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THE USE OF MITOTIC ORDINALS IN CARDINAL ARITHMETIC

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In this paper, based on the properties of mitotic ordinals, some results of the cardinal arithmetic are obtained in a rather natural way.

In what follows, any reference to order among ordinal numbers is made with respect to their usual order. Thus, if u and v are ordinals then $u \leqq v$ if and only if $u \subseteqq v$ if and only if "$u \in v$ *or* $u = v$".

DEFINITION. *A nonzero ordinal w is called mitotic if and only if it can be partitioned into $\overline{\overline{w}}$ pairwise disjoint subsets each of type w. Such a partition is called a mitotic partition of w.*

For instance, ω is a mitotic ordinal since ω can be partitioned into denumerably many pairwise disjoint denumerable subsets R_i with $i = 0, 1, 2, \cdots$, where the elements of R_i are precisely the ordinals appearing in the i-th row of the following table:

$$\begin{array}{cccccccc} 0 & 1 & 3 & 6 & \cdot & \cdot & \cdot \\ 2 & 4 & 7 & \cdot & \cdot & \cdot & \cdot \\ 5 & 8 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \, . \end{array}$$

Clearly, each R_i is of type ω.

LEMMA 1. *Let w be a mitotic ordinal. Then w is a limit ordinal. Moreover, for every element S_i of a mitotic partition $(S_i)_{i \in w}$ of w we have*:

$$\cup S_i = \sup S_i = w \, . \tag{1}$$

Proof. Since S_i is of type w we see that S_i is similar to w. Let f_i be a similarity mapping from w onto S_i. But then by [1, p. 302] we have $x \leqq f_i(x)$ for every $x \in w$. Now, assume on the contrary that w is not a limit ordinal and let k be the last element of w. But then clearly, $k = f_i(k)$ and therefore $k \in S_i$. However, since 1 is not a mitotic ordinal, we see that the mitotic partition of w must have at least two distinct elements, S_0 and S_1. But then $k \in S_0$ and $k \in S_1$ which contradicts the fact that S_0 is disjoint from S_1. Thus, our assumption is false and w is a limit ordinal.

Next, since the similarity of w to S_i implies the existence of a one-to-one mapping f_i from w onto S_i such that $x \leqq f_i(x)$ for every $x \in w$, we see that $\cup w \leqq S_i$ and therefore $\cup w = \cup S_i$ since $S_i \subseteqq w$. On the other hand, since w is a limit ordinal by [1, p. 323] we have $\cup w = w$. Hence, (1) is established.

Based on the *natural expansion* [1, p. 355] of ordinals we prove the following lemma.

LEMMA 2. *Let w be a mitotic ordinal and let $\omega^e n$ be the last term of the normal expansion of w. Then*

$$\overline{\overline{w}} = \overline{\overline{\omega^e n}} \tag{2}$$

Proof. Let $w = u + \omega^e n$ and let $(S_i)_{i \in w}$ represent a mitotic partition of w. From (1) it follows that for every $i \in w$, we must have $(u + v) \in S_i$ for some $v < \omega^e n$. But then (2) follows from the fact that $(S_i)_{i \in w}$ is a family of pairwise disjoint elements S_i.

LEMMA 3. *For every nonzero ordinal e the ordinal ω^e is mitotic.*

Proof. Since $\omega < \omega^e$ we see that there is a mitotic ordinal of type ω^h such that $h \leqq e$. Let P be the set of all mitotic partitions of mitotic ordinals of type ω^h which are less than or equal to ω^e. Partial order P by $\leqq^*$ as follows:

$$(S_{u_i})_{i \in \omega^u} \leqq^* (S_{v_i})_{i \in \omega^v}$$

if and only if $S_{u_i} \subseteqq S_{v_i}$ for every $i \in (\omega^u \cap \omega^v)$.

Let $((S_{u_i})_{i \in \omega^u})_{u \in A}$ be a simply ordered subset of $(P, \leqq^*)$. But then it is easy to verify that $(\bigcup_{u \in \cup A} S_{u_i})_{i \in \omega^{\cup A}}$ is a mitotic partition of the ordinal $\omega^{\cup A}$. Hence every simply ordered subset of the nonempty partially ordered set $(P, \leqq^*)$ has a least upper bound. Consequently, $(P, \leqq^*)$ has a maximal element $(M_i)_{i \in \omega^k}$ where ω^k is a mitotic ordinal such that $k \leqq e$.

Let (M_i) denote the mitotic partition $(M_i)_{i \in \omega^k}$ of ω^k, i.e.,

$$(M_i) = (M_i)_{i \in \omega^k} \, . \tag{3}$$

To prove the lemma it is sufficient to show that $k = e$. Assume on the contrary that $k < e$. Thus $\omega^k \omega \leqq \omega^e$.

For every $n \in \omega$, let $(M_i)n$ denote the mitotic partition given by (3) where each entry is augmented on the left by $\omega^k n$. But then

$$\begin{array}{cccc} (M_i)0 & (M_i)1 & (M_i)3 & \cdot \\ (M_i)2 & (M_i)4 & \cdot & \cdot \\ (M_i)5 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

is clearly a mitotic partition of $\omega^k\omega = \omega^{k+1}$. But since $\omega^k \leqq \omega^k\omega < \omega^{k+1} \leqq \omega^e$ we arrive at a contradiction. Thus, our assumption is false and $k = e$.

