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EXTENSIONS OF AN INEQUALITY BY PÓLYA AND SCHIFFER FOR VIBRATING MEMBRANES

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The inequality by Pólya and Schiffer considered in this paper is concerned with the sume of the n first reciprocal eigenvalues of the problem $\Delta u+\lambda u=0$ in G, $u=0$ on ∂G. First we extend this inequality to the problem of an inhomogeneous membrane $\Delta u+\lambda\rho u=0$ in G, $u=0$ on ∂G. Then we prove a sharper form of it for a class of homogeneous membranes with partially free boundary. The proofs are based on a variational characterization for the eigenvalues and use conformal mapping and transplantation arguments.

The inequality by Pólya and Schiffer considered in this paper is concerned with the eigenvalue problem $\Delta\varphi+\lambda\varphi=0$ in a Jordan domain $G, \varphi=0$ on ∂G. It can be stated as follows: *Among all domains with given maximal conformal radius $\dot{r}$, the circle yields the minimum of the expression* $\sum_{i=1}^{n}\lambda_i^{-1}$. This theorem is related to the geometrical inequality

$$\pi\dot{r}^2 \leqq A\ , \tag{1}$$

where A denotes the total area of G. The aim of this paper is (i) to extend the inequality by Pólya and Schiffer to the problem of an inhomogeneous membrane fixed on the boundary, (ii) to sharpen it for certain kinds of elastically supported, homogeneous membranes. Instead of considering the problem of an inhomogeneous membrane we will study the equivalent eigenvalue problem $Lu+\lambda u=0$ where $L=\Delta/\rho$ is the Beltrami operator of an abstract surface with the line element $ds^2=\rho(dx^2+dy^2)$. With the help of inequalities by Alexandrow [1], we will derive first some relations between $\dot{r}$, ρ and the Gaussian curvature of the surface. These results will be needed for the theorem concerning the eigenvalue problem. Its proof is essentially based on a method indicated by Hersch in [6] which uses conformal mapping and transplantation arguments. In the last part, we give an isoperimetric inequality for a class of plane membranes. The extremal domain is in this case the circular sector.

1. Geometrical preliminaries.

DEFINITIONS 1.1. Let Σ be an abstract surface given by a Jordan domain G in the z-parameter plane ($z=x+iy$), and by the metric $ds^2=\rho(z)|dz|^2$ where $\rho(z)$ is an arbitrary positive function in C^2.

$A(B) = \iint_B \rho dxdy$ is the area of a domain $B \subseteqq \Sigma$ and

$$L(\gamma) = \int_\gamma \sqrt{\rho}\,|dz|$$

is the length of an arc $\gamma \subseteqq \Sigma$. The Gaussian curvature has the form

$$K_G = (-\Delta_z \ln \rho)/2\rho\left[\Delta_z = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right].$$

We shall assume that the inequality $K_G \leqq K_0$ holds in G. Consider a surface $\mathscr{M}_{K_0}$ of constant curvature K_0 given in the following isothermic representation:

(i) w-plane ($w = u + iv$) with the metric

$$ds^2 = \frac{4c^2}{(1 + |w|^2)^2}|dw|^2 \quad \text{if} \quad K_0 = 1/c^2$$

(ii) interior of the unit circle $\{w; |w| < 1\}$ with the metric

$$ds^2 = \frac{4c^2}{(1 - |w|^2)^2}|dw|^2 \quad \text{if} \quad K_0 = -1/c^2 .$$

(iii) w-plane with the metric $ds^2 = |dw|^2$ if $K_0 = 0$.

We shall define the metric of $\mathscr{M}_{K_0}$ by $ds^2 = g(w)|dw|^2$, where $g(w)$ depends on K_0 and is determined by one of the preceding formulas. Let $f_a(z)$ be the conformal mapping from G onto the unit circle $\{w; |w| < 1\}$ with $f_a(a) = 0$ and $f'_a(a) > 0$. The *conformal radius* of the point a with respect to G is then defined as $r_a(G) = 1/f'_a(a)$ [9, p. 16]. We set

$$R_a(G) = \begin{cases} \frac{1}{2}\sqrt{\rho(a)|K_0|}\, r_a(G) & \text{if} \quad K_0 \neq 0 \\ \sqrt{\rho(a)}\, r_a(G) & \text{if} \quad K_0 = 0 . \end{cases} \tag{2}$$

EXAMPLE. If G is a circle with the radius r_0, the center in the origin and $\rho(z) = g(z)$, then $R_0(G) = r_0$. $w_a(z) = R_a(G)f_a(z)$ maps G onto the circle $\{w; |w| < R_a(G)\}$, and $z_a(w)$ denotes its inverse. We shall denotes the circle $\{w; |w| < \varepsilon\}$ by C_ε. $R_a(G)$ has been chosen in such a way that

$$\iint_{C_\varepsilon} g(w)dudv = \iint_{z_a(C_\varepsilon)} \rho(z)dxdy + o(\varepsilon^2) . \tag{3}$$

Since

$$\iint_{C_\varepsilon} g(w)dudv = \begin{cases} 4\pi c^2\varepsilon^2 + o(\varepsilon)^2 & \text{if} \quad K_0 \neq 0 \\ \pi\varepsilon^2 & \text{if} \quad K_0 = 0 , \end{cases}$$

it follows that

$$(4)\qquad \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\iint_{C_\varepsilon} g(w)dudv = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\iint_{z_a(C_\varepsilon)}\rho(z)dxdy$$

exists and is different from zero.

