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# ON BOREL PRODUCT MEASURES

W. W. BLEDSOE AND C. E. WILKS

**It has been known for many years that the product of two regular borel measures on compact hausdorff topological spaces may not be borel in the product topology. The problem of defining a new product measure that extends the classical product measure and carries over this borel property has been approached in different ways by Edwards, by Bledsoe and Morse (Product Measures, Trans. Amer. Math. Soc. 79 (1955), 173–215; called PM here.) and by Johnson and Berberian. Godfrey and Sion and Hall have shown that all three of these methods are equivalent for the case of Radon measures on locally compact hausdorff spaces.**

**Elliott has extended the results of PM by defining a product measure for a pair, the first of which is a (generalized) borel measure and the second a continuous regular conditional measure (generalization of conditional probability), and proving a corresponding Fubini-type theorem.**

**The purpose of this paper is to extend the results of PM in a manner similar to Elliott's, but with his continuity condition replaced by an absolute continuity condition and by a "separation of variables" condition. It is still an open question whether Elliott's continuity condition is necessary.**

1. DEFINITIONS AND NOTATION[1]. By a *measure* (outer measure) $\mu$ on a space $M$ is meant a nonnegative countably subadditive function on $2^M$, the subsets of $M$. In a topological space $(M, m)$, an *m-borel measure on M* is any measure on $M$ for which the open sets are (Caratheodory) measurable, and the *borel sets* of $(M, m)$ are the members of the smallest $\sigma$-algebra containing $m$. If $G$ is any family of sets, let $\sigma G$ be the union $\bigcup_{a \in G} a$ of the family $G$. If $H \subseteqq 2^M$ and $g$ is a nonnegative function on $H$, then $mss\, gMH$ is defined to be the function on $2^M$ such that $mss\, gMH(A) = \inf_G \sum_{a \in G} g(a)$ where $G$ varies over all countable subsets of $H$ for which $A \subseteqq \sigma G$. $\psi = mss\, gMH$ is called the *measure generated by the gauge g*, and $H$ is called the *basis of* $\psi$.

2. Product measures. If $\mu$ measures $M$ and $\nu$ measures $N$, we call a subset $D$ of $M \times N$ a *nilset* (more correctly a $\mu\nu$*-nilset*) if

$$\iint Cr_D(x, y)\mu dx \nu dy = 0 = \iint Cr_D(x, y)\nu dy \mu dx ,$$

where $Cr_D$ is the characteristic function on $D$. The *ordinary product measure* of $\mu$ *and* $\nu$ *is given by*

[1] Most of the notation used here is taken from [2] and [4].

$$\psi = mss\, g(M \times N)R$$

where $R$ is the set of $\mu\nu$-measurable rectangles, and $g(a \times b) = \mu(a)\cdot\nu(b)$ for $a \times b \in R$. The *extended product measure* of $\mu$ and $\nu$ given in [2] is

$$\phi = mss\, g(M \times N)(R \cup Z)$$

where $Z$ denotes the set of $\mu\nu$-nilsets, and

$$g(D) = \iint Cr_D(x, y)\mu dx \nu dy$$

for $D \in (R \cup Z)$. (i.e., $g(D) = 0$ for $D \in Z$, and $g(a \times b) = \mu(a)\cdot\nu(b)$ for $a \times b \in R$).

It was shown in [2] that

$$\phi \text{ is an extension of } \psi\ ,$$

(i.e., if $A$ is $\psi$-measurable then $A$ is $\phi$-measurable and $\phi(A) = \psi(A)$). In cases where the nilsets are $\psi$-measurable then, of course, $\phi$ and $\psi$ are identical; but the $\psi$-measurability of nilsets is still an open question in the interesting case when $\mu$ and $\nu$ are regular borel measures on compact hausdorff spaces.