LEMMA 4. *The sum of finitely many pairwise equipollent mitotic ordinals is a mitotic ordinal.*

Proof. Obviously, it is sufficient to prove that the sum of two equipollent mitotic ordinals is a mitotic ordinal. Let $(R_i)_{i\in\overline{\overline{u}}}$ and $(S_i)_{i\in\overline{\overline{v}}}$ represent respectively mitotic partitions of mitotic ordinals u and v where $\overline{\overline{u}} = \overline{\overline{v}} = c$. Now, let

$$R_i = (r_0, r_1, r_2, \cdots) \text{ and } S_i = (s_0, s_1, s_2, \cdots) .$$

Consider

$$H_i = (r_0, r_1, r_2, \cdots, (\cup R_i) + s_0, (\cup R_i) + s_1, (\cup R_i) + s_2, \cdots) .$$

Clearly, $H_i \subseteqq (u + v)$ and H_i is of type $u + v$ for every $i \in c$. But then observing that $\overline{\overline{u + v}} = c$ we see that $(H_i)_{i\in c}$ is a mitotic partition of the ordinal $u + v$. Thus, $u + v$ is mitotic, as desired.

THEOREM 1. *An infinite ordinal is mitotic if and only if it is equipollent to the last term of its normal expansion.*

Proof. Let w be an infinite ordinal. Without loss of generality we may assume that the normal expansion of w has two terms and is given by:

$$w = \omega^a m + \omega^e n . \tag{4}$$

Now, if w is mitotic then by (2) we see that w is equipollent to the last term of its normal expansion. Conversely, let w be equipollent to the last term of its normal expansion. But then clearly,

$$\overline{\overline{w}} = \overline{\overline{\omega^a m}} = \overline{\overline{\omega^e n}} . \tag{5}$$

However, since $\omega^a m$ is a finite sum of summands each equal to ω^a, in view of Lemmas 3 and 4, we see that $\omega^a m$ is mitotic. Similarly, $\omega^e n$ is mitotic. But then again, from (5), (4) and Lemma 4, we see that w is mitotic, as desired.

From Theorem 1 it follows that each of the following ordinal numbers is mitotic:

$$\omega^\omega,\ \omega^\omega + \omega,\ \omega_1^\omega + \omega_1,\ \omega_2^\omega + \omega_2\omega_1\omega,\ \cdots .$$

Also, since the normal expansion of every infinite cardinal has one term, from Theorem 1, we have:

COROLLARY 1. *Every infinite cardinal is mitotic.*

Next, based on the properties of mitotic ordinals we derive some results pertaining to the cardinal arithmetic.

THEOREM 2. *Let w be a mitotic ordinal and $(c_i)_{i\in w}$ a nondecreasing sequence of type w of cardinals c_i. Then*

$$\prod_{i\in w} c_i = \left(\prod_{i\in w} c_i\right)^{\overline{\overline{w}}} . \tag{6}$$

Proof. Let $(S_i)_{i\in w}$ be a mitotic partition of w. Since $(c_i)_{i\in w}$ is nondecreasing, we have

$$\prod_{i\in w} c_i \leqq \prod \{c_i \mid c_i \in S_j\} \text{ for every } j \in w$$

and since the right side of the above inequality is a subproduct of the left side, we have

$$\prod_{i\in w} c_i = \prod \{c_i \mid c_i \in S_j\} \text{ for every } j \in w . \tag{7}$$

On the other hand, in view of the general commutativity and associativity of the infinite product of cardinal numbers, we have

$$\prod_{i\in w} c_i = \prod_{j\in w} (\prod \{c_i \mid c_i \in S_j\} . \tag{8}$$

But then (6) follows readily from (7) and (8).

Based on Theorem 2, we prove a theorem which extends a result of Tarski-Hausdorff [2, p. 14] to the case of a nondecreasing sequence of cardinals.

THEOREM 3. *Let w be a mitotic ordinal and $(c_i)_{i\in w}$ a nondecreasing sequence of type w of nonzero cardinals c_i. Then*

$$\prod_{i\in w} c_i = (\sup_{i\in w} c_i)^{\overline{\overline{w}}} . \tag{9}$$

Proof. Since $c_i \leqq \sup_{i\in w} c_i$ for every $i \in w$, we have

$$\prod_{i\in w} c_i \leqq (\sup_{i\in w} c_i)^{\overline{\overline{w}}} . \tag{10}$$

On the other hand, for establishing (9), we may assume without loss of generality, that $c_i > 1$ for every $i \in w$. But then we have:

$$(\sup_{i\in w} c_i)^{\overline{\overline{w}}} \leqq (\sum_{i\in w} c_i)^{\overline{\overline{w}}} \leqq (\sum_{i\in w} c_i)^{\overline{\overline{w}}} \tag{11}$$

and then (9) follows readily from (6), (10) and (11).