1.2. *Some Properties of* $R_a(G)$.

(a) $R_a(G)$ *is invariant under conformal mapping.*

Proof. Let $\xi(z)\colon \Sigma \Rightarrow \hat{\Sigma}$ be a conformal mapping and let $z(\xi)$ be its inverse. We set $\xi(a) = a$ and $\xi(G) = \hat{G}$. The line element of $\hat{\Sigma}$ is $d\hat{s}^2 = \hat{\rho}(\xi)|d\xi|^2$ with $\hat{\rho}(\xi) = \rho(z(\xi))|dz/d\xi|^2$. Since K_G is a conformal invariant, we have

$$(5)\qquad R_{\hat{a}}(\hat{G}) = \frac{1}{2}\sqrt{\hat{\rho}(\hat{a})|K_0|}\, r_{\hat{a}}(\hat{G}) = \frac{1}{2}\sqrt{\rho(a)|K_0|}\left|\frac{dz}{d\xi}\right|_{\xi=\hat{a}} r_{\hat{a}}(\hat{G})\ .$$

Because of the relation $|dz/d\xi|_{\xi=\hat{a}} r_{\hat{a}}(\hat{G}) = r_a(G)$ [9], it follows that $R_{\hat{a}}(\hat{G}) = R_a(G)$.

(b) *If* $K_0 < 0$, *then* $R_a(G) < 1$ *for any* $a \in G$

Proof. The function $\hat{\rho}(w) = \rho(z_a(w))|dz_a/dw|^2$ satisfies in

$$C = \{w; |w| \leqq R_a(G)\}$$

the inequality $\Delta_W \ln\hat{\rho}(w) \geqq (2/c^2)\hat{\rho}(w)$. By a theorem of Osserman [7]

$$(6)\qquad \hat{\rho}(w) \leqq \frac{4c^2R_a^2}{(R_a^2 - r^2)^2}\quad (r = |w|, R_a = R_a(G)\ \text{ for any }\ w \in C)\ .$$

Since $\hat{\rho}(0) = 4c^2$, (6) implies $R_a \leqq 1$.

(c) Let $\mu_\varepsilon = \mu(\partial C, \partial C_\varepsilon)$ be the modulus of the annulus $D = C\backslash C_\varepsilon$ [$C = \{w; |w| < R_a\}$, ∂C the boundary of C; $C_\varepsilon = \{w; |w| < \varepsilon\}$, ∂C_ε the boundary of C_ε]. Let h be the solution of the Dirichlet problem $\Delta h = 0$ in $C\backslash C_\varepsilon$, $h = 0$ on ∂C_ε, $h = 1$ on ∂C and let $D(h)$ denote the Dirichlet integral of h. Then $\mu_\varepsilon = \{D(h)\}^{-1}$. In an analogous may we define $\mu(\Gamma, \Gamma_\varepsilon)$, where Γ and Γ_ε are boundaries of G and $z_a(C_\varepsilon)$. Since the modulus is invariant under conformal mapping, we conclude that

$$\mu_\varepsilon = \mu(\Gamma, \Gamma_\varepsilon) = \frac{1}{2\pi}\ln\frac{R_a}{\varepsilon}\ ,$$

and thus

$$(7)\qquad R_a = \varepsilon e^{2\pi\mu_\varepsilon} = \lim_{\varepsilon\to 0}\varepsilon e^{2\pi\mu_\varepsilon}\qquad [10,\ p.\ 45]\ .$$

If G is contained in G', then it follows from (7) and the Dirichlet principle that $R_a(G) \leqq R_a(G')$.