A result from [2], (Thms. 5.11–5.13), is

THEOREM 2.1. *If $\mu$ measures $M$ and $\nu$ measures $N$ and $\phi$ is the extended product measure of $\mu$ and $\nu$, then*

.1 *$\phi$ is an extension of the ordinary product measure of $\mu$ and $\nu$,*
.2 *$\mu\nu$-measurable rectangles are $\phi$-measurable,*
.3 *$\mu\nu$-nilsets have $\phi$ measure 0, and*
.4 *(Fubini) if $f$ is $\phi$-integrable, then*

$$\iint f(x, y)\mu dx \nu dy = \int f(z)\phi dz = \iint f(x, y)\nu dy \mu dx\ .$$

Elliott, in [4], has generalized this result by replacing the measure $\nu$ by a regular conditional measure, defined as follows. Let $\mathscr{R}$ be a family of subsets of $N$ for which $\sigma\mathscr{R} \in \mathscr{R}$. $\nu$ is called a *regular conditional measure on $M \times \mathscr{R}$* if $\nu$ is such a function on $M \times \mathscr{R}$ that

(i) for each $x \in M$, $\nu_x = \nu(x, \cdot)$ is a measure for which members of $\mathscr{R}$ are $\nu_x$-measurable, and

(ii) for each $b \in \mathscr{R}$, the function $\nu(\cdot, b)$ is $\mu$-integrable $\left(\text{i.e., } \int \nu(x, b)\mu dx \leqq \infty\right)$.

A rectangle $a \times b$ is a *$\mu\nu\mathscr{R}$-basic rectangle* if $a$ is $\mu$-measurable, $b \in \mathscr{R}$, and

$$\int Cr_a(x)\nu(x, b)\mu dx < \infty \ .$$

A set $C \subseteqq M \times N$ is called a *$\mu\nu$-nilset* if

$$\iint Cr_C(x, y)\nu_x dyx = 0 \ .$$

The $\mu\nu\mathscr{R}$-product measure (of Elliott) is defined as $mss\, g(M \times N)F$, where $F$ is the family consisting of all $\mu\nu\mathscr{R}$-basic rectangles and $\mu\nu$-nilsets, and

$$g(C) = \iint Cr_C(x, y)\nu_x dy \mu dx$$

for $C \in F$. A corresponding *ordinary (conditional) product measure* for $\mu$ and $\nu$ can be defined on $M \times N$ using only $\mu\nu$-basic rectangles [9].

Elliott [4, Thms 1.0, 1.4], generalized 2.1 as follows:

THEOREM 2.2. *If $\mu$ measures $M$, $\mathscr{R}$ is a $\sigma$-algebra of subsets of $N$, $\sigma\mathscr{R} = N \in \mathscr{R}$, $\nu$ is a regular conditional measure on $M \times \mathscr{R}$, and $\phi$ is the $\mu\nu\mathscr{R}$-product measure, then*

.1 *$\phi$ is an extension of the ordinary (conditional) product measure of $\mu$ and $\nu$,*

.2 *$\mu\nu\mathscr{R}$-basic rectangles are $\phi$-measurable,*

.3 *$\mu\nu$-nilsets have $\phi$-measure 0, and*

.4 *(Fubini-like) if $f$ is $\phi$-integrable, then*

$$\int f(z)\phi dz = \iint f(x, y)\nu_x dy \mu dx \ .$$

**3. Topological measures.** Let $(M, m)$ be a topological space with a measure $\mu$ on $M$. $\mu$ is said to be an *$m$-borel measure* if members of $m$ (open sets) are $\mu$-measurable. If, additionally, for each $A \in m$,

$$\mu(A) = \sup_C \mu(C) \ ,$$

where $C$ varies over all closed subsets of $A$ for which $\mu(C) < \infty$, then $\mu$ is called an *$m$-inner regular* borel measure. $m$ is said to be *$\mu$-almost lindelof* if for each subfamily $H$ of $m$ for which $M = \sigma H$, and for each $S \subseteqq M$ for which $\mu(S) < \infty$, there is a countable subfamily $G$ of $H$ for which $\mu(S - \sigma G) = 0$.

