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BOUNDARY RESPECTING MAPS OF 3-MANIFOLDS

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This paper is about maps of compact 3-manifolds which map the boundary of the domain (possibly nonhomeomorphically) into the boundary of the range. F. Waldhausen has shown that such a map between compact, orientable, irreducible 3-manifolds with nonempty, incompressible boundary is homotopic to a homeomorphism if and only if the map induces an isomorphism at the fundamental group level. The main theorem of this paper states that the above theorem remains valid if the assumption of incompressible boundary is dropped.

A study of disk sums of bounded 3-manifolds will be required in order to prove the above-mentioned theorem. This investigation involves theorems about disk sums of bounded 3-manifolds analogous to the classical Kneser theorem for closed 3-manifolds.

The reader may wish to consult [11] for a proof of Waldhausen's theorem mentioned above and [4] and [7] for variations of Waldhausen's theorem related to the theorems proved in this paper.

All spaces and maps in this paper are assumed to belong to the precise linear category, and each subspace that we shall discuss is taken to be piecewise linearly embedded. If A is a subcomplex of the simplicial complex X, we use the notation $U(A, X)$ to denote a regular neighborhood of A in a second derived subdivision of X.

If X is a manifold, we use the notation $\operatorname{bd} X$ and $\operatorname{int} X$ to denote the boundary of X and the interior of X respectively.

A 3-manifold M is said to be *irreducible* if each 2-sphere in M is the boundary of some 3-cell in M.

A compact 2-manifold F embedded in a manifold M is *properly embedded* in M if $F \cap \operatorname{bd} M = \operatorname{bd} F$. A compact 2-manifold F properly embedded in a 3-manifold M is *incompressible* in M if for each disk D in M such that $D \cap F = \operatorname{bd} D$, there exists a disk D' in F such that $\operatorname{bd} D = \operatorname{bd} D'$.

Let F denote a 2-manifold properly embedded in a 3-manifold M, and let J denote a loop in M that meets F transversely. We define the symbol $[J, F]$ to be 0 if J meets F an even number of times, and $[J, F] = 1$ if J meets F an odd number of times. Observe that if J^* is a loop in M that is homotopic to J, then $[J, F] = [J^*, F]$.

Let M and N denote compact, orientable n-manifolds, and let $f: (M, \mathrm{bd}\, M) \rightarrow (N, \mathrm{bd}\, N)$ be a map. Having chosen generators α and β for $H_n(M, \mathrm{bd}\, M)$ respectively, we say that f has *degree* k if and only if $f_*(\alpha) = k\beta$. In general we shall not be concerned with the sign of the degree of a map. We note for future reference that if $f: (M, \mathrm{bd}\, M) \rightarrow (N, \mathrm{bd}\, N)$ has degree ± 1, then $f_*: \prod_1(M) \rightarrow \prod_1(N)$ is an epimorphism [2].

A compact 3-manifold M is a *cube-with-handles* if and only if M is homeomorphic with $U(\Gamma, R^3)$ where Γ is some connected graph embedded in R^3.

If α and β are loops in a manifold M based at a common point $*$, we use the notation $\alpha \sim \beta$ to indicate that α is homotopic to β relative to the base point $*$. Where there is no danger of confusion, we shall also use this notation to denote homotopic maps.

II. Disk sum decompositions. Kneser's theorem [10] for connected sums of closed 3-manifolds with fundamental group a free product is not in general true for disk sums of bounded 3-manifolds [5]. However as we shall shortly show, each bounded 3-manifold has a natural decomposition into disk sums determined by its fundamental group. This is made more precise in Theorem 2.4. I wish to thank William Jaco for collaborating with me on the proofs of the theorems of this section.

A group G is said to be *indecomposable* if G cannot be written as a nontrivial free product. Let G be a finitely generated group. If $A_1, \cdots, A_k$ are indecomposable groups which are not infinite cyclic, and if F is a free group such that $A_1 * \cdots * A_k * F$ is isomorphic with G, then $A_1 * \cdots * A_k * F$ is said to be a *free decomposition* for G. Each finitely generated group has a free decomposition which is unique up to isomorphism and order of factors. (see page 245 of [6].)

LEMMA 2.1. *Let M be a compact 3-manifold with nonempty boundary and suppose $\pi_1(M) \approx G*Z$ for some group G. Then there exists a properly embedded disk D in M which does not separate M. Futhermore $\pi_1(M - D) \approx G$.*

Proof. Let K denote a $C - W$ complex such that $\pi_1(K) \approx G$ and $\pi_i(K) = 0$ for $i > 1$. Let A denote a simple closed curve. Let $K \vee A$ denote the space obtained by attaching a point of A to a point of K. Then by Van Kampen's theorem, $\pi_1(K \vee A) \approx G*Z$. Observe that the universal cover of $K \vee A$ is contractible. Hence $\pi_i(K \vee A) = 0$ for each $i > 1$.

Since $\pi_1(K \vee A) \approx \pi_1(M)$ and $\pi_i(K \vee A) = 0$ for $i > 1$, we can construct a map $f: M \to K \vee A$ such that $f_*: \pi_1(M) \to \pi_1(K \vee A)$ is an isomorphism. Let x denote a point on $A - K$. We may assume that the map f is transverse with respect to x. Then $f^{-1}(x) = F_1 \cup \cdots \cup F_r$ consists of a collection of mutually exclusive, 2-sided, properly embedded 2-manifolds in M. We wish to change f by a homotopy in such a way that each component of $f^{-1}(x)$ becomes either a disk or a 2-sphere.

Let $i: F_j \to M$ denote the inclusion map. Then $f_* \circ i_*: \pi_1(F_j) \to \pi_1(K \vee A)$ is the trivial map. Thus $i_*: \pi_1(F_j) \to \pi_1(M)$ is also the zero homomorphism. If some F_i is not simply connected, then there is a loop l on F_i such that l is not contractible in F_i, but l does contract in M. By the usual surgery on a map that represents a contraction of l, we are able to find a loop l^* on F_j (possibly $i \neq j$) such that l^* does not contract in F_j, but * has a contraction in M that meets $\bigcup_{i=1}^r F_i$ only in l^*. Then since each F_i is 2-sided, we may apply the loop theorem [9] to $\mathrm{cl}\,(M - \bigcup_{i=1}^r U(F_i, M))$ to obtain a disk D in M such that $D \cap \bigcup_{i=1}^r F_i = D \cap F_j = \mathrm{bd}\, D$, and $\mathrm{bd}\, D$ is not contractible in F_j.

We consider $U(D, M)$ as a product $D' \times [0, 1]$ with $D \subset D' \times 1/2$. Observe that $F_j \cap U(D, M)$ is an annulus R. R separates $D' \times [0, 1]$ into two components, one of which we identify with $D \times [0, 1]$. Define $g: M \to K \vee A$ as follows. Set $g \equiv f$ on $\mathrm{cl}\,(M - D' \times [0, 1] \cup \mathrm{bd}\, D \times [0, 1/4] \cup \mathrm{bd}\, D \times [3/4, 1]$. Set $g(D \times 1/4 \cup D \times 3/4) = x$. Since $\pi_2(K \vee A) = 0$, we can extend g over $D \times [0, 1/4]$, $D \times [3/4, 1]$, and $\mathrm{cl}\,(D' \times [0, 1] - (D \times [0, 1/4] \cup D \times [3/4, 1]))$ in such a way that $g^{-1}(x) \cap D' \times [0, 1] = \mathrm{bd}\, D \times [0, 1/4] \cup D \times 1/4 \cup D \times 3/4 \cup \mathrm{bd}\, D \times [3/4, 1]$. Observe that g differs from f only on the interior of the 3-cell $D' \times [0, 1]$. Hence since $\pi_i(K \vee A) = 0$ for $i > 1$, it follows that g is homotopic to f.