(d) Let $A = A(G) = \iint_G \rho dxdy$ be the total area of G with respect to the metric $ds^2 = \rho|dz|^2$, and let $A_C = \iint_C g(w)dudv$ be the total area of C with respect to the metric $ds^2 = g(w)|dw|^2$. A_C takes the values

$$(8) \qquad A_C = \begin{cases} 4\pi c^2 R_a^2/(1 + R_a^2) & \text{if } K_0 = c^{-2} \\ 4\pi c^2 R_a^2/(1 - R_a^2) & \text{if } K_0 = -c^{-2} \\ \pi R_a^2 & \text{if } K_0 = 0 . \end{cases}$$

The following result is an extension of a classical theorem [9, problem 125 IV]. We have

$$(9) \qquad A \geqq A_C .$$

Equality holds in (9) *if and only if* G *is a geodesic circle on a surface of constant curvature* K_0. *(If* $K_0 > 0$ *we have to assume that* $A < 4\pi/K_0$.*)*

Proof. Let $A_1(\varepsilon)$ and $A_1'(\varepsilon)$ denote the area of $z_a(C_\varepsilon)$ and C_ε. By (7) and Corollary 2 [3] it follows that

$$(10) \qquad \mu(\Gamma, \Gamma_\varepsilon) = \frac{1}{2\pi}\ln\frac{R_a}{\varepsilon} \leqq \frac{1}{4\pi}\left\{\ln\frac{A}{4\pi - K_0 A} - \ln\frac{A_1(\varepsilon)}{4\pi - K_0 A_1(\varepsilon)}\right\} .$$

Equality holds only if Γ and Γ_ε are two "concentric" circles on a surface of constant curvature K_0. Suppose that $K_0 \neq 0$. From (8) we have $A_1'(\varepsilon) = 4\pi c^2\varepsilon^2 + o(\varepsilon^2)$. Substituting this expression in (10), we obtain

$$\frac{4\pi c^2 R_a^2}{A_1'(\varepsilon) + o(\varepsilon^2)} \leqq \frac{A(4\pi - K_0 A_1(\varepsilon))}{A_1(\varepsilon)(4\pi - K_0 A)} = \Phi(\varepsilon) .$$

Since $\lim_{\varepsilon\to 0}(A_1'(\varepsilon)/A_1(\varepsilon)) = 1$ (cf. (3), (4)), it follows that

$$(11) \qquad R_a^2 = \frac{A_C}{c^2(4\pi - K_0 A_C)} \leqq \lim_{\varepsilon\to 0}\frac{A_1'(\varepsilon) + o(\varepsilon^2)}{4\pi c^2}\Phi(\varepsilon) = \frac{1}{c^2}\frac{A}{(4\pi - K_0 A)} .$$

This inequality implies $A_C \leqq A$. The case $K_0 = 0$ can be treated in exactly the same way and will therefore be omitted.

REMARKS. (1) Let $g_z(z, a)$ be the Green's function defined by $\Delta_z g_z(z, a) = -\delta_a(z)$ in G, $g_z(z, a) = 0$ on Γ. $g_w(w, 0)$ is the corresponding Green's function in C. We shall use the following notations $G(t) = \{z \in G; g_z(z, a) > t\}$, $C(t) = \{w \in C; g_w(w, 0) > t\}$; $A_z(t) = \iint_{G(t)} \rho dxdy$ and $A_w(t) = \iint_{C(t)} g(w)dudv$. By the same reasoning as before we can show that

$$A_z(t) \geqq A_w(t) \ . \tag{12}$$

Equality holds if and only if G is a geodesic circle on a surface of constant curvature K_0. If $K_0 > 0$, we have, of course, to assume that $A_z(t) < 4\pi/K_0$.

(2) We define $\dot{R}(G) = \max_{a \in G} R_a(G)$. If G is a circle of radius r with the center at the origin and the metric $ds^2 = g(w)|dw|^2$, then $R_a(G) = (r^2 - |a|^2)/(1 \pm |a|^2)r$ [9]. In this case, $\dot{R}(G) = R_0(G)$. Because of (11) we have the isoperimetric inequality: *Among all domains with given total area A and with given K_0, the geodesic circles on a surface of constant curvature K_0 have the largest value of $\dot{R}(G)$.* From (11) it follows that

$$r_a^2(G) \leqq \frac{4A}{\rho(a)(4\pi - K_0 A)} \ . \tag{13}$$

If $\rho \equiv 1$, then (13) reduces to $\pi r_a^2(G) \leqq A$.

2. Bounds for the eigenvalues of an inhomogeneous membrane. Let Σ be an abstract surface given in an isothermic representation (cf. §1.1). We consider the following eigenvalue problem

$$\text{I} \qquad \frac{\Delta_z}{\rho}\varphi(x, y) + \lambda\varphi(x, y) = 0 \quad \text{in} \quad G$$

$$\varphi = 0 \quad \text{on} \quad \Gamma \text{ (boundary of } G) \ .$$

Δ_z/ρ represents the Beltrami operator of Σ. Suppose that a countable number of eigenvalues $0 < \lambda_1 < \lambda_2 \leqq \cdots$ exists. $R[v] = D(v)\Big/\iint_G v^2 \rho dxdy$ $\left[D(v) = \iint_G \text{grad}^2 v dxdy\right]$ is the Rayleigh quotient of Problem I. Let L_n be an n-dimensional linear space of continuously differentiable functions which vanish on Γ, and let $v_1, \cdots, v_n$ be an orthogonal basis in L_n with respect to the Dirichlet metric, i.e.,

$$D(v_i, v_j) = \iint_G \text{grad}\, v_i \,\text{grad}\, v_j dxdy = 0 \quad \text{if} \quad i \neq j \ .$$