A regular conditional measure on $M \times \mathscr{R}$ is said to be *$m$-continuous* if, for each $b \in \mathscr{R}$, $\nu(\cdot, b)$ is an $m$-continuous function.

3.0. Throughout the remainder of this paper we shall assume that

.1 $(M, m)$ and $(N, n)$ are topological spaces and that $(P, p)$ is their topological product.

.2 $\mu$ is a finite,[2] $m$-inner regular borel measure on $M$, and $m$ is $\mu$-almost lindelof. $\nu'$ is a finite,[2] $n$-inner regular borel measure on $N$, and $n$ is $\nu'$-almost lindelof.

.3 $\mathscr{R}$ is a $\sigma$-algebra of subsets of $N$ such that $n \subseteqq \mathscr{R}$.

.4 $\nu$ is a regular conditional measure on $M \times \mathscr{R}$, with the properties that for each $x \in M$, $\nu_x$ is a finite $n$-inner regular borel measure, $n$ is $\nu_x$-almost lindelof and each member of $\mathscr{R}$ is $\nu_x$-measurable.

It should be noted that conditions 3.0 are satisfied if $(M, m)$ and $(N, n)$ are locally compact hausdorff spaces, and $\mu$ and $\nu_x$ are finite regular borel measures on $M$ and $N$ respectively, for $x \in M$.

One of the principal results (Th. 7.7) of [2] is the following:

THEOREM 3.1. *If $\mu$ and $\nu'$ are measures satisfying conditions* 3.0.1 *and* 3.0.2, *and $\phi'$ is the extended product measure of $\mu$ and $\nu'$, then*

.1 *$\phi'$ is a $p$-inner regular borel measure on $P$, and*

.2 *$p$ is $\phi'$-almost lindelof.*

Elliott, in 2.3 of [4], generalized this result as follows:

THEOREM 3.2. *If $\mu$ and $\nu$ satisfy conditions* 3.0, *and $\phi$ is the $\mu\nu\mathscr{R}$-product measure, and if $\nu$ is $m$-continuous, then*

.1 *$\phi$ is a $p$-inner regular borel measure on $P$, and*

.2 *$p$ is $\phi$-almost lindelof.*

The continuity condition of 3.2 can be replaced by an absolute continuity condition, as follows:

THEOREM 3.3. *If $\mu$, $\nu$, and $\nu'$ satisfy the conditions of* 3.0, *$\mathscr{R}$ is the set of $\nu'$ measureable sets, $\phi$ is the $\mu\nu\mathscr{R}$-product measure, and if*

$$\nu' << \nu_x << \nu'$$

*for each $x \in M$, then*

.1 *$\phi$ is a $p$-inner regular borel measure, and*

.2 *$p$ is $\phi$-almost lindelof.*

*Proof.* Let $\nu''(x, B) = \nu'(B)$ for each $x \in M, B \in \mathscr{R}$. Then $\nu''$ is a regular conditional measure which is constant in $x$ and therefore continuous. Let $\phi'$ be the (Elliott) $\mu\nu''\mathscr{R}$-product measure and complete the proof in Parts V, VI and VII below.

[2] Many of the results that follow hold also for non-finite measures.

*Part* I. $\phi(A) = 0 \leftrightarrow \phi'(A) = 0$
$\leftrightarrow A$ is a $\mu\nu$-nilset $\leftrightarrow A$ is a $\mu\nu''$-nilset.