By deformations of the above type, we can construct a map $h: M \to K \vee A$ such that $h_*: \pi_1(M) \to \pi_1(K \vee A)$ is an isomorphism and each component of $h^{-1}(x)$ is a simply connected 2-manifold.

Let l denote a simple closed curve in M such that $h(l)$ is homotopic to the simple closed curve A. Then $[h(l), x] = [A, x] = 1$. Hence $1 = [l, h^{-1}(x)] \equiv \sum_{i=1}^r [l, F_i] \pmod 2$. It follows that for some i, $[l, F_i] = 1$. (In particular, $h^{-1}(x) \neq \emptyset$.) If F_i is a disk, we set $D = F_i$. If F_i is a 2-sphere, let P be an arc joining F_i to $\mathrm{bd}\, M$. We then construct D from F_i by first removing the interior of the disk $U(P, M) \cap F_i$ from F_i and then adjoining the obvious annulus in $\mathrm{bd}\, U(P, M)$.

Then by Van Kampen's theorem, $\pi_1(M) \approx \pi_1(\mathrm{cl}\,(M - U(D, M))_* Z$. It follows from uniqueness of a free decomposition that $G \approx \pi_1(M - D)$.

Lemma 2.1 provides an easy proof of the following well known theorem.

COROLLARY 2.2: *Let M be a compact, orientable, irreducible 3-manifold. Then $\pi_1(M)$ is free if and only if M is a cube-with-handles.*

The proof of Corollary 2.2 is a straightfoward application of Lemma 2.1 and is omitted.

LEMMA 2.3: *Let M be a compact 3-manifold with nonempty boundary such that $\pi_1(M) \approx A_1 * A_2$ with $A_1 \neq 1 \neq A_2$. If each properly embedded disk in M separates M, then there exists a disk D in M that separates M into two nonsimply connected components.*

Proof. Construct $C - W$ complexes K_1, K_2 such that $\pi_1(K_i) \approx A_i$, $\pi_j(K_i) = 0$, $i = 1, 2$, $j > 1$. Let L be obtained from K_1 and K_2 by adjoining one end point of an arc A to K_1 and attaching the other end point to K_2. Let x denote the midpoint of A. By Van Kampen's theorem, $\pi_1(L) \approx \pi_1(K_1) * \pi_1(K_2)$. Since L is aspherical, there exists a map $f\colon M \to L$ such that $f_*\colon \pi_1(M) \to \pi_1(L)$ is an isomorphism.

Using the same techniques as in Lemma 2.1, we can change f by a series of homotopies so that $f^{-1}(x) = E_1, \cdots, E_r$ consists of properly embedded disks and 2-spheres. Since each properly embedded disk (and hence each 2-sphere) in M separates M, the 2-manifold E_1 separates M into two components whose closures we denote by M_1 and M_2. Suppose M_1 is simply connected. Let $M_2^* = \mathrm{cl}\,(M_2 - U(E_1, M))$, $M_1^* = M_1 \cup U(E_1, M)$. Define $g\colon M \to L$ as follows. Let $g \mid M_2^* = f \mid M_2^*$. We may assume that g is a level preserving map on $U(E_1, M)$ so that $g(M_1^* \cap M_2^*)$ is a single point p on A. Thus we may let $g(M_1^*) = p$.

Observe first of all that E_1 does not occur as a component of $g^{-1}(x)$. Secondly since $i_*\colon \pi_1(M_2^*) \to \pi_1(M)$ is an isomorphism, it follows from the following commutative diagram that $g_*\colon \pi_1(M) \to \pi_1(L)$ is an isomorphism.

$$\begin{array}{ccc} \pi_1(M_2^*) & \xrightarrow{i_*} & \pi_1(M) \\ \Big\downarrow{\scriptstyle i_*} & & {\scriptstyle g_*}\Big\downarrow \\ \pi_1(M) & \xrightarrow{f_*} & \pi_1(L)\,. \end{array}$$

Thus if each 2-manifold E_i $(1 \leqq i \leqq r)$ separates M into two components, one of which is simply connected, then altering f as above we can construct a map $h\colon M \to L$ such that $h_*\colon \pi_1(M) \to \pi_1(L)$ is an isomorphism and $h^{-1}(x) = \varnothing$. But then $h(M) \subset K_1$.

$$\begin{array}{ccc} \pi_1(M) & \xrightarrow{h_*} & \pi_1(K_1) \\ & {\scriptstyle h_*}\searrow & \Big\downarrow{\scriptstyle i_*} \\ & & \pi_1(L) \end{array}$$

Then from the above diagram we see that $i_*: \pi_1(K_1) \to \pi_1(L)$ is an epimorphism. It follows that $\pi_1(K_2) = 1$ contrary to the hypothesis of the lemma. Hence some component E_i separates M into two components neither of which is simply connected. If E_i is a 2-sphere, we change it to a properly embedded disk by the same technique used at the end of Lemma 2.1.

THEOREM 2.4. *Let M be a compact 3-manifold with nonempty boundary. Suppose $\pi_1(M) \approx A_1^* \cdots * A_k * F_r$ is a free decomposition for $\pi_1(M)$ where r is the rank of F_r. Then there exist disks $D_1, \cdots, D_{k+r-1}$ separating M into k components $M_1, \cdots, M_k$. And there exists an isomorphism $\psi: \pi_1(M) \to A_1 * \cdots * A_k * F_r$ such that $\psi i_* \pi_1(M_i) = A_j$ $(1 \leqq i \leqq k)$.*

Outline of proof. Let $G = \pi_1(M)$. If G has a free factor of Z, we write $G = G_1 * Z$. We apply Lemma 2.1 to find a disk D_1 such that $M^1 = \mathrm{cl}\,(M - U(D_1, M))$ and $\pi_1(M^1) \approx G_1$. If G_1 has a free factor of Z, we apply Lemma 2.1 again to construct M^2. Since M is compact, G is finitely generated. Hence Grushko's theorem (page 191 of [6]) assures us that this process must eventually yield a compact 3-manifold M^r such that each properly embedded disk in M^r separates M^r. If $\pi_1(M^r)$ is decomposable, we apply Lemma 2.3 to find a properly embedded disk that separates M^r into two nonsimply connected components M_1, M_2. We apply Lemma 2.3 to each of the components M_1, M_2. Again by Grushko's theorem this process must terminate.

We have a collection $D_1, \cdots, D_p$ of properly embedded disks in M separating into components $M_1, \cdots, M_q$ such that $\pi_1(M_i)$ is neither decomposable nor infinite cyclic for each $i (1 \leqq i \leqq q)$. By Van Kampen's theorem, $\pi_1(N) \approx \pi_1(M_1) * \cdots * \pi_1(M_q) {}_* F$ where F is a free group. The theorem now follows from uniqueness of the free decomposition for $\pi_1(M)$.