Thus, Theorem 3 is proved.

Let us observe that the formula analogous to (9) for the sum of an (not necessarily nondecreasing) infinite sequence $(c_i)_{i\in v}$ of type v (not necessarily mitotic) of nonzero cardinals c_i is given by:

$$\sum_{i\in v} c_i = \overline{\overline{v}} \sup_{i\in v} c_i \,. \tag{12}$$

REMARK. In the arithmetic of ordinal numbers infinite sums and products of ordinals are respectively equal to the limit of their partial sums and partial products. In fact, in ordinal arithmetic, evaluation of the result of an infinite operation as the limit of those of partial ones is a general method. In contrast to this, in the arithmetic of cardinal numbers infinite sums and products of cardinals are not equal, in general, to the limit of their partial sums and the limit of their partial products respectively. However, as shown below, in cardinal arithmetic, infinite sums of cardinals and products of nondecreasing cardinals are respectively equal to the sum of their partial sums and to the product of their partial products (this, in general, is not true in ordinal arithmetic).

The statement concerning an infinite sum of cardinals can be given as a corollary of (12).

COROLLARY 2. *Let $(c_i)_{i\in v}$ be an infinite sequence of type v of nonzero cardinals c_i. Then*

$$\sum_{i<v} c_i = \sum_{u<v} \Big(\sum_{i\leqq u} c_i\Big) \,. \tag{13}$$

Proof. From (12) it follows:

$$\sum_{u<v} \Big(\sum_{i\leqq u} c_i\Big) = \sum_{u<v} \overline{\overline{u}}\cdot c_u = \overline{\overline{v}}\cdot\overline{\overline{v}} \sup c_i = \overline{\overline{v}} \sup c_i = \sum_{i<v} c_i \,.$$

Next, based on the properties of mitotic ordinals we prove the following theorem.

THEOREM 4. *Let u be limit ordinal and $(c_i)_{i\in u}$ a nondecreasing sequence of type u of cardinals c_i. Then*

$$\prod_{i<u} c_i = \prod_{j<u} \Big(\prod_{i<j} c_i\Big) \,. \tag{14}$$

Proof. Without loss of generality, we may assume that the normal expansion of u has two terms and is given by

$$u = \omega^e p + \omega^h q \,.$$

Hence, by Lemma 3, without loss of generality, we may assume that u is a sum of two mitotic ordinals w and r, i.e.

$$u = w + r \text{ with } \overline{\overline{w}} \geqq \overline{\overline{r}} \geqq \aleph_0 \,. \tag{15}$$

Thus, to prove (14), it is enough to show that

$$\prod_{i<w+r} c_i = \prod_{i<w+r} \Big(\prod_{i<j} c_i\Big) \,. \tag{16}$$

However, since u is a limit ordinal and $c_j \leqq \prod_{i<j+1} c_i$ for every $j < u$, we see that the left side of the equality sign in (16) is less than or equal to the right side. Thus, it is enough to show that the right side is less than or equal to the left side.

Since w and r are both mitotic ordinals, in view of (15) and (9) we have:

$$\begin{aligned}
\prod_{j<w+r} \Big(\prod_{i<j} c_i\Big) &= \prod_{i<w} \Big(\prod_{i<j} c_i\Big) \cdot \prod_{j<r} \Big(\prod_{i<w+j} c_i\Big) \\
&\leqq \big(\sup_{i<w} c_i\big)^{\overline{\overline{w}} \cdot \overline{\overline{w}}} \cdot \prod_{j<r} \Big(\prod_{j<w} c_i \cdot \prod_{i<j} c_{w+i}\Big) \\
&\leqq \big(\sup_{i<w} c_i\big)^{\overline{\overline{w}}} \cdot \big(\sup_{i<w} c_i\big)^{\overline{\overline{w}} \cdot \overline{\overline{r}}} \cdot \big(\sup_{i<r} c_{w+i}\big)^{\overline{\overline{r}} \cdot \overline{\overline{r}}} \\
&= \big(\sup_{i<w} c_i\big)^{\overline{\overline{w}}} \cdot \big(\sup_{i<r} c_{w+i}\big)^{\overline{\overline{r}}} \\
&= \prod_{i<w} c_i \cdot \prod_{i<r} c_{w+i} = \prod_{i<w+r} c_i
\end{aligned}$$

as desired.

Finally, based on (14) we obtain the formula analogous to (13) for the product of cardinals.

THEOREM 5. *Let $(c_i)_{i<v}$ be an infinite nondecreasing sequence of type v of cardinals c_i. Then*

$$\prod_{i<v} c_i = \prod_{j<v} \Big(\prod_{i \leq j} c_i\Big) \,. \tag{17}$$

Proof. As the proof indicates, without loss of generality we may assume $v = u + 1$ where u is a limit ordinal. But then from (14) it follows:

$$\prod_{i<u+1} c_i = \Big(\prod_{i<u} c_i\Big) c_u = \prod_{j<u} \Big(\prod_{i<j} c_i\Big) \cdot c_u = \prod_{j<u+1} \Big(\prod_{i \leq j} c_i\Big) \,.$$

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