*Proof*. $\phi(A) = 0$
$\leftrightarrow A$ is a $\mu\nu$-nilset (by 2.2.4, .3)
$\leftrightarrow \iint Cr_A(x, y)\nu_x dy \mu dx = 0$
$\leftrightarrow \nu_x(A_x) = 0$ for $\mu$-almost all $x \in M$, where $A_x = \{y \mid (x, y) \in A\}$
$\leftrightarrow \nu'(A_x) = 0$ for $\mu$-almost all $x$ (since $\nu' << \nu << \nu'$)
$\leftrightarrow \nu_x''(A_x) = 0$ for $\mu$-almost all $x$
$\leftrightarrow \iint Cr_A(x, y)\nu_x'' dy \mu dx = 0$
$\leftrightarrow A$ is a $\mu\nu''$-nilset
$\leftrightarrow \phi'(A) = 0$. (by 2.2.4, .3)

*Part* II. $\phi' << \phi << \phi'$

*Proof*. Use Part I.

*Part*. III. If $A \in p$ (an open set) then for some countable subfamily $G$ of $mn$-open rectangles, $\sigma G \subseteqq A$, and $\phi'(A - \sigma G) = 0$.

*Proof*. This is Th. 2.2.3 of [4].

*Part* IV. $mn$-open rectangles are $\phi$-measurable.

*Proof*. This follows from 2.2.2.

*Part* V. $\phi$ is a borel measure.

*Proof*. Let $A \in p$. By Parts III and IV we can find a family $G$ for which $\sigma G \subseteqq A, \sigma G$ is $\phi$-measurable and $\phi'(A - \sigma G) = 0$. But by Part I, $\phi(A - \sigma G) = 0$, and hence $A$ is $\phi$-measurable.

*Part* VI. $p$ is $\phi$-almost lindelof.

*Proof*. Use 3.2.2 and Part I.

*Part* VII. $\phi$ is $p$-inner regular.

*Proof*. Let $A \in p$ and $\varepsilon > 0$. Check that $\phi'(A) \leqq \mu(M) \cdot \nu'(N) < \infty$, and use 3.2.1 to secure such a sequence $c$ of $p$-closed sets that $c_i \subseteqq c_{i+1} \subseteqq A$, and

$$\phi'(A) = \lim_{i\to\infty} \phi'(c_i) .$$

Let $C = \bigcup_{i \in \omega} c_i$, and using Parts I and V, observe that $\phi'(A - C) = 0$, $\phi(A - C) = 0$, $\phi(A) = \lim_{i \to \infty} \phi(c_i)$.

Consequently, $\phi$ is $p$-inner regular.

It is interesting to consider whether the conclusions of Theorem 3.3 remain valid when the condition $\nu' << \nu_x << \nu'$ is lessened to $\nu_x << \nu'$, or when this condition is removed altogether. The authors have been unable to settle these questions.

Theorems 3.4 and 3.5 below are further results in the spirit of 3.2 and 3.3, in which the continuity hypothesis has been replaced by a "separation of variables" condition.

THEOREM 3.4. *If $\mu$ and $\nu$ satisfy conditions* 3.0, *and $\phi$ is the $\mu\nu\mathscr{R}$-product measure, and if there exist a $\mu$-integrable function $f$ and a measure $\psi$ on $N$ such that*

$$\nu(x, b) = f(x) \cdot \psi(b)$$

*for each $x \in M$ and $b \in \mathscr{R}$, then*

.1 *$\phi$ is a $p$-inner regular borel measure on $P$, and*

.2 *$p$ is $\phi$-almost lindelof.*

*Proof.* For each set $C$, let

$$C_x = \{y \mid (x, y) \in C\} .$$

If $f(x) = 0$ for each $x \in M$, or if $\psi(N) = 0$, then the conclusions hold trivially. So we assume that $f(x_0) > 0$ for some $x_0 \in M$ and that $f(x) < \infty$ for all $x \in M$. Hence $\psi(b) = \nu(x_0, b)/f(x_0)$, for all $b \in \mathscr{R}$, and by 3.0.4, we conclude that $\psi$ is an $n$-inner regular borel measure and $n$ is $\psi$-almost lindelof. Let $G$ be the family of $\mu$-measurable sets, and for each $A \in G$, let

$$h(A) = \int_A f(x)\mu dx \quad \text{and} \quad \gamma = mss\, hMG .$$

This defines a measure $\gamma$ on $M$, and $\gamma(A) = h(A)$ for $A \in G$. Since $\psi$ is $m$-continuous (indeed it is *constant* on $M$) we can define the $\gamma\psi\mathscr{R}$-product measure, $\phi'$, and conclude from 3.2 that $\phi'$ has the desired properties .1 and 2. We complete the proof by showing that $\phi' = \phi$.