Following [6] we define $T\,\text{Rinv}\,[L_n] = \sum_{i=1}^n \{R[v_i]\}^{-1}$. For the sums of the reciprocal eigenvalues we have the variational characterization [5, 6]

$$\sum_{i=1}^n \lambda_i^{-1} = \max_{L_n} T\,\text{Rinv}\,[L_n] \ . \tag{14}$$

The maximum is attained if $v_i = \varphi_i$ $i = 1, \cdots, n$ are the first n eigenfunctions of Problem I. *Assume that* $(-\Delta_z \ln \rho)/2\rho \leqq K_0 = \pm c^{-2}$ *in* G, *where K_0 is any real number.* In addition to Problem I we consider the auxiliary problem

$$\text{II}\qquad \frac{\Delta_w}{(gw)}\hat{\varphi} + \hat{\lambda}\hat{\varphi} = 0 \quad \text{in} \quad C = \{w; |w| < R_a\}$$
$$\hat{\varphi} = 0 \quad \text{on} \quad \partial C = \{w; |w| = R_a\} .$$

$g(w)$ depends on K_0 and was defined in § 1.1; and $R_a = \sqrt{\rho(a)}\, r_a/2c$ or $R_a = \sqrt{\rho(a)}\, r_a$ (cf. § 1.1). The eigenfunctions of this problem are either of the form

$$\hat{\varphi}_k(r, \theta) = R_0(\hat{\lambda}_k, r) \tag{15}$$

or

$$\hat{\varphi}_k(r, \theta) = R_m(\hat{\lambda}_k, r) \cos m\theta \quad \text{and} \quad \hat{\varphi}_{k+1}(r, \theta) = R_m(\hat{\lambda}_k, r) \sin m\theta \quad m = 1, 2, \cdots . \tag{16}$$

In $(0, R_a)$, $R_m(\hat{\lambda}_k, r)$ satisfies the differential equation

$$(rR')' - \frac{m^2 R}{r} + \frac{4\hat{\lambda}_k c^2 r R}{(1 \pm r^2)^2} = 0 \qquad \left(' = \frac{d}{dr}\right) \tag{17}$$

if $K_0 = \pm c^{-2}$, and

$$(rR')' - \frac{m^2 R}{r} + \hat{\lambda}_k R = 0 \quad \text{if} \quad K_0 = 0 . \tag{18}$$

The boundary conditions are

$$R'(0) < \infty \quad \text{and} \quad R(R_a) = 0 . \tag{19}$$

We shall call m the order of R. By introducing the new variable

$$z = \begin{cases} (r^2 - 1)/(1 + r^2) & \text{if} \quad K_0 > 0 \\ (r^2 + 1)/(1 - r^2) & \text{if} \quad K_0 < 0 \end{cases}$$

(17) is transformed into the Legendre equation

$$\pm \frac{d}{dz}\left[(z^2 - 1)\frac{d}{dz} y(z)\right] \mp \frac{m^2 y(z)}{z^2 - 1} + \hat{\lambda}_k c^2 y(z) = 0 .$$

The following result is a generalization of a theorem of Pólya-Schiffer [8]. We shall use a method of proof devised by Hersch [6].

THEOREM 1. *If* $(-\Delta \ln \rho)/2\rho \leqq K_0$, $2\pi - K_0 A > 0$, *and* n *is a natural number, then we have the isoperimetric inequality*

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_n} \geqq \frac{1}{\hat{\lambda}_1} + \frac{1}{\hat{\lambda}_2} + \cdots + \frac{1}{\hat{\lambda}_n} \tag{20}$$

where $\hat{\lambda}_i$ *is the* i^{th} *eigenvalue of Problem* II.

Proof. Let $\hat{\varphi}_1(w), \cdots, \hat{\varphi}_n(w)$ be the first n eigenfunctions of Problem II and let $U_1(z), \cdots, U_n(z)$ be the transplanted functions $U_i(z) = \hat{\varphi}_i(w_a(z))$. Because of the invariance of the Dirichlet integral under conformal transformation, we have $D_G(U_i, U_j) = D_C(\hat{\varphi}_i, \hat{\varphi}_j) = 0$ if $i \neq j$. $U_i(z)$ $i = 1, \cdots, n$ can therefore be used as trial functions for the variational characterization (14). Thus,

$$\sum_{i=1}^{n} \lambda_i^{-1} \geqq \sum_{i=1}^{n} \{R[U_i]\}^{-1} = \sum_{i=1}^{n} \frac{\iint_C \hat{\varphi}_i^2 \left| \frac{dz_a}{dw} \right|^2 \rho(z_a(w)) du dv}{D_C(\hat{\varphi}_i)} . \tag{21}$$