III. Boundary respecting maps.

THEOREM 3.1. *Let M and N denote compact, orientable, irreducible 3-manifolds with nonempty boundaries. Let $f(M, \mathrm{bd}\, M) \to (N, \mathrm{bd}\, N)$ be a degree 1 map such that $f_*: \pi_1(M) \to \pi_1(N)$ is a monomorphism. Then there exists a homotopy $h_t: (M, \mathrm{bd}\, M) \to (N, \mathrm{bd}\, N)$ such that $h_0 = f$ and h_1 is a homeomorphism.*

Proof. Since f is a degree 1 map, it follows that $f_*: \pi_1(M) \to \pi_1(N)$ is an epimorphism and hence is an isomorphism. By Lemma 2 of [7] it suffices to show that f is homotopic to map $g: (M, \mathrm{bd}\, M) \to (N, \mathrm{bd}\, N)$ such that $g|_{\mathrm{bd}\, M}: \mathrm{bd}\, M \to \mathrm{bd}\, N$ is a homeomorphism. We proceed to show this.

It follows from the sphere theorem [3] that M and N are aspherical manifolds. Thus, $f: M \to N$ is a homotopy equivalence. Since f is a degree 1 map, we have for each q that the following diagram commutes where λ is the usual Lefschetz duality isomorphism.

$$\begin{array}{ccc} H_q(M, \operatorname{bd} M) & \xrightarrow{\lambda} & H^{n-q}(M) \\ \downarrow{\scriptstyle f_*} & & \uparrow{\scriptstyle f^*} \\ H_q(N, \operatorname{bd} N) & \xrightarrow{\lambda} & H^{n-q}(N) \, . \end{array}$$

Hence $f_*: H_q(M, \operatorname{bd} M) \to H_q(N, \operatorname{bd} N)$ is an isomorphism for each q. Thus we may apply the five lemma to the following diagram to conclude that $f_*: H_q(\operatorname{bd} M) \to H_q(\operatorname{bd} N)$ is an isomorphism for each q.

$$\begin{array}{ccccccccc} H_{q+1}(M) & \to & H_{q+1}(M, \operatorname{bd} M) & \to & H_q(\operatorname{bd} M) & \to & H_q(M) & \to & H_q(M, \operatorname{bd} M) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ H_{q+1}(N) & \to & H_{q+1}(N, \operatorname{bd} N) & \to & H_q(\operatorname{bd} N) & \to & H_q(N) & \to & H_q(N, \operatorname{bd} N) \, . \end{array}$$

In particular since each boundary component of M and N is orientable and $f_*: H_2(\operatorname{bd} M) \to H_2(\operatorname{bd} N)$ is an isomorphism, it follows that each boundary component of N has a unique boundary component of M in its preimage.

Let $X_1, \cdots, X_r, Y_1, \cdots, Y_r$ denote the boundary components of M and N respectively; assume notation has been chosen so that $f(X_i) \subset Y_i$ $(1 \leqq i \leqq r)$. Put $f_i = f|_{W_i}$. It follows from the above remarks that f_i is a degree 1 map for each i $(1 \leqq i \leqq r)$. Thus $f_{i_*}: \pi_1(X_i) \to \pi_1(Y_i)$ is an epimorphism. Furthermore

$$f_*: H_1(X_1) \oplus \cdots \oplus H_1(X_r) \to H_1(Y_1) \oplus \cdots \oplus H_1(Y_r)$$

is an isomorphism with $f_*(H_1(X_i)) \subset H_1(Y_i)$ for each i $(1 \leqq i \leqq r)$. It follows that $f_{i_*}: H_1(X_i) \to H_1(Y_i)$ is an isomorphism for each i $(1 \leqq i \leqq r)$. Since the rank of the first homology of a closed orientable 2-manifold determines the genus of the 2-manifold, it follows that X_i is homeomorphic to Y_i for each i $(1 \leqq i \leqq r)$. Then since fundamental group of a closed 2-manifold is necessarily Hopfian [1], it follows that $f_{i_*}: \pi_1(X_i) \to \pi_1(Y_i)$ is an isomorphism for each i $(1 \leqq i \leqq r)$. Neilsen's theorem [8] now assures us that $f_i: X_i \to Y_i$ is in fact homotopic to a homeomorphism.

Thus we are able to construct a map $g: (M, \operatorname{bd} M) \to (N, \operatorname{bd} N)$ such that g is homotopic to f, and $g|_{\operatorname{bd} M}: \operatorname{bd} M \to \operatorname{bd} N$ is a homeomorphism. This completes the proof of Theorem 3.1.

THEOREM 3.2. *Let M and N denote compact, orientable, irredu-*

cible 3-manifolds with nonempty boundary. Let $f: (M, \mathrm{bd}\, M) \to (N, \mathrm{bd} N)$ *be a map such that* $f_*: \pi_1(M) \to \pi_1(N)$ *is an isomorphism. Then f is homotopic to a homeomorphism. Furthermore, if M is not the product of a compact 2-manifold with the unit interval, the homotopy above can be chosen so that it maps* $\mathrm{bd}\, M$ *into* $\mathrm{bd} N$.

Proof. Suppose N is a cube-with-handles. Then since the fundamental group of M is free of the same rank as $\pi_1(N)$, it follows that M is also a cube-with-handles, and there exists a homeomorphism $g: N \to M$. The map gf then induces an automorphism α on the fundamental group of M. Then applying a theorem of Zieschang [12], there exists a homeomorphism $h: M \to M$ such that $h_* = \alpha$. Then the maps f and $g^{-1}h$ induce identical isomorphisms from the fundamental group of M onto the fundamental group of N. Then since N is an aspherical manifold, if follows that f is homotopic to the homeomorphism $g^{-1}h$. This completes the proof of Theorem 3.2 when N is a cube-with-handles.

We continue the proof assuming N is not a cube-with-handles. Let $\pi_1(N) \approx B_1 * \cdots * B_n * F$ denote a free decomposition for $\pi_1(N)$. Let $D_1, \cdots, D_d$ denote the collection of properly embedded disks in N whose existence is guaranteed by Theorem 2.4. Let

$$\mathrm{cl}\left(N - \bigcup_{j=1}^{d} U(D_j, N)\right) = N_1, \cdots, N_n ,$$

and let $\psi: \pi_1(N) \to B_1 * \cdots * B_n * F$ be an isomorphism such that

$$\psi i_*(\pi_1(N_j)) = B_j \quad (1 \leqq j \leqq n) .$$

Since N is not a cube-with-handles, we have the $n \geqq 1$ so that no component N_j $(1 \leqq j \leqq n)$ is simply connected.

Let $f^{-1}(\bigcup_{j=1}^{d} D_j) = \bigcup_{i=1}^{e} E_i$ where for each i, E_i is a properly embedded 2-manifold in M. As a consequence of the sphere theorem [3], both M and N are aspherical manifolds. Hence applying the techniques of §2, we may assume that each component E_i is simply connected. Furthermore, if E_i is a 2-sphere, then E_i bounds a 3-cell R in M. It is easy then to change f on a regular neighborhood of R so that E_i no longer occur as a component of $f^{-1}(\bigcup_{j=1}^{d} D_j)$. Thus we may assume that each component E_i $(1 \leqq i \leqq e)$ is a disk properly embedded in M.