Let $F$ be the family of $\mu\nu\mathscr{R}$-basic rectangles and $\mu\nu$-nilsets, let $F'$ be the family consisting of $\gamma\psi\mathscr{R}$-basic rectangles and $\gamma\psi$-nilsets, and let

$$g(C) = \iint Cr_C(x, y)\nu_x dy \mu dx ,$$

$$g'(C) = \iint Cr_C(x, y)\psi dy \gamma dx ,$$

for $C \in F$.

First, if $a \times b$ is a $\mu\nu\mathscr{R}$-basic rectangle, then

$$\begin{aligned} g'(a \times b) &= \iint Cr_{a\times b}(x, y)\psi dy\gamma dx \\ &= \psi(b) \cdot \gamma(a) \\ &= \psi(b)\int_a f(x)\mu dx \\ &= \int_a \nu(x, b)\mu dx \\ &= \int_a\int_b 1\nu_x dy\mu dx \\ &= \iint Cr_{a\times b}(x, y)\nu_x dy\mu dx \\ &= g(a \times b)\ . \end{aligned}$$

Secondly, if $C$ is a $\mu\nu$-nilset, then for $\mu$-almost all $x$,

$$0 = \int Cr_C(x, y)\nu_x dy = \nu(x, C_x) = f(x) \cdot \psi(C_x)\ ,$$

and hence,

$$\begin{aligned} g'(C) &= \iint Cr_C(x, y)\psi dy\gamma dx \\ &= \int \psi(C_x)\gamma dx \\ &= \int \psi(C_x) \cdot f(x)\mu dx \\ &= 0 = g(C)\ . \end{aligned}$$

Thus $g'(C) = g(C)$, for $C \in F$.

Now let $Z = \{x \in M \mid f(x) = 0\}$ and observe that $Z$ is $\mu$-measurable, and

(1) $$Z \times N \in F' \text{ and } g'(Z \times N) = 0.$$

Let

$$F_1 = \{(a \cup z) \times b \mid a \text{ is } \mu\text{-measurable, } z \subseteqq Z, \text{ and } b \in \mathscr{R}\}\ ,$$

and check that $F_1 = F'$.

Therefore,

$$\begin{aligned} \phi' &= mss\, g'PF' \\ &= mss\, g'PF_1 \\ &= mss\, g'PF && \text{(using (1))} \\ &= mss\, gPF \\ &= \phi\ . \end{aligned}$$

The result in 3.4 leads to the following

THEOREM 3.5. *If $\mu$ and $\nu$ satisfy conditions* 3.0 *and $\phi$ is the $\mu\nu\mathscr{R}$-product measure, and if there exist, for each $i \in \omega$, $\mu$-integrable functions $f_i$ and measures $\psi_i$ on $N$ such that*

$$\nu(x, b) = \sum_{i \in \omega} f_i(x) \cdot \psi_i(b) \tag{2}$$

*for each $x \in M$ and $b \in \mathscr{R}$, then*

.1 *$\phi$ is a $p$-inner regular borel measure on $P$, and*

.2 *$p$ is $\phi$-almost lindelof.*

*Proof.* For each $i \in \omega$, $x \in M$, and $b \in \mathscr{R}$, let

$$\begin{aligned} \nu_i(x, b) &= f_i(x) \cdot \psi_i(b) , \\ \phi_i &= (\text{the } \mu\nu_i\mathscr{R}\text{-product measure}) , \\ \phi' &= \sum_{i \in \omega} \phi_i . \end{aligned}$$

By 3.4 learn that, for each $i \in \omega$ ,

$\phi_i$ is a $p$-inner regular borel measure on $P$, and
$p$ is $\phi_i$-almost lindelof ,

and hence, since these two properties carry over to countable sums, we have

$\phi'$ is a $p$-inner regular borel measure on $P$, and
$p$ is $\phi$-almost lindelof .