Let $\hat{\varphi}_k(w)$ and $\hat{\varphi}_{k+1}(w)$ be two functions of the type (16). In this case

$$\{R[U_k]\}^{-1} + \{R[U_{k+1}]\}^{-1} = \frac{\int_0^{R_a}\int_0^{2\pi} (\hat{\varphi}_k^2 + \hat{\varphi}_{k+1}^2) g(w) \left| \frac{dz_a}{dw} \right|^2 \frac{\rho}{g} r dr d\theta}{D_C(\hat{\varphi}_k)} . \tag{22}$$

We observe that

$$\hat{\varphi}_k^2(w) + \hat{\varphi}_{k+1}^2(w) = \Phi(r) \tag{23}$$

is independent of θ. By the Schwarz inequality,

$$\int_0^{2\pi} \left| \frac{dz_a}{dw} \right|^2 \frac{\rho(z_a(w))}{g(w)} r d\theta \geqq \left(\int_0^{2\pi} \left| \frac{dz_a}{dw} \right| \sqrt{\rho}\, r d\theta \right)^2 \Big/ \int_0^{2\pi} g(w) r d\theta . \tag{24}$$

We note that for fixed r

$$\int_0^{2\pi} \left| \frac{dz_a}{dw} \right| \sqrt{\rho}\, r d\theta = L_z(t) ,$$

where $L_z(t)$ is the length of the level line $g_z(z, a) = t = (1/2\pi) \ln (R_a/r)$ in the metric of Σ. We also observe that $\int_0^{2\pi} g(w) r d\theta = L_W^2(t)/2\pi r$, where $L_w(t)$ is the length of the level line $g_w(w, 0) = t$ with respect to the metric of $\mathscr{M}_{K_0}$.

In order to estimate $L_z^2(t)$, we use the following geometrical isoperimetric inequality of Alexandrow [1]: If G is a domain on Σ homeomorphic to a circle, and if $K_G \leqq K_0$, then the following relation holds between the area A of G and the length L of the boundary ∂G:

$$L^2 \geqq A(4\pi - K_0 A) . \tag{25}$$

Equality holds iff G is isometric to a geodesic circle on a surface of constant curvature K_0.[1] From this inequality we conclude that

$$L_z^2(t) \geqq A_z(t)(4\pi - K_0A_z(t)) = f(A_z) . \tag{26}$$

$A_z(t)$ has been defined in §1.2. If $K_0 \leqq 0$, then $f(A_z)$ is a monotone increasing function; if K_0 is positive then $f(A_z)$ is monotone increasing in the interval $[0, 2\pi/K_0]$. By (26), (12) and our assumption on A, it follows that

$$L_z^2(t) \geqq A_w(t)(4\pi - K_0A_W(t)) = L_w^2(t) .$$

This implies

$$\int_0^{2\pi} \left|\frac{dz_a}{dw}\right|^2 \frac{\rho(z_a(w))}{g(w)} r d\theta \geqq 2\pi r .$$

From this inequality and from (22) and (23)

$$\{R[U_k]\}^{-1} + \{R[U_{k+1}]\}^{-1} \geqq 2\hat{\lambda}_k^{-1} .$$

If $\hat{\varphi}_n$ and $\hat{\varphi}_{n+1}$ belong to the same order m [cf. (16)], we denote by $\hat{\varphi}_n(w)$ the function for which

$$\{R[U_n]\}^{-1} \geqq \hat{\lambda}_n^{-1} . \tag{27}$$

By the same arguments as before, (27) holds also for the functions $\hat{\varphi}_k(w)$ of order 0 [cf. (15)]. This establishes the theorem.

REMARKS. If ρ is constant we obtain the theorem of Pólya-Schiffer [8, 6]. It is easy to see that (20) is optimal if we choose a such that $R_a(G) = \max_{p \in G} R_p(G)$.

3. Generalization. Let Σ' be a piece of an abstract surface with the line element $ds^2 = |z - a|^{-\omega/\pi}\nu(z)|dz|^2$ where $\nu(z) \in C^2$ and $0 \leqq \omega < 2\pi$. Σ' includes the regular surfaces in the usual sense which have at the point a corner of curvature ω [cf. 1]. *We assume that* $(-\Delta_z \ln \nu)/2\nu \leqq K_0$. In this case we define

$$R_a(G) = \begin{cases} \dfrac{1}{2 - \omega/\pi}\sqrt{\nu(a)|K_0|}\, r_a(G) & \text{if} \quad K_0 \neq 0 \\ \dfrac{2}{2 - \omega/\pi}\sqrt{\nu(a)}\, r_a(G) & \text{if} \quad K_0 = 0 . \end{cases} \tag{28}$$

We consider a circular cone $\mathscr{C}_{K_0}$ in a three-dimensional space of constant curvature K_0 with the curvature ω at the corner [1]. It can be represented by

[1] This inequality is valid for more general surfaces. A brief summary can be found in [1, pp. 509, 514].

