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INTERPOLATION SETS FOR UNIFORM ALGEBRAS

ARNE STRAY

Let A be a uniform algebra on a compact Hausdorff space X and let $E \subset X$ be a closed subset which is a G_δ. Denote by B_E all functions on $X \backslash E$ which are uniform limits on compact subsets of $X \backslash E$ of bounded sequences from A.

It is proved that a relatively closed subset S of $X \backslash E$ is an interpolation set and an intersection of peak sets for B_E if and only if each compact subset of S has the same property w. r. t. A. In some special cases the interpolation sets for B_E are characterized in a similar way. A method for constructing infinite interpolation sets for A and B_E whenever $x \in E$ is a peak point for A in the closure of $X \backslash \{x\}$, is presented.

With X as above let $S \subset X$ be a topological subspace. Then $C_b(S)$ denotes all bounded continuous complexvalued functions on S and we put $||f|| = \sup\{|f(x)|: x \in S\}$ if $f \in C_b(S)$.

A subset S of $X \backslash E$ closed in the relative topology is called an interpolation set for B_E if any $f \in C_b(S)$ has an extension to $X \backslash E$ which belongs to B_E. If there exists $f \in B_E$ such that $f = 1$ on S and $|f| < 1$ on $(X \backslash E) \backslash S$, we call S a peak set for B_E. If S has both this properties it is called a peak interpolation set for B_E. Peak and interpolation sets for A are defined in the same way.

It is easy to see that B_E is a Banach algebra with the norm $N(f) = \inf\{\sup_n ||f_n||: \{f_n\} \subset A, f_n \to f$ uniformly on compact subsets of $X \backslash E\}$. It is an interesting problem in itself when this norm coincides with sup norm on $X \backslash E$.

In case $X = \{z: |z| \leqq 1\}$ and A is the classical disc algebra of all continuous functions on X which are analytic in $D = \{z: |z| < 1\}$ the interpolation sets for B_E (where E is a closed subset of ∂X) are characterized by that $S \cap \partial X$ has zero linear measure and that $S \cap D$ is an interpolation set for $H^\infty(D)$, the algebra of all bounded analytic functions on D. This result was obtained in [8] by E. A. Heard and J. H. Wells.

Their work has been generalized in different ways. Various authors have considered more general subsets E of $\{z: |z| \leqq 1\}$ and more general algebras of analytic functions. ([2], [3], [4], [6], [9] and [10]).

In this note we wish to generalize the results of Heard and Wells to the setting of uniform algebras. We start with an extension of Theorem 2 in [8].

THEOREM 1. *Let $S \subset X \backslash E$ be closed in the relative topology. Assume X is the maximal ideal space of A. The following statements are equivalent*:

(i) *Given $g \in C_b(S)$, $\varepsilon > 0$ and an open set $U \supset S$, there exists $f \in B_E$ such that $f = g$ on S, $\|f\| = \|g\|$, $|f| < \varepsilon$ on $(X \backslash E) \backslash U$ and $N(f) \leqq \|g\|(1 + \varepsilon)$.*

(ii) *There exists a constant M such that if $g \in C_b(S)$, $\varepsilon > 0$ and $U \supset S$ is open we can find $f \in B_E$ such that $f = g$ on S, $|f| < \varepsilon$ on $(X \backslash E) \backslash U$ and $N(f) \leqq M\|g\|$.*

(iii) *Each compact subset of S is an interpolation set and an intersection of peak sets for A.*

Proof. That (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii). Choose $g \in C(K)$ with $\|g\| = 1$.

Let $K \subset S$ be compact, U and W open sets such that $K \subset W \subset \bar{W} \subset U \subset \bar{U} \subset X \backslash E$ and choose $\varepsilon > 0$. By hypothesis there exists $g_1 \in B_E$ equal to g on K such that $|g_1| < \varepsilon/2$ on $\bar{U} \backslash W$ and $N(g_1) \leqq M$.

Hence we can find $g_2 \in A$ with $\|g_2\| \leqq M$, $|g - g_2| < \varepsilon$ on K, $|g_2| < \varepsilon$ on $\bar{U} \backslash W$ and $\|g_2\| \leqq M$. By ([8], Lemma 2) applied to the restriction map $B_E \to C(K)$ we get that any $g \in C(K)$ we get that any $g \in C(K)$ has an extension f to X such that $f \in A$, $\|f\| \leqq M/(1 - \varepsilon)$ and $|f| < \varepsilon/(1 - \varepsilon)$ on $\bar{U}/W$. Essentially by Bishops "1/4 – 3/4-Theorem" (See [5], Th. 11.1 p. 52) we can use what is proved until now to find a compact set K_1 and $f_1 \in A$ such that $f_1 = 1$ on K_1, $|f_1| < 1$ on $U \backslash K_1$ and $K \subset K_1 \subset W$. By "Rossis Local Peak Set Theorem" ([5], p. 91) K_1 is a peak set for A and (iii) is proved.