We identify distinguished regular neighborhoods $U(E_i, M)$ and $U(D_j, N)$ with $E_i \times [0, 1]$ and $D_j \times [0, 1]$ respectively. We may assume that these neighborhoods have been chosen so that

$$f^{-1}\left(\bigcup_{j=1}^{d} U(D_j, N)\right) = \bigcup_{i=1}^{e} U(E_i, M) ,$$

and that if $f(E_i) \subset D_j$, then $f|_{E_i \times [0,1]}: E_i \times [0, 1] \to D_j \times [0, 1]$ is a level preserving map.

Let $\mathrm{cl}(M - \bigcup_{i=1}^{e} U(E_i, M)) = \bigcup_{i=1}^{m} M_i$. Let x (resp. y) denote the base point of M (resp. N), and let x_i (resp. y_j) denote the base point of M_i $(1 \leqq i \leqq m)$ (resp. N_j $(1 \leqq j \leqq n)$). We choose the base points so that $f(x) = y$, and $f(\{x_i\}) = \{y_j\}$. Let α_i (resp. β_j) denote arcs in M (resp. N) joining x_i to x (resp. y_j to y), and let the inclusion maps at the fundamental group level be defined along these arcs.

Finally, we assume that f $(\mathrm{int}\, M) \subset \mathrm{int}\,(N)$, and we put $f_i = f|_{M_i}$ $(1 \leqq i \leqq m)$.

By Van Kampen's theorem, there is an isomorphism $\varphi: \pi_1(M) \to A_1 * \cdots * A_m * F'$ such that $\varphi i_* \pi_1(M_i) = A_i$ $(1 \leqq i \leqq m)$. Note that some of the groups A_i may be trivial.

LEMMA A. *Let N_j be a component of* $\mathrm{cl}\,(N - \bigcup_{j=1}^{d} U(D_i, N))$. *Then there exists a unique component M_i of* $\mathrm{cl}\,(M - \bigcup_{i=1}^{e} U(E_i, M))$ *such that $f_{i_*}: \pi_1(M_i) \to \pi_1(N_j)$ is an isomorphism. If M_p is any other component of* $\mathrm{cl}\, M - \bigcup_{i=1}^{e} U(E_i, M))$ *such that $f(M_p) \subset N_j$, then M_p is a 3-cell.*

Proof. Observe first of all that as a consequence of Van Kampen's theorem, the inclusion induced homomorphisms $i_*: \pi_1(M_i) \to \pi_1(M)$ and $i_*: \pi_1(N_j) \to \pi_1(N)$ are monomorphisms. For each i such that $f(M_i) \subset N_j$ we have a commutative diagram.

$$\begin{array}{ccc} \pi_1(M_i) & \xrightarrow{f_{i_*}} & \pi_1(N_j) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ \pi_1(M) & \xrightarrow{f_*} & \pi_1(N) \end{array} .$$

We conclude from this diagram that $f_{i_*}: \pi_1(M_i) \to \pi_1(N_j)$ is a monomorphism.

Let $\lambda: A_1 * \cdots * A_m * F' \to B_1 * \cdots * B_n * F$ be the isomorphism defined by the composition $\psi f_* \varphi^{-1}$. Consider the factor $B_j = \psi i_*(\pi_1(N_j))$ of $B_1 * \cdots * B_n * F$. The group $B_1 * \cdots * B_n * F$ can be written as a free product $\lambda(A_1) * \cdots * \lambda(A_m) * \lambda(F')$. By the Kurosh subgroup theorem (Corollary 4.9.1 of [6]), B_j is a free product of conjugates of subgroups of $\lambda(A_1), \cdots, \lambda(A_m)$ and a free group. But B_j is neither decomposable nor free. Thus for some element t, and for some i, $t^{-1}B_j t < \lambda(A_i)$. Since $B_j \neq 1$, it follows that $A_i \neq 1$.

Let z denote an element of A_i. Choose a loop $\bar{z}$ in M_i such that $\varphi i_*(\bar{z}) = z$. Let $f(M_i) \subset N_q$.

$$\lambda(z) = \psi f_*(\alpha_i^{-1} \bar{z} \alpha_i)$$
$$\lambda(z) = \psi(f(\alpha_i^{-1}) f(\bar{z}) f(\alpha_i))$$

$$\lambda(z) = \psi(f(\alpha_i^{-1})\beta_q i_* f(\bar{z})\beta_q^{-1} f(\alpha_i))$$
$$\lambda(z) = ghg^{-1}$$

where $g = \psi(f(\alpha_i^{-1})\beta_q)$ and $h = \psi i_* f(\bar{z}) \in B_q$. Thus $g^{-1}\lambda(A_i)g < B_q$. Hence $g^{-1}t^{-1}B_j tg < g^{-1}\lambda(A_i)g < B_q$. But no nontrivial factor of a free product can be conjugate to a subgroup of any other factor. It follows that $j = q$, $tg \in Bj$, and $g^{-1}\lambda(A_i)g = B_j$. Recalling that $g = \psi(f(\alpha_i)B_j^{-1})$, it is now straightforward to show that $f_{i_*}: \pi_1(M_i) \to \pi_1(N_j)$ is an epimorphism and hence an isomorphism.

Finally if $f(M_p) \subset N_j$, and if A_p is not the trivial group, then the above argument applied to A_p yields $C^{-1}\lambda(A_p)C = B_j = g^{-1}\lambda(A_i)g$ (for some C). Then $gC^{-1}\lambda(A_p)Cg^{-1} = \lambda(A_i)$. It follows that $i = p$. Hence if $f(M_p) \subset N_j$ and if $p \neq i$, then $\pi_1(M_p)$ is the trivial group. Furthermore, since each component of $\mathrm{cl}\,(M - \bigcup_{i=1}^{e} U(E_i, M))$ is irreducible, it follows that M_p is a 3-cell. This completes the proof of Lemma A.

It is our aim to show that f is a degree 1 map except possibly in the case that M is homeomorphic with the product of a closed 2-manifold with the unit interval. (In which case Waldhausen's theorems [11] will be applicable to complete the proof of Theorem 3.2.) Theorem 3.1 will then apply to prove Theorem 3.2. A study of the map $f_*: H_2(\mathrm{bd}\, M) \to H_2(\mathrm{bd}\, N)$ seems to be the only accessible route to this end. As will become clear toward the end of the proof, the crucial obstruction to gaining the information we need about

$$f_*: H_2(\mathrm{bd}\, M) \to H_2(\mathrm{bd}\, N)$$

is the possibility that distinct boundary components of some component M_i of $\mathrm{cl}\,(M - \bigcup_{i=1}^{e} U(E_i, M))$ may be mapped under f into a single boundary component of some component N_j of $\mathrm{cl}\,(N - \bigcup_{j=1}^{d} U(D_j, N))$. The succeeding three lemmas (particularly Lemma D) are concerned with eliminating this possibility. They constitute the most important (and easily the most difficult) steps in the proof of Theorem 3.2.