(i) sector $0 < \theta < 2\pi - \omega$ (θ, r polar coordinates of the w-plane) the lines $\theta = 0$ and $\theta = 2\pi - \omega$ identified, and the metric

$$ds^2 = \frac{4c^2}{(1 + |w|^2)^2}|dw|^2 \quad (K_0 = 1/c^2)$$

(ii) sector $0 < \theta < 2\pi - \omega, 0 < r < 1$ with the lines $\theta = 0$ and $\theta = 2\pi - \omega$ identified, and the metric

$$ds^2 = \frac{4c^2}{(1 - |w|^2)^2}|dw|^2 \quad (K_0 = -1/c^2)$$

(iii) wedge $0 < \theta < 2\pi - \omega$ with the lines $\theta = 0$ and $\theta = 2\pi - \omega$ identified and the metric $ds^2 = |dw|^2 \quad (K_0 = 0)$.

With the help of the function $\xi = w^{2\pi/(2\pi-\omega)}$, the sector $0<\theta<2\pi - \omega$ is mapped into the ξ-plane. $g(w)$ is then transformed into $\tilde{g}(\xi) = g(w(\xi))|dw/d\xi|^2$ which is $\tilde{g}(\xi) = c^2(2 - \omega/\pi)^2|\xi|^{-\omega/\pi}/(1 \pm |\xi|^{2-\omega/\pi})^2$ if $K_0 = \pm c^{-2}$ or $\tilde{g}(\xi) = ((2\pi - \omega)/2\pi)^2|\xi|^{-\omega/\pi}$ if $K_0 = 0$.

EXAMPLE. Let G be a circle with the radius r_0, the center in the origin and the metric $ds^2 = \tilde{g}(\xi)|d\xi|^2$. In this case $R_0(G) = r_0$. Let $C = \{\xi; |\xi| < R_a(G)\}$ be a circle on the cone $\mathscr{C}_{K_0}$. The line element is then $ds^2 = \tilde{g}(\xi)|d\xi|^2$. In this metric

$$A_C = \iint_C \tilde{g}(\xi)d\xi_1 d\eta = \begin{cases} 2(2\pi - \omega)c^2 R_a^{2-\omega/\pi}/(1 \pm R_a^{2-\omega/\pi}) & \text{if} \quad K_0 = \pm c^{-2} \\ \dfrac{2\pi - \omega}{2} R_a^{2-\omega/\pi} & \text{if} \quad K_0 = 0 \end{cases}$$

$$(\xi = \xi_1 + i\eta)$$

is the total area of C. $A = \iint_G |z - a|^{-\omega/\pi}\nu(z)dxdy$ represents the total area of G. All properties (a), (b), (c) and (d) remain valid in this case. The proofs are the same as in §1.2 except for (d) where we use Theorem 2 [3] instead of Corollary 2 [3].

We now consider on Σ' the eigenvalue problem I, and on $C \in \mathscr{C}_{K_0}$ the auxiliary problem II (cf. §2). By transplanting the last into the w-plane, it becomes equivalent to the following eigenvalue problem:

$$\frac{\Delta w}{g(w)}\hat{\varphi} + \hat{\lambda}\hat{\varphi} = 0 \quad \text{in} \quad \{w; |w| < R_a^{1-\omega/2\pi} \quad \text{and} \quad 0 < \arg w < 2\pi - \omega\}$$

$$\hat{\varphi} = 0 \quad \text{on} \quad |w| = R_a^{1-\omega/2\pi} ,$$

$$\hat{\varphi}|_{\theta=0} = \hat{\varphi}|_{\theta=2\pi-\omega} .$$

By a separation of the variables it follows that $\hat{\varphi}(r, \theta)$ is either of the type $\hat{\varphi}_k = R_0(\hat{\lambda}_k, r)$, or else $\hat{\varphi}_k = R_m(\hat{\lambda}_k, r)\cos m\theta$ and $\hat{\varphi}_{k+1} = R_m(\hat{\lambda}_k, r)\sin m\theta$ with $m = 2\pi n/(2\pi - \omega)$ $(n = 1, 2, \cdots)$. In $(0, R_a^{1-\omega/2\pi})$ $R_m(\hat{\lambda}_k, r)$ satisfies the differential equation (17) with the boundary conditions (18). In the same way as in §2 we can prove

THEOREM I′. *If* $(-\Delta \ln \nu)/2\nu \leqq K_0$ *and* $2\pi - \omega - K_0 A > 0$, *then*

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_n} \geqq \frac{1}{\hat{\lambda}_1} + \frac{1}{\hat{\lambda}_2} + \cdots + \frac{1}{\hat{\lambda}_n}.$$

This inequality is valid for arbitrary n.

