{cases}$$

Now write $\tau_k = \tau_k^r$ for the (C, r) mean of the sequence $\{ka_k\}$; thus the assumption that $\sum_{n=0}^{\infty} a_n$ is summable $|C, r|$ is equivalent to

$$\sum_{n=0}^{\infty} \frac{|\tau_n|}{n} < \infty . \tag{2}$$

If we take $((n-1)!)^2/(n-k!)(n+k)!$ as meaning 0 whenever $k > n$, we deduce from (1) by n partial summations that, for $n \geqq 1$,

$$v_n = \sum_{k=1}^{n} \Delta_k^r \left\{ \frac{((n-1)!)^2 k}{(n-k)!(n+k)!} \right\} \binom{k+r}{k} \tau_k .$$

Now it is well-known that in order that the series-to-series transformation

$$b_n = \sum_{k=0}^{\infty} \alpha_{nk} a_k$$

should be that $\sum_{n=0}^{\infty} |b_n|$ converges whenever $\sum_{n=0}^{\infty} |a_k|$ does so, it is necessary and sufficient that

$$\sum_{n=0}^{\infty} |\alpha_{nk}|$$

should be bounded. Thus it is enough to show that, for $k \geqq 1$,

$$\sum_{n=k}^{\infty} \left| \Delta_k^r \left\{ \frac{((n-k)!)^2 k}{(n-k)!(n+k)!} \right\} \right| = O(k^{-r-1}) . \tag{3}$$

It is easily seen by induction on r that

$$\Delta_k^r \left\{ \frac{((n-1)!)^2 k}{(n-k)!(n+k)!} \right\} = \frac{A^r(n, k)((n-1)!)^2}{(n-k)!(n+k+r)!} ,$$

where $A^r(n, k)$ is defined inductively by

$$\begin{cases} A^0(n, k) = k , \\ A^{r+1}(n, k) = (n+k+r+1)A^r(n, k) - (n-k)A^r(n, k+1) . \end{cases} \tag{4}$$

Write $P_j(k)$ for a polynomial in k of degree not exceeding j, possibly different at each occurrence (thus $P_0(k)$ denotes a constant). We deduce from (4) by induction that

$$A^{2s}(n, k) = \sum_{j=0}^{s} P_{2j+1}(k) n^{s-j} ,$$

$$A^{2s+1}(n, k) = \sum_{j=0}^{s+1} P_{2j}(k) n^{s+1-j} .$$

Hence, uniformly in the ranges stated

$$A^r(n, k) = \begin{cases} O(n^{(r+1)/2}) & (1 \leqq k \leqq n^{1/2}) , \\ O(K^{r+1}) & (n^{1/2} < k \leqq n) . \end{cases}$$

Next, for large n uniformly in $k \leqq n^{2/3}$ we have, by Stirling's formula

$$\frac{(n!)^2}{(n-k)!(n+k)!} = O(H(n, k)) ,$$

where

$$H(n, k) = \left(1 - \frac{k}{n}\right)^{-n+k-1/2}\left(1 + \frac{k}{n}\right)^{-n-k-1/2} .$$

We have

$$\log H(n, k) = -\frac{k^2}{n} + O\left(\frac{k^3}{n^2}\right) .$$

Now since we supposing that $k \leqq n^{2/3}$ we have

$$\exp\left\{O\left(\frac{k^3}{n^2}\right)\right\} = O(1) \tag{5}$$

so that

$$\frac{(n!)^2}{(n-k)!(n+k)!} = O\left\{\exp\left(-\frac{k^2}{n}\right)\right\} .$$

This will not apply if $k > n^{2/3}$. Since we cannot then assert (5). However, for fixed n, $(n!)^2/(n-k)!(n+k)!$ is a decreasing function of k so that, for $k > n^{2/3}$,

$$\frac{(n!)^2}{(n-k)!(n+k)!} = O\{\exp(-n^{1/3})\} .$$

Also, it is trivial that

$$\frac{((n-1)!)^2}{(n-k)!(n+k+2)!} = \frac{(n!)^2}{(n-k)!(n+k)!}O(n^{-r-2}) .$$

Combining these results, we find that

$$\Delta_k^r\left\{\frac{((n-1)!)^2k}{(n-k)!(n+k)!}\right\} = \begin{cases} O(n^{-(r+3)/2}) & (1 \leqq k \leqq n^{1/2}) , \\ O\left(\frac{k^{r+1}}{n^{r+2}}\exp\left(-\frac{k^2}{n}\right)\right) & (n^{1/2} < k \leqq n^{2/3}) , \\ O(n^{-1}\exp(-n^{-(1/3)})) & (n^{2/3} < k \leqq n) . \end{cases}$$

Thus the sum (3) is

$$O\left\{\sum_{k\leqq n<k^{3/2}}\frac{1}{n}\exp\left(-n^{-(1/3)}\right)\right\}+O\left\{\sum_{k^{3/2}\leqq n<k^2}\frac{k^{r+1}}{n^{r+2}}\exp\left(-\frac{k^2}{n}\right)\right\}+O\left\{\sum_{n\geqq k^2}\frac{1}{n^{(r+3)/2}}\right\}$$
$$= O(I_1) + O(I_2) + O(I_3) ,$$

say. It is clear that

$$I_1 = O(k^{-r-1}) ,$$
$$I_3 = O(k^{-r-1})$$

so we need consider only I_2. Now for fixed k

$$\frac{k^{r+1}}{y^{r+2}}\exp\left(-\frac{k^2}{y}\right)$$

is increasing for $y < y_0$ and decreasing for $y > y_0$, where $y_0 = y_0(k) = k^2/(r + 2)$. Hence

$$I_2 \leqq k^{r+1}\int_{k^{3/2}-1}^{k^2+1}\frac{1}{y^{r+2}}\exp\left(-\frac{k^2}{y}\right)dy + \frac{k^{r+1}}{y_0^{r+2}}\exp\left(-\frac{k^2}{y_0}\right) . \tag{6}$$

The second term on the right of (6) is a constant multiple of k^{-r-3}. The first does not exceed

$$k^{r+1}\int_0^\infty\frac{1}{y^{r+2}}\exp\left(-\frac{k^2}{y}\right)dy .$$

Putting $y = k^2/w$, this becomes

$$k^{-r-1}\int_0^\infty w^r e^{-w}dw = \Gamma(r + 1)k^{-r-1} ,$$

hence the result.

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Received May 28, 1971.

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