A component M_i of $\mathrm{cl}\,(M - \bigcup_{i=1}^{e} U(E_i, M))$ is *f-singular* if there exist components X_1 and X_2 of $\mathrm{bd}\, M_i$ such that $f(X_1) \cup f(X_2)$ is contained in a single boundary component of a component N_j of $\mathrm{cl}\,(N - \bigcup_{j=1}^{d} U(D_j, N))$. Consider an arbitrary component M_i of $\mathrm{cl}\,(M - \bigcup_{i=1}^{e} U(E_i, M))$, and let $f(M_i) \subset N_j$. If M_i is f-singular then M_i has disconnected boundary and so cannot be a 3-cell. Thus according to Lemma A, $f_{i_*}: \pi_1(M_i) \to \pi_1(N_j)$ is an isomorphism. Furthermore, since $\pi_1(N_j)$ is neither infinite cyclic nor a nontrivial free product, it follows that M_i and N_j each have incompressible boundary. Hence we may apply the theorems of Waldhausen [11] to conclude that $f_i: M_i \to N_j$ is homotopic to a homeomorphism. Further, if M_i is not the product of a closed 2-manifold with the unit interval, then the homotopy can be taken to respect the boundary of M_i and N_j. It follows that if M_i is an f-singular component of $\mathrm{cl}\,(M - \bigcup_{i=1}^{e}(E_i, M))$ such that $f(M_i) \subset N_j$, then M_i and N_j are each homeomorphic with the product of a closed 2-manifold with the unit interval.

We require one further definition before beginning a more serious analysis of f-singular components of $\mathrm{cl}\,(M - U_{i=1}^{e}(E_i, M))$.

We shall think of loops and arcs as maps with $[0, 1]$ as domain. A loop Θ in M is *f-closed* if Θ can be divided into arcs $\Theta = \sigma_1\tau_1 \cdots \sigma_k\tau_k$ with the following properties:

(i) Each σ_i is an arc in $\bigcup_{i=1}^{e} U(E_i, M)$.

(ii) Each τ_i is an arc in $\mathrm{cl}\,(M - \bigcup_{i=1}^{e} U(E_i, M))$.

(iii) For each i, $f\circ\tau_i$ is a loop in N.

LEMMA B. *Let $M_1 = G \times [0, 1]$ be an f-singular component of $\mathrm{cl}\,(M - U_{i=1}^{e} U(E_i, M))$ with $f(M_1) \subset N_1 = H \times [0, 1]$ where G and H are homeomorphic 2-manifolds. If $f(G \times \{0, 1\}) \subset H \times 0$, then $f^{-1}(H \times 1) = \phi$.*

Proof. First of all we note that since $f(\mathrm{int}\, M) \subset \mathrm{int}\, N$, we have that $f^{-1}(H \times 1) \subset \bigcup_{i=1}^{m} \mathrm{bd}\, M_i$. Suppose w is a point in $\bigcup_{i=1}^{m} \mathrm{bd}\, M_i$ such that $f(w) \in H \times 1$. Let $f^{-1}(N_1) = M_1, M_{i_1}, \cdots, M_{i_l}$ where $M_{i_1}, \cdots, M_{i_l}$ are 3-cells. Define a map $g\colon M \to N$ as follows. Let g and f coincide outside $\mathrm{int}\,(M_1 \cup M_{i_l} \cdots \cup M_{i_l})$. Define g on M_1 so that $g|_{M_1}$ is homotopic relative to the boundary of M_1 to $f|_{M_1}$, and $g(M_1) \subset H \times 0$. On each 3-cell M_{i_j} $(1 \leqq j \leqq l)$ let g be a contraction of $f|_{\mathrm{bd} M_{i_j}}$ in $\mathrm{bd}\, N_1$. Observe that $g^{-1}(H \times 1/2) = \phi$. Since N is aspherical, it is easy to see that g is homotopic to f relative to $\bigcup_{i=1}^{m} \mathrm{bd}\, M_i$. Thus $g_*\colon \pi_1(M) \to \pi_1(N)$ is an isomorphism.

Let γ be an arc in M joining w to a point in $G \times 0$. Then $g\circ\gamma(0) \in H \times 1$, $g\circ\gamma(1) \in H \times 0$, and $g\circ\gamma([0, 1]) \cap H \times 1/2 = \phi$. Let γ^* be an arc in $H \times [0, 1]$ joining $g\circ\gamma(1)$ with $g\circ\gamma(0)$, and let μ be a loop in M such that $g\circ\mu \sim (g\circ\gamma)\gamma^*$. Then since $g^{-1}(H \times 1/2) = \phi$, we have that $0 = [g(\mu), H \times 1/2] = [(g\circ\gamma)\gamma^*, H \times 1/2] = 1$. This contradiction completes the proof of Lemma B.

LEMMA C. *If Θ is an f-closed loop in M, then $[\Theta, E_i] = 0$ for each i $(1 \leqq i \leqq e)$.*

Proof. Let $\Theta = \sigma_1\tau_1 \cdots \sigma_k\tau_k$ where the arcs σ_i and τ_i satisfy the conditions (i), (ii), (iii) in the definition of an *f-closed* loop. Observe that since each $f\circ\tau_i$ is a loop in N, there are at most two components N_1, N_2 of $\mathrm{cl}\,(N - \bigcup_{j=1}^{d} U(D_j, N))$ and a single disk D_1 such that $f\circ\Theta \subset$

$N_1 \cup N_2 \cup D_1 \times [0, 1]$. (Possibly $N_1 = N_2$.) Let M_1 and M_2 be the unique components of $\mathrm{cl}\,(M - \bigcup_{i=1}^{e} U(E_i, M))$ such that $f_{i_*}\colon \pi_1(M_i) \to \pi_1(N_i)$ $(i = 1, 2)$ is an isomorphism.

Observe that as a consequence of Lemma B and the theorems of Waldhausen [11], it is the case that for each r such that $f(\tau_r(0)) \in N_1$ (resp. $f(\tau_r(0)) \in N_2$), there is a point v_r in M_1 (resp. M_2) such that $f(v_r) = f(\tau_r(0))$ $(1 \leqq r \leqq k)$. (If M_1 is f-singular, then by Lemma B $f(\mathrm{bd}\,M_1)$ and $f(\tau_r(0))$ must lie in the same component of $\mathrm{bd}\,N_1$. If M_1 is not f-singular, then Waldhausen's theorems apply.)

Let Θ_r be an arc in M joining $\tau_r(0)$ to v_r $(1 \leqq r \leqq k)$. Then $f \circ \Theta_r$ is a loop in N. Since $f_*\colon \pi_1(M) \to \pi_1(N)$ is an isomorphism, we may take $f \circ \Theta_r$ to be a contradictible loop. (If necessary replace Θ_r by $\Theta_r \Theta_r^*$ where Θ_r^* is a loop in M based at v_r such that $(f \circ \Theta_r)^{-1} \sim f \circ \Theta_r^*$.) For each s $(1 \leqq s \leqq k)$ such that $f \circ \tau_s \subset N_1$ (resp. N_2) let τ_s^* be a loop in M_1 (resp. M_2) based at v_s such that $f \circ \tau_s^* \sim (f \circ \tau_s)^{-1}$.