4. **Bounds for the eigenvalues of plane membranes with partially free boundary.** Let G be a Jordan domain in the z-plane. Suppose that its boundary consists of three analytic arcs $\widehat{OA}$, $\widehat{AB}$ and $\widehat{BO}$ where $\widehat{OA}$ and $\widehat{BO}$ are concave with respect to G. We assume further that $\widehat{OA}$ and $\widehat{BO}$ meet in 0 at an angle $\alpha (0 < \alpha \leqq \pi)$.

There exists a conformal mapping $f(z)$ from G into the circular sector $0 \leqq \theta \leqq \alpha, r \leqq 1$. ($r, \theta$ polar coordinates of the w-plane) such that $f(0) = 0, f(A) = 1, f(B) = e^{i\alpha}$ and $f'(0) > 0$ [4, p. 378]. If we put $r_0 = \{f'(0)\}^{-1}$, then $w(z) = r_0 f(z) = z + a_2 z^2 + \cdots$. Its inverse will be called $z(w)$. We consider the following eigenvalue problem of the membrane with partially free boundary:

$$\text{(A)} \qquad \begin{aligned} \Delta_z \varphi + \lambda\varphi &= 0 \quad \text{in} \quad G \\ \varphi &= 0 \quad \text{on} \quad \widehat{AB} \\ \frac{\partial\varphi}{\partial n} &= 0 \quad \text{on} \quad \widehat{OA} \cup \widehat{BO}\,. \end{aligned}$$

These eigenvalues will be compared with the eigenvalues $\hat{\lambda}$ of the problem

$$\text{(B)} \quad \begin{aligned} \Delta_w \hat{\varphi} + \hat{\lambda}\hat{\varphi} &= 0 \quad \text{in} \quad \hat{G} = \{w; |w| < r_0 \quad \text{and} \quad 0 < \arg w < \alpha\} \\ \hat{\varphi} &= 0 \quad \text{on} \quad r = r_0 \\ \hat{\varphi}|_{\theta=0} &= \hat{\varphi}|_{\theta=\alpha}\,. \end{aligned}$$

The solutions of (B) are

$$\hat{\varphi}_k(r, \theta) = J_0(\sqrt{\hat{\lambda}_k}\, r)$$

or

$$\hat{\varphi}_k(r, \theta) = J_{\frac{2\pi m}{\alpha}}(\sqrt{\hat{\lambda}_k}\, r)\cos\frac{2\pi m}{\alpha}\theta$$

and

$$\hat{\varphi}_{k+1}(r,\theta) = J_{\frac{2\pi m}{\alpha}}(\sqrt{\hat{\lambda}_k}\, r)\sin\frac{2\pi m}{\alpha}\theta \quad m = 1, 2, \cdots .$$

$J_\beta(r)$ is the Bessel function of order β. (B) can be interpeted as the problem of a vibrating membrane on a circular cone.

THEOREM II. *For an arbitrary integer n we have*

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_n} \geqq \frac{1}{\hat{\lambda}_1} + \frac{1}{\hat{\lambda}_2} + \cdots + \frac{1}{\hat{\lambda}_n} .$$

Proof. Let $f(w) = f(r)$ be a function depending only on r. We first show that every function $F(z) = f(w(z))$ satisfies the inequality

$$\begin{aligned}\iint_G F^2(z)dxdy &= \int_0^{r_0} f^2(r)rdr\int_{\theta=0}^{\alpha}\left|\frac{dz}{dw}\right|^2 d\theta \\ &\geqq \alpha\int_0^{r_0} f^2(r)rdr = \iint_{\hat{G}} f^2 dudv .\end{aligned} \tag{29}$$

By the Schwarz inequality, we have

$$\int_0^\alpha \left|\frac{dz}{dw}\right|^2 d\theta \geqq \frac{1}{\alpha t^2}\left(\int_0^\alpha\left|\frac{dz}{dw}\right| t d\theta\right)^2 . \tag{30}$$

We observe that $L(t) = \int_0^\alpha |dz/dw|\, t d\theta$ is the length of the arc $z(C_t)$ where C_t is the circular arc $w = te^{i\theta}$ $0 \leqq \theta \leqq \alpha$. Let $A(t)$ denote the area of the domain $z(\hat{G}_t)$, where $\hat{G}_t$ is the circular sector $0 \leqq r \leqq t$, $0 \leqq \theta \leqq \alpha$. Because of the concavity of the arcs $\widehat{OA}$ and $\widehat{BO}$ it follows from a reflection argument and an isoperimetric inequality by Alexandrow [1] that

$$L^2(t) \geqq 2\alpha A(t) .$$[2]

The function $\xi = w^{2\pi/\alpha}$ maps the sector $0 \leqq \theta \leqq \alpha$ onto the ξ-plane. Let $\tilde{\theta}$ and $\tilde{r}$ be the polar coordinates of the ξ-plane. We have

$$\begin{aligned}A(t) &= \int_0^t\int_0^\alpha \left|\frac{dz}{dw}\right|^2 rdrd\theta \\ &= \left(\frac{\alpha}{2\pi}\right)^2\int_0^{t^{2\pi/\alpha}} \tilde{r}^{(\alpha-2\pi)/\pi}\tilde{r}d\tilde{r}\int_0^{2\pi}\left|\frac{dz}{dw}(w(\xi))\right|^2 d\tilde{\theta} \\ &= \frac{\alpha t^2}{2}\cdot\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{dz}{dw}(w(\xi))\right|^2 d\tilde{\theta} .\end{aligned} \tag{31}$$