It remains to prove (iii) $\Rightarrow$ (i). We only indicate how to modify our proof of Lemma 2.1 in [10] to apply to the present situation. As in that lemma we construct a sequence $\{f_n\}_{n=1}^{\infty} \subset A$ with the properties listed there. Let $t \in \langle 0, 1 \rangle$. The sum $\sum_1^{\infty} f_n = f \in B_E$ and the proof of Lemma 2.1 gives (i) if we can show that $N(f) \leqq 1 + t$. This is obtained by constructing $\{f_n\}$ such that $\|f_n + f_{n+1}\| \leqq 1 + 1/2 \cdot t$ for $n = 0, 1, \cdots$.

This can be obtained if when constructing f_{n+1} we arrange it so that $|f_n + f_{n+1}| = |f_n| + |f_{n+1}|$ on $K_{n+1} \cup K_{n+2}$ (K_{n+1}, K_{n+2} as in [10]) and then if needed, modify f_{n+1} to $h \cdot f_{n+1}$ where $h \in A$ equals $1 = \|h\|$ on $K_{n+1} \cup K_{n+2} \cup K_{n+3}$, is small where $|f_n + f_{n+1}|$ may be large and has a small imaginary part.

We now state a lemma which is due to A. M. Davie:

LEMMA 1. *There exists a sequence $\{Q_k\}_{k=1}^{\infty}$ of polynomials with the following properties*:

(1) $\sum_1^n Q_k(z) \to 1$ *uniformly on compact subset of* $\{z: |z| < 1\}$

(2) $Q_k(1) = 0$ *for* $k = 1, 2, \cdots$ *and* $\sum_1^{\infty} |Q_k(z)| \leqq 3$ *if* $|z| \leqq 1$.

For a construction of $\{Q_k\}$ see the proof of Theorem 2.4 in [1]. We now have:

THEOREM 2. *Let E be a peak set for A and let $S \subset X \backslash E$ be closed in the relative topology. The following statements are equivalent*:

(i) *S is an interpolation set for B_E.*

(ii) *There exists $M > 0$ such that if $K \subset S$ is compact and $g \in C(K)$ we can find $f \in A$ equal to g an K and with $\|f\| \leqq M\|g\|$.*

Proof. (ii) follows from (i) as in the first part of the proof that (ii) $\Rightarrow$ (iii) in Theorem 1. For the converse an argument used by Davie in [1] works: Choose $h \in A$ peaking on E and put $E_k = S \cap \{x: |Q_k \circ h(x)| \geqq \varepsilon \cdot h^{-k}\}$ where $\varepsilon > 0$ is given in advance. Let $g \in C_b(S)$ with $\|g\| = 1$. Choose by hypothesis $g_k \in A$ equal to g on E_k with $\|g_k\| \leqq M$ and put $G = \sum_{k=1}^{\infty} (Q_k \circ h) \cdot g_k$. Then by Lemma 1 $G \in B_E$, $\|G\| \leqq 3M$ and if $x \in S$ we have

$$|G(x) - g(x)| = \left|\sum_1^\infty (g_k(x) - g(x)) Q_k \circ f(x)\right|$$
$$\leqq \sum_1^\infty \in 2^{-k} = \varepsilon .$$

By Lemma 2 in [8] (i) follows.

The hypothesis that E is a peak set for A seems unnecessary, but we needed it to apply Lemma 1. It would be of interest to get some examples where Theorem 2 holds without assuming E to be a peak set.

A case which deserves investigation is when A is an algebra of generalized analytic functions ([5], Ch VII) viewed as a uniform algebra on its maximal ideal space. Then B_E is very easy to describe whenever E is a closed subset of the Šilov boundary of A. In particular the norm $N(f)$ coincides with sup norm on $X \backslash E$ in this case.

We want to give two examples where a more detailed description of the interpolation sets for B_E can be given.

(a) Let $U \subset \boldsymbol{C}^n$ be a strictly pseudoconvex domain with C^2 boundary and let X be the closure of U. Let A be the algebra $A(U) = \{f \in C(X): f|_U \text{ is analytic}\}$.