Consider the loop $\Theta^* = \sigma_1(\Theta_1 \tau_1^* \Theta_1^{-1} \tau_1) \cdots \sigma_k(\Theta_k \tau_k^* \Theta_k^{-1} \tau_k)$. Observe first of all that since $[\Theta_r \tau_r^* \Theta_r^{-1}, E_i] = 0$ for each r and i $(1 \leqq r \leqq k, 1 \leqq i \leqq e)$, it follows that $[\Theta^*, E_i] = [\Theta, E_i]$ for each i $(1 \leqq i \leqq e)$. Also, $f \circ (\Theta_r \tau_r^* \Theta_1^{-1} \tau_r)$ is a contractible loop in N for each r $(1 \leqq r \leqq k)$. Hence $f \circ \Theta^* \sim (f \circ \sigma_1)(f \circ \sigma_2) \cdots (f \circ \sigma_k)$. Since $(f \circ \sigma_k)(f \circ \sigma_2) \cdots (f \circ \sigma_k)$ is contained in the 3-cell $U(D_1, N)$, it follows that $f \circ \Theta^* \sim 1$. Thus $\Theta^* \sim 1$. Hence $0 = [\Theta^*, E_i] = [\Theta, E_i]$ for each i $(1 \leqq i \leqq e)$. This completes the proof of Lemma C.

Lemma D. *If M is not the product of a closed 2-manifold with the unit interval, then no component of* $\mathrm{cl}\,(M - \bigcup_{i=1}^{e} U(E_i, M))$ *is f-singular.*

Proof. Suppose M_1 is an f-singular component of

$$\mathrm{cl}\left(M - \bigcup_{i=1}^{e} U(E_i, M)\right),$$

and suppose notation has been chosen so that $f(M_1) \subset N_1$. Let $M_1 = G \times [0, 1]$, $N_1 = H \times [0, 1]$, and $f(G \times \{0, 1\}) \subset H \times 0$ where G and H are homeomorphic 2-manifolds.

If $\bigcup_{j=1}^{d} U(D_j, N) \cap H \times 0 = \phi$, then $\bigcup_{i=1}^{e} U(E_i, M) \cap M_1 = \phi$. Since M is connected and $f^{-1}(D_j) \neq \phi$ for any j $(1 \leqq j \leqq d)$, it would follow that $e = d = 0$, and $M = M_1$ completing the proof of the lemma.

Thus, we assume that there is at least one disk D_1 such that $D_1 \times 0 \subset H \times 0$. This together with the assumption that M_1 is f-singular shall eventually lead us to a contradiction.

If $f(E_i) \subset D_j$, then $f|_{\mathrm{bd}\,E_i}$ induces a homeomorphism $f_*^i\colon \pi_1(\mathrm{bd}\,E_i) \to \pi_1(\mathrm{bd}\,D_j)$. We say that f is *nondegenerate* at E_i if f_*^i is not the trivial homeomorphism. Observe that if f were degenerate at each component of $f^{-1}(D_1 \times 0) \cap G \times 0$, then we could construct a map $g: G \times 0 \to H \times 0$ such that g is homotopic to $f|_{G\times 0}$ and $g^{-1}(\operatorname{int} D_1 \times 0) = \phi$. This is not possible since $f|_{G\times 0}$ is homotopic to a homeomorphism from $G \times 0$ onto $H \times 0$. Thus there exists a disk $E_2 \times 0$ in $G \times 0$ and similarly there exists a disk $E_3 \times 0$ in $G \times 1$ such that $f(E_2 \times 0) = f(E_3 \times 0) = D_1 \times 0$, and f is nondegenerate at E_2 and E_3. Let $D_1 \times 1 \subset N_2$, $E_2 \times 1 \subset M_2^1$, $E_3 \times 1 \subset M_3^1$ (possibly $N_2 = N_1$ in which case it may also occur that $M_2^1 = M_1$ or $M_3^1 = M_1$).

We shall now begin construction of a somewhat complicated (in the sense that there are numerous technicalities involved in its construction) f-closed loop Θ in M which will enable us to reach a contradiction to the assumption that M_1 is f-singular and $M \neq M_1$. The loop Θ is constructed from two basic arcs Θ_2 with $\Theta_2(0) \in E_2 \times 0$ and Θ_3 with $\Theta_3(0) \in E_3 \times 0$.

Specifically, we wish Θ_2 to be divisible into arcs $\Theta_2 = \sigma_0\tau_1\sigma_1 \cdots \tau_k\sigma_k$ (possibly $k = 0$) with the following properties. (For notational convenience, put $E_2^0 = E_2$.)

(1) Each σ_i is an arc in $E_2^i \times [0, 1]$ such that the endpoints of σ_i lie in distinct components of $E_2^i \times \{0, 1\}$.

(2) If $i \neq j$, then $E_2^i \neq E_2^j$.

(3) Each τ_i is an arc in a component M_2^i of $\operatorname{cl}(M - \bigcup_{i=1}^{e} U(E_i, M))$, and $f \circ \tau_i$ is a loop in N.

(4) Each M_2^i $(1 \leqq i \leqq k)$ is a 3-cell.

(5) If $M_2^i = M_2^j$, then either $|i - j|$ is odd or $i = j$.

(6) f is nondegenerate at each disk E_2^i and $f(E_2^i) \subset D_1$.

(7) $\Theta_2(1)$ lies in a component M_2^{k+1} of $\operatorname{cl}(M - \bigcup_{i=1}^{e} U(E_i, M))$, and M_2^{k+1} is not a 3-cell.

(8) $f(\Theta_2(1)) \in D_1 \times 1$.

Suppose we have constructed the arcs Θ_2 with properties (1) through (8) above and Θ_3 with similar properties except that $\Theta_3(0) \in E_3 \times 0$. Then the proof of Lemma D can be completed as follows. Let $\Theta_3(1)$ lie in M_3^{l+1} where M_3^{l+1} is not a 3-cell. Since $f \circ \Theta_3(1) \cup f \circ \Theta_2(1) \subset D_1 \times 1$, $f \circ \Theta_3(0) \cup f \circ \Theta_2(0) \cup f \circ \Theta_2(0) \subset D_1 \times 0$, and f is nondegenerate at the disks E_2^0, E_2^k, E_3^0, E_3^l we can move the endpoints of Θ_2 and Θ_3 slightly so that $f \circ \Theta_2(0) = f \circ \Theta_3(0)$ and $f(\Theta_2(1)) = f(\Theta_3(1))$.

Since $f(\Theta_2(1)) \in D_1 \times 1 \subset N_2$, $f(\Theta_3(1)) \in D_1 \times 1 \subset N_2$, $\Theta_2(1) \in M_2^{k+1}$, and $\Theta_3(1) \in M_3^{l+1}$, it follows that $f(M_2^{k+1}) \cup f(M_3^{l+1}) \subset N_2$. But neither M_2^{k+1} nor M_3^{l+1} is a 3-cell. It follows from Lemma A that $M_2^{k+1} = M_2^{l+1}$.

Let τ_0 be an arc in M_1 joining $\Theta_3(0)$ with $\Theta_2(0)$, and let τ_* be an arc in M_2^{k+1} joining $\Theta_2(1)$ with $\Theta_3(1)$ (possibly τ_* is a loop). Then $\Theta = \Theta_2\tau^*\Theta_3^{-1}\tau_0$ is an f-closed loop in M. It follows from (2) that Θ_2 meets E_2 exactly once, and it follows from (4) and (8) that Θ_3 does not meet E_2. Hence Θ is an f-closed loop in M such that $[\Theta, E_2] = 1$. This is not consistent with the conclusion of Lemma C.