We complete the proof by showing that $\phi = \phi'$.

Let $F$ be the family consisting of all $\mu\nu\mathscr{R}$-basic rectangles and $\mu\nu$-nilsets. For each $i \in \omega$ and $C \in F$, let

$$g_i(C) = \iint Cr_C(x, y)\nu_{ix}dy\mu dx$$

$$g(C) = \iint Cr_C(x, y)\nu_x dy\mu dx .$$

Thus

$$g(C) = \sum_{i \in \omega} g_i(C)$$

for $C \in F$, and

$$\begin{aligned} \phi(A) &= mss\, g(M \times N)F(A) \\ &= mss\, (\sum_{i \in \omega} g_i)(M \times N)F(A) \end{aligned}$$

$$= \sum_{i \in \omega} mss\, g_i(M \times N)F(A)$$
$$= \sum_{i \in \omega} \phi_i(A)$$
$$= \phi'(A) ,$$

for $A \subseteqq M \times N$. The third step above follows from Theorem 3.8 below and the fact that $F$ is disjunctive (See Definition 3.7), and the fact that the $g_i$ are nonnegative and countably additive on disjointed subsets of $F$.

The desired conclusion is at hand.

Questions that naturally arise are: when can a representation of the type (2) be obtained? How useful therefore is Theorem 3.5? No satisfactory answer to these questions is known to the authors at this time.

Theorem 3.2 (Elliott) generalizes Theorem 3.1 (Morse-Bledsoe) in yet another way, in that it uses *one-sided* nilsets $C$, where

$$\int\int Cr_C(x, y)\nu_x dy \mu dx = 0 ,$$

instead of *two-sided* nilsets $D$, where

$$\int\int Cr_D(x, y)\nu dy \mu dx = 0 = \int\int Cr_D(x, y)\mu dx \nu dy .$$

Thus Theorem 3.1 is equally valid if the definition of $\mu\nu$-nilset given in §2 is amended to read: a subset $D$ of $M \times N$ is a $\mu\nu$-left nilset if

$$\int\int Cr_D(x, y)\nu dy \mu dx = 0 .$$

Similarly, we could use $\mu\nu$-right nilsets.

The remainder of the paper gives results needed in the proof of Theorem 3.5.

LEMMA 3.6. *If $T$ is a directed set with respect to the relation $\prec$ and if $0 \leqq A_{it'} \leqq A_{it}$ whenever $i \in \omega$, $t \in T$, $t' \in T$, and $t \prec t'$, and if $\sum_{i \in \omega} A_{it_0} < \infty$ for some $t_0 \in T$, then*

$$\sum_{i \in \omega} \inf_{t \in T} A_{it} = \inf_{t \in T} \sum_{i \in \omega} A_{it} .$$

*Proof.* Let $\varepsilon > 0$ and select $N \in \omega$ so that

$$\sum_{i=N}^{\infty} A_{it0} < \varepsilon/2 ,$$

and then choose $\bar{t} \in T$ so that

$$A_{it} \leqq \inf_{t' \in T} A_{it'} + \frac{\varepsilon}{2N}, \text{ for } \bar{t} \prec t, i < N.$$