In this case Theorem 2 is valid if E is any closed subset ∂U and the interpolation set S can then also be characterized by the following:

(I): Each compact subset of $S \cap \partial U$ is a peak interpolation set for A,

and

(II): $S \cap U$ is an interpolation set for $H^\infty(U)$, the algebra of all bounded analytic functions in U.

For a proof of this note that (i) $\Rightarrow$ (ii) in Theorem 2 holds whenever E is a closed G_δ. That (ii) $\Rightarrow$ (II) is a simple normal family argument and I also follows from (ii) by a result of N. H. Varopoulos [11] and since each $x \in \partial U$ is a peak point for $A(U)$ in this special case.

To obtain (i) from (I) and (II) one can argue as in the proof of Theorem 2.2 in [10]. To use that proof one needs an approximation result similar to Theorem 2.1 in [10]. This nontrivial result is contained in a recent work of R. M. Range [9].

(b) Assume A is a Dirichlet algebra on its Šilov boundary Y.

Let E be a peak interpolation set for A and let $S \subset X \backslash E$ be closed in the relative topology and assume $S \backslash Y$ countable. Then one can prove that S is an interpolation set for B_E if each compact subset of $S \cap Y$ is an interpolation set for A and if for some constant C the following result holds: If P is a nontrivial Gleason part for A and $S \cap P = z_1, z_2, \cdots$ and $\alpha_1, \alpha_2, \cdots$ are numbers such that $|\alpha_k| \leqq 1$ for $k = 1, 2, \cdots$ there exists $f \in H^\infty(P)$ such that $f(z_k) = \alpha_k$ for $k = 1, 2, \cdots$ and $|f| \leqq C$ on P. (For the necessary definitions see [5] on page 34, 142 and 161).

Using this hypothesis and the Wermer-Glicksberg decomposition ([5], Thm. 7.11, p. 45) we can prove that $S \cup E$ is an interpolation set for A. This is done in the same way as Glicksberg proves Theorem 4.1 in [7]. But then S is an interpolation set for B_E by Theorem 2.

In [8] Heard and Wells described an explicit method for constructing infinite interpolation sets for $B_{\{x\}}$ if $x \in X$ is a non-isolated peak point for A. Their method didn't depend on Carlesons characterization of the interpolating sequences for $H^\infty(D)$.

We indicate here how the polynomials $\{Q_k\}$ can be used for a similar construction avoiding an unnecessary hypothesis about connectedness which Heard and Wells assumed. ([8], Theorem 3).

THEOREM 3. *Let $x \in X$ be a peak point for A and $P \subset X \backslash \{x\}$ a set which contains x in its closure. Then an infinite interpolation set for $B_{\{x\}}$ contained in P can be constructed.*

Proof. Choose $\varepsilon > 0$ and $f \in A$ peaking at x. For $k = 1, 2, \cdots$ choose numbers n_k and m_k such that $n_k < m_k < n_{k+1}$ and put $H_k = \sum_{n_k}^{m_k} Q_j \circ f$. Using Lemma 1 it is easy to see that we can arrange it such that the sets $E_k = \{x\colon |H_k(x)| \geqq \varepsilon 2^{-k}\}$ and

$$B_k = P \cap \{x\colon |H_k(x) - 1| < \varepsilon 2^{-k}\}$$

are nonempty for $k = 1, 2, \cdots$ and that $E_i \cap E_j = \emptyset$ if $i \neq j$.

If we choose $x_k \in B_k$ for $k = 1, 2, \cdots$ then $S = \{x_k\}_{k=1}^{\infty}$ is an interpolation set for $B_{\{x\}}$. For if $g \in C_b(S)$ and we put $G = \sum_1^\infty g(x_k)H_k$ then $G \in B_{\{x\}}$, $||G|| \leqq 3||g||$ by Lemma 1 and $|G - g| < \varepsilon||g||$ on S.

Comments on Theorem 2:

We want to point out that the hypothesis that E be a peak set cannot be omitted. If A is any uniform algebra for which there exists an infinite interpolation set F not meeting the Šilov boundary, one obtains a counterexample by taking E to be a limit point of F and $S = F\backslash E$. For an example of such an algebera A we refer to Theorem 2.8. in [1]. On the other hand A. M. Davie has recently proved (private communication) that in case A is the algebra $R(X)$ and X is a compact plane set, Theorem 2 is valid without assuming E to be a peak set.

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UNIVERSITY OF OSLO