Thus, in order to prove Lemma D, we need only show that Θ_2 can be constructed with the properties (1) through (8) above. (The construction of Θ_3 will be similar to that of Θ_2.)

Let $\mathscr{A}$ denote the collection of all arcs μ in M such that μ can be divided into arcs $\mu = \sigma_0\tau_1\sigma_1 \cdots \tau_s\sigma_s$ satisfying properties (1) through (6) above. If $\mu \in \mathscr{A}$ and $\mu = \sigma_0\tau_1\sigma_1 \cdots \tau_s\sigma_s$, we shall say that μ has length s.

The set $\mathscr{A}$ is not empty since the arc σ_0 (of length zero) in $E_2 \times [0, 1]$ that joins $E_2 \times 0$ with $E_2 \times 1$ is a member of $\mathscr{A}$. Note also that as a consequence of (2), each element of $\mathscr{A}$ has length not greater than e. Thus there is an element of maximal length in $\mathscr{A}$. We choose $\Theta_2 = \sigma_0\tau_1\sigma_1 \cdots \tau_k\sigma_k$ to be an element of maximal length in $\mathscr{A}$. In order to complete the proof of Lemma D, we must show that Θ_2 satisfies properties (7) and (8).

Suppose Θ_2 does not satisfy (7); that is, suppose M_2^{k+1} is a 3-cell. We have two cases to consider; each of which shall lead to a contradiction.

Case 1. There exists an i such that $M_2^{k+1} = M_2^i$ and $|k + 1 - i|$ is even.

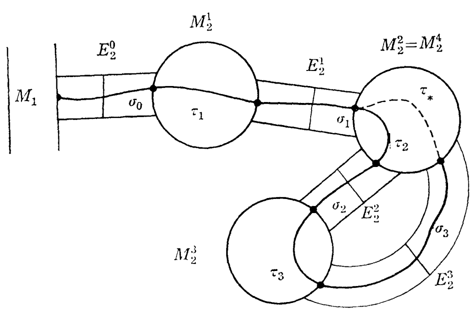

Figure 1

Observe that since $f|_{E_i \times [0,1]}: E \times [0, 1] \rightarrow D_j \times [0, 1]$ is a level preserving map for each i and j such that $f(E_i) \subset D_j$, it follows from (1) and (3) above that $f(\sigma_i(1))$ and $f(\sigma_j(1))$ lie in the same component of $D_1 \times \{0, 1\}$ if and only if $|i - j|$ is even.

Since $|k + 1 - i|$ is even, $f(\sigma_j(1))$ and $f(\sigma_{i-1}(1))$ lie in the same component of $D_1 \times \{0, 1\}$. Thus we can move the endpoint of σ_k so that $f \circ \sigma_k(1) = f \circ \sigma_{i-1}(1)$. Let τ_* be an arc in M_2^i joining $\sigma_k(1)$ with $\sigma_{i-1}(1)$. Then the loop $\Theta^* = \sigma_0\tau_1\sigma_1 \cdots \tau_k\sigma_k\tau_*\sigma_{i-1}^{-1}\tau_{i-1}^{-1} \cdots \tau_1^{-1}\sigma_0^{-1}$ is an f-closed loop. But $[\Theta^*, E_2^k] = 1$ contrary to the conclusion of Lemma C.

Case 2. If $M_2^{k+1} = M_2^i$ for any i, then $|k + 1 - i|$ is odd.

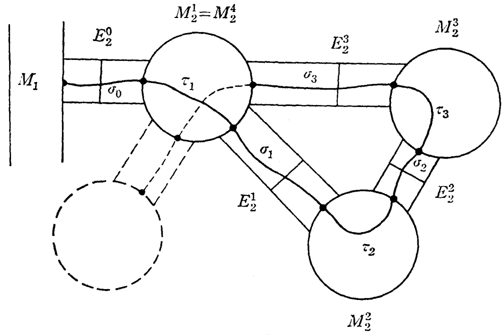

Figure 2

With no loss of generality in discussing this case, we assume that $f(\sigma_k(1)) \in D_1 \times 0$. There is a map g: $(\mathrm{bd}\, M_2^{k+1}) - \mathrm{int}\, E_2^k \times 0 \rightarrow \mathrm{bd}\, N_1$ such that $g|_{\mathrm{bd}E_2^k \times 0} = f|_{\mathrm{bd}E_2^k \times 0}$ and $g^{-1}(D_1 \times 0)$ consists of $\mathrm{bd}\, E_2^k \times 0$ and disks $E_i \times 0$ ($E_i \neq E_2^k$) on which f is nondegenerate. If $g^{-1}(D_1 \times 0) = \mathrm{bd}\, E_2^k \times 0$, then g is a contraction of $f|_{\mathrm{bd}E_2^k \times 0}$ in $\mathrm{bd}\, N_2 - \mathrm{int}\, D_1 \times 0$ contrary to the fact that f is nondegenerate at E_2^k. Thus there exists a disk $E_2^{k+1} \times 0$ in $\mathrm{bd}\, M_2^{k+1}$ such that $E_2^{k+1} \neq E_2^k$, and f is nondegenerate at E_2^{k+1}.

Let τ_{k+1} be an arc in M_2^{k+1} joining the endpoint of σ_k with a point in $E_2^{k+1} \times 0$ such that $f \circ \tau_{k+1}$ is a loop in N. Let σ_{k+1} be an arc in $E_2^{k+1} \times [0, 1]$ joining $\tau_{k+1}(1)$ with a point in $E_2^{k+1} \times 1$. We shall show that $\Theta_2^* = \sigma_0 \tau_1 \sigma_1 \cdots \tau_{k+1} \sigma_{k+1}$ satisfies the conditions (1) through (6) contradicting the maximality of Θ_2. The only nontrivial verification required is that (2) holds for Θ_2^*. If $M_2^{k+1} \neq M_2^i$ for each $i < k + 1$, then (2) follows immediately. Thus suppose $M_2^{k+1} = M_2^i$, $i < k + 1$. Since $k + 1 - i$ is odd, the disks $E_2^{i-1} \times 1, E_2^i \times 1, E_2^k \times 0, E_2^{k+1} \times 0$ meet M_2^i. No other disks in the sequence $E_2^0 \times 0, E_2^0 \times 1, \cdots, E_2^k \times 0, E_2^k \times 1$ meet M_2^i. For suppose $E_2^{j-1} \times \varepsilon$ and $E_2^j \times \varepsilon$ meet M_2^i where ε is either 0 or 1 and $i \neq j \neq k + 1$. Then $M_2^i = M_2^j$. If $\varepsilon = 1$, then since $f(\sigma_{i-1}(1))$ and $f(\sigma_{j-1}(1))$ lie in the same component of $D_1 \times \{0, 1\}$, it follows that $|i - j|$ is even. But this is inconsistent with property (5) for Θ_2. On the other hand, if $\varepsilon = 0$, then arguing just as above $M_2^j = M_2^{k+1}$ and $|k + 1 - j|$ is even contrary to our assumptions in the case we are considering. This proves property (2) for Θ_2^* and so contradicts the maximality of Θ_2. We conclude that M_2^{k+1} is not a 3-cell.