Thus for $\bar{t} \prec t$, $t_0 \prec t$, we have

$$\begin{aligned}\sum_{i \in \omega} A_{it} &= \sum_{i=0}^{N-1} A_{it} + \sum_{i=N}^{\infty} A_{it} \\ &\leqq \sum_{i=0}^{N-1} A_{it} + \sum_{i=N}^{\infty} A_{it_0} \\ &\leqq \sum_{i=0}^{N-1} A_{it} + \varepsilon/2 \\ &\leqq \sum_{i=0}^{N-1} \left(\inf_{t' \in T} A_{it'} + \frac{\varepsilon}{2N}\right) + \varepsilon/2 \\ &= \sum_{i=1}^{N-1} \inf_{t' \in T} A_{it'} + \varepsilon \\ &\leqq \sum_{i \in \omega} \inf_{t' \in T} A_{it'} + \varepsilon .\end{aligned}$$

Therefore

$$\inf_{t \in T} \sum_{i \in \omega} A_{it} \leqq \sum_{i \in \omega} \inf_{t \in T} A_{it} .$$

Since the reversed inequality is well known, the desired conclusion is at hand.

DEFINITION 3.7. We say that a family $H$ is *disjunctive* if for each $G_1$ and each $G_2$ which are countable subfamilies of $H$, there is a countable pairwise disjointed subfamily $G$ of $H$ which is a refinement of both $G_1$ and $G_2$ and such that $\sigma G = \sigma G_1 \cap \sigma G_2$.

THEOREM 3.8. *If*

.1 *$H$ is disjunctive,*

.2 *for each $i \in \omega$, $g_i \geqq 0$,*
*$g_i$ is subadditive*[3] *on $H$,*
*$g_i$ is countably additive on disjointed subfamilies of $H$,*

.3 *$A \subseteqq S$, and*

.4 *$mss\,(\sum_{i \in \omega} g_i)SH\ (A) < \infty$,*

*then*

$$mss\,(\textstyle\sum_{i \in \omega} g_i)SH\ (A) = \sum_{i \in \omega} mss\, g_{iSH}(A) .$$

*Proof.* Let

[3] A function $f$ is said to be subadditive on $H$ if for each $B$ and each countable subfamily $G$ of $H$ for which $B \subseteqq \sigma G$, we have $f(B) \leqq \sum_{\alpha \in G} f(\alpha)$.

$$H_A = \{G \mid G \text{ is a countable subfamily of } H \text{ for which } A \subseteqq \sigma G\} .$$

Since $H$ is disjunctive it follows that $H_A$ is a directed set with respect to the refinement relation. Also using .2 it follows that

$$0 \leqq \sum_{\alpha \in G'} g_i(\alpha) \leqq \sum_{\alpha \in G} g_i(\alpha)$$

whenever $i \in \omega$, $G \in H_A$, $G' \in H_A$, and $G'$ is a disjointed refinement of $G$. Furthermore, from .4 we know that for some $G_0 \in H_A$,

$$\sum_{\alpha \in G_0} \sum_{i \in \omega} g_i(\alpha) < \infty .$$

Thus by Lemma 3.6 (identifying $H_A$ with $T$, and $A_{it}$ with $\sum_{\alpha \in G} g_i(\alpha)$), we conclude that

$$\sum_{i \in \omega} \inf_{G \in H_A} \sum_{\alpha \in G} g_i(\alpha) = \inf_{G \in H_A} \sum_{i \in \omega} \sum_{\alpha \in G} g_i(\alpha) .$$

Consequently,

$$\begin{aligned} mss\, (\sum_{i \in \omega} g_i) SH(A) &= \inf_{G \in H_A} \sum_{\alpha \in G} \sum_{i \in \omega} g_i(\alpha) \\ &= \inf_{G \in H_A} \sum_{i \in \omega} \sum_{\alpha \in G} g_i(\alpha) \\ &= \sum_{i \in \omega} \inf_{G \in H_A} \sum_{\alpha \in G} g_i(\alpha) \\ &= \sum_{i \in \omega} mss\, g_i SH(A) . \end{aligned}$$

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THE UNIVERSITY OF TEXAS
AND
PEPPERDINE UNIVERSITY