Since

[2] A detailed proof with more general results can be found in [2].

$$\Delta_\xi \left| \frac{dz}{dw}(w(\xi)) \right|^2 = 4 \frac{\partial}{\partial \xi} \frac{\partial}{\partial \bar{\xi}} \left| \frac{dz}{dw}(w(\xi)) \right|^2 \geqq 0 \,,$$

it follows that

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{dz}{dw} \right|^2 d\tilde{\theta} \geqq \left| \frac{dz}{dw} \right|^2_{w=0} = 1$$

and hence

$$A(t) \geqq \frac{\alpha t^2}{2} \tag{32}$$

(32) and (30) imply

$$\int_1^\alpha \left| \frac{dz}{dw} \right|^2 d\theta \geqq \alpha \tag{33}$$

which proves (29).

The remaining part of the proof proceeds as in Theorem I (§ 2). We transplant the eigenfunction $\hat{\varphi}_i$ into the z-plane. $U_i(z) = \hat{\varphi}_i(w(z))$ are admissible for the variational characterization (14), and we thus have

$$\sum_{i=1}^n \lambda_i^{-1} \geqq \mathrm{Trin}\, v\, [L(U_1, \cdots, U_n)] = \sum_{i=1}^n \frac{\iint_G \hat{\varphi}_i^2 \left| \frac{dz}{dw} \right|^2 dudv}{D_{\hat{G}}(\hat{\varphi}_i)} \,. \tag{34}$$

If

$$\hat{\varphi}_k(r, \theta) = J_{\frac{2\pi m}{\alpha}}(\sqrt{\hat{\lambda}_k} r) \cos \frac{2\pi m \theta}{\alpha}$$

and

$$\hat{\varphi}_{k+1}(r, \theta) = J_{\frac{2\pi m}{\alpha}}(\sqrt{\hat{\lambda}_k} r) \sin \frac{2\pi m}{\alpha} \theta \,,$$

then (29) implies

$$\{R[U_k]\}^{-1} + \{R[U_{k+1}]\}^{-1} \geqq 2 \hat{\lambda}_k^{-1} \,. \tag{35}$$

For functions $\hat{\varphi}_k$ which depends only on r we have $\{R[U_k]\}^{-1} \geqq \hat{\lambda}_k$. It is always possible to choose $\hat{\varphi}_n(r, \theta)$ such that the last inequality remains true for $k = n$. These relations together with (34) establish the theorem.

The first eigenvalue $\hat{\lambda}_1$ of problem (B) is the same as the first eigenvalue ν_1, of the problem $\Delta_w \tilde{\varphi} + \nu \tilde{\varphi} = 0$ in G, $\tilde{\varphi} = 0$ on $r = r_0$, $\partial \tilde{\varphi} / \partial n = 0$ on $\theta = 0$ and $\theta = \alpha$. Theorem II and Theorem III in [2] yield the

COROLLARY. *If A denotes the total area of G and $j_0 = 2{,}4048\cdots$ is the first zero of the Bessel function $J_0(r)$, then*

$$\frac{\alpha}{2A} j_0^2 \leqq \lambda_1 \leqq \left(\frac{j_0}{r_0}\right)^2 . \tag{36}$$

Equality holds in both cases if and only if G is a circular sector.

The right-hand side of (36) is a generalization of an inequality by pólya and Szegö [8]. The following charaterization of r_0 is based on the one indicated in [8] for the conformal radius. Let $\mu(\widehat{AB}, \Gamma_\varepsilon)$ be the modulus of the domain $G_\varepsilon \subseteqq G$ bounded by $\widehat{AB}, \widehat{BO}, \widehat{OA}$ and $\Gamma_\varepsilon = \{z; |z| = \varepsilon\}$. It is defined as $\mu(\widehat{AB}, \Gamma_\varepsilon) = 1/D(h)$ where $\Delta h = 0$ in G_ε, $h = 1$ on Γ_ε and $h = 0$ on $\widehat{AB}$. An easy computation (cf. §1 (c)) yields

$$r_0 = \lim_{\varepsilon \to 0} \varepsilon e^{\alpha\mu(AB, \Gamma_\varepsilon)} . \tag{37}$$

Let D denote the shortest distance from the arc $\widehat{AB}$ to the origin 0. By (37) and the monotonicity of $\mu(\widehat{AB}, \Gamma_\varepsilon)$ it follows that $D \leqq r_0$. This inequality together with the corollary implies $\lambda_1 \leqq (j_0/D)^2$.

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