Property (8) remains. Suppose $f(\Theta_2(1)) \in D_1 \times 0 \subset N_1$. Then

$$f(M_2^{k+1}) \subset N_1 ,$$

and it follows from Lemma A that $M_2^{k+1} = M_1$. We can move $\Theta_2(1)$ slightly so that $f(\Theta_2(0) = f(\Theta_2(1))$. Let τ_* denote an arc in M_1 joining $\Theta_2(1)$ with $\Theta_2(0)$. Then $\tau_* \Theta_2$ is an f-closed loop with $[\tau_* \Theta_2, E_2] = 1$.

It follows that $f \circ \Theta_2(1) \in D_1 \times 1$.

Thus Θ_2 satisfies the conditions (1) through (8) above, and as already noted this is sufficient to prove Lemma D.

We are now prepared to complete the proof of Theorem 3.2 by showing that if M is not the product of a compact 2-manifold with the unit interval, then f is a degree 1 map. To this end, let Y denote a boundary component of N. Let Δ denote a 2-simplex in $\mathrm{cl}\,(Y - \bigcup_{j=1}^{d} U(D_j, N))$. Then Δ lies in a boundary component Y_0 of a component N_j of $\mathrm{cl}\,(N - \bigcup_{j=1}^{d} U(D_j, N))$. By Lemma A, $f^{-1}(N_j) = M_{i_0}, M_{i_1}, \cdots, M_{i_s}$ where $f_{i_0*}: \pi_1(M_{i_0}) \to \pi_1(N_j)$ is an isomorphism and $M_{i_1}, \cdots, M_{i_s}$ are all 3-cells. By Lemma D, there is a unique component X_0 of $\mathrm{bd}\, M_{i_0}$ such that $f(X_0) \subset Y_0$.

Observe that $f^{-1}(\Delta)$ consists of a disjoint collection of simplices $\Delta_0^1, \cdots, \Delta_0^{t_0}, \Delta_1^1, \cdots, \Delta_1^{t_1}, \cdots, \Delta_s^1, \cdots, \Delta_s^{t_s}$ in $\mathrm{bd}\, M$ where $\Delta_r^1, \cdots, \Delta_r^{t_r} \subset \mathrm{bd}\, M_{i_r}$ $(0 \leqq r \leqq s)$. For a 2-simplex σ in a 2-manifold F, we use the notation $*\sigma$ to denote the generator of the infinite cyclic summand of the simplicial chain group of F associated with σ. Observe that as a consequence of Waldhausen's theorems [10], $f|_{X_0}: X_0 \to Y_0$ is homotopic to a homeomorphism. Also, since the higher homotopy of a closed 2-manifold of genus greater than zero is trivial, we have that $f|_{\mathrm{bd} M_{i_r}}: \mathrm{bd}\, M_{i_r} \to Y_0$ is homotopic to a constant map for each r $(1 \leqq r \leqq s)$. It follows that

$$\sum_{i=1}^{t_r} f(*\Delta_r^i) = \begin{cases} \pm *\Delta, & r = 0 \\ 0, & r > 0 . \end{cases}$$

Let X be the unique component of $\mathrm{bd}\, M$ which meets X_0. Then assuming notation has been properly chosen,

$$X \cap f^{-1}(\Delta) = \{\Delta_0^1, \cdots, \Delta_0^{t_0}, \cdots, \Delta_p^1, \cdots, \Delta_p^{t_p}\} .$$

Put $A = X \cap f^{-1}(\Delta)$. Then

$$\sum_{\sigma \in A} f(*\sigma) = \sum_{i=1}^{t_0} f(*\Delta_0^i) + \sum_{j=1}^{p} \sum_{i=1}^{t_j} f(*\Delta_j^i) = \pm *\Delta .$$

Since the degree of a map between closed manifolds can be computed locally, it follows that $f|_X: X \to Y$ is a degree 1 map. The same argument applies to show that if W is any component of $\mathrm{bd}\, M$ other than X such that $f(W) \subset Y$, then $f|_W: W \to Y$ is a degree zero map.

According to the above remarks, the homeomorphism

$$f_*: H_2(\mathrm{bd}\, M) \to H_2(\mathrm{bd}\, N)$$

may be described as follows. Let $X_1, \cdots, X_h, Y_1, \cdots, Y_q$ denote the boundary components of M and N respectively. Assume notation has been chosen so that $f|_{X_1}$ is a degree 1 map of X_i onto Y_i for each $i \leqq q$. Then $f|_{X_j}$ is a degree zero map of X_j into some boundary component of N for each $j > q$. Let α_i (resp. β_j) denote the generator of $H_2(X_i)$ (resp. $H_2(Y_j)$) $(1 \leqq i \leqq h, 1 \leqq j \leqq q)$. Then if f_* is the homomorphism of $H_2(\text{bd } M)$ to $H_2(\text{bd } N)$ induced by f, we have

$$f_*(\alpha_i) = \begin{cases} \pm\beta_i, \ i \leqq q \\ 0, \ i > q \ . \end{cases}$$

Let ∂: $H_3(M, \text{bd } M) \to H_2(\text{bd } M)$ denote the usual boundary homomorphism, and let γ denote the generator of $H_3(M, \text{bd } M)$. Then $\partial(\gamma) = \varepsilon_1\alpha_1 + \cdots + \varepsilon_h\alpha_h$ where $\varepsilon_i = \pm 1$ $(1 \leqq i \leqq h)$. Thus with the above description of f_*: $H_2(\text{bd } M) \to H_2(\text{bd } N)$, we observe the following two important facts.

(i) The image of the map ∂: $H_3(M, \text{bd } M) \to H_2(\text{bd } M)$ meets the kernel of f_*: $H_2(\text{bd } M) \to H_2(\text{bd } N)$ only in the trivial element.

(ii) f_*: $H_2(\text{bd } M) \to H_2(\text{bd } N)$ is an epimorphism.

Since M and N are aspherical manifolds, it follows that f is a homotopy equivalence. Thus f_*: $H_2(M) \to H_2(N)$ is an isomorphism. Thus from the following commutative diagram we conclude that the kernel of f_*: $H_2(\text{bd } M) \to H_2(\text{bd } N)$ is subset of the image of

$$\partial\colon H_3(M, \text{bd } M) \to H_2(\text{bd } M) \ .$$

$$\begin{array}{ccccccc} 0 \longrightarrow & H_3(M, \text{bd } M) & \xrightarrow{\partial} & H_2(\text{bd } M) & \longrightarrow & H_2(M) \\ & \downarrow{f_*} & & \downarrow{f_*} & & \downarrow{f_*} \\ 0 \longrightarrow & H_3(N, \text{bd } N) & \longrightarrow & H_2(\text{bd } N) & \longrightarrow & H_2(N) \ . \end{array}$$

But we have already observed that these two subgroups meet only in the trivial element. It follows that f_*: $H_2(\text{bd } M) \to H_2(\text{bd } N)$ has trivial kernel and hence is an isomorphism. Finally, we apply the five lemma to the above diagram to conclude that f_*: $H_3(M, \text{bd } M) \to H_3(N, \text{bd } N)$ is an isomorphism. An application of Theorem 3.1 now completes the proof of Theorem 3.2.

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