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A COMPLETE COUNTABLE $L^Q_{\omega_1}$ THEORY WITH MAXIMAL MODELS OF MANY CARDINALITIES

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Because of the compactness of first order logic, every structure has a proper elementarily equivalent extension. However, in the countably compact language $L^Q_{\omega_1}$ obtained from first order logic by adding a new quantifier Q and interpreting Qx as "there are at least ω_1 x's such that...," the situation is radically different. Indeed there are structures of countable type which are maximal in the sense of having no proper $L^Q_{\omega_1}$-extensions, and the class S of cardinals admitting such maximal structures is known to be large. Here it is shown that there is a countable complete $L^Q_{\omega_1}$ theory T having maximal models of cardinality κ for each $\kappa \geqq \beth_1$ which is in S. The problem of giving a complete characterization of the maximal model spectra of $L^Q_{\omega_1}$ theories T remains open: what classes of cardinals have the form $\mathrm{Sp}(T) = \{\kappa$: there is a maximal model of T of cardinality $\kappa\}$ for T a (complete, countable) $L^Q_{\omega_1}$ theory.

That S is large is shown in [4]. Assuming the *GCH*, it is particularly simple to describe: S is the set of uncountable cardinals which are less than the first uncountable measurable cardinal and not weakly compact. Here we will need the fact that $\beth_1 \in S$; this is proved in [4] without assuming the *GCH*. The countable compactness of $L^Q_{\omega_1}$ is shown in Fuhrken [2]. For additional results and references on the model theory of $L^Q_{\omega_1}$ see Kiesler [3].

1. Notation and preliminaries.

1.1. *Relatively common notation.* We identify cardinals with initial ordinals, and each ordinal with the set of smaller ordinals. We use α, β, γ for ordinals, κ, λ, μ for cardinals, and m, n for finite cardinals. $S(X) = \{t: t \subseteqq X\}$; cX is the cardinality of X; $\beth_1$ is the cardinality of the continuum; ω_1 the first uncountable cardinal; $\prod_{i \in Y} X_i$ the cartesian product; ${}^Y X$ the set of all functions on Y into X, $f \mid x$ the restriction of the function f to x.

The type $\tau\Sigma$ of a set Σ of formulas is the set of non-logical symbols occurring Σ.

In this paper all structures will be relational structures. Capital german letters are used for structures, and the corresponding roman letters for their universes. Alternatively we may write $|\mathfrak{A}|$ for the universe of $\mathfrak{A}$. The type $\tau\mathfrak{A}$ of $\mathfrak{A}$ is the set of non-logical symbols having denotations in $\mathfrak{A}$, so that $\mathfrak{A} = \langle A, \mathsf{s}^{\mathfrak{A}}\rangle_{s\in\tau\mathfrak{A}}$. We use sans serif letters for non-logical symbols, and if $\mathfrak{A}$ is understood we may use roman letters for the corresponding denotations, so $S = \mathsf{s}^{\mathfrak{A}}$. If S is a relation with rank $n+1$, and the last argument is a function of the first n places, we call S a function. If R_i $(i \in I)$ are relations, then $(\mathfrak{A}, R_i)_{i\in I}$ is a structure $\mathfrak{B}$ which results from $\mathfrak{A}$ by extending the type of $\mathfrak{A}$ to include new relation symbols R_i $(i \in I)$, where $\mathsf{R}_i^{\mathfrak{B}}$ is the relation R_i (appropriately restricted to A).

The phrase "κ admits a structure such that..." means "there is a structure $\mathfrak{A}$ such that $c\,|\mathfrak{A}| = \kappa$ and...."

1.2. *Less common notation, special sums and products.* As usual $\mathfrak{A} \prec \mathfrak{B}$ and $\mathfrak{A} \equiv \mathfrak{B}$ mean respectively that $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$, $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$. Similarly $\mathfrak{A} \equiv_{\omega_1} \mathfrak{B}$ means that $\mathfrak{A}, \mathfrak{B}$ are $L^Q_{\omega_1}$-equivalent, i.e. that $\mathfrak{A}, \mathfrak{B}$ have the same true $L^Q_{\omega_1}$-sentences, and $\mathfrak{A} \prec_{\omega_1} \mathfrak{B}$ means that $\mathfrak{A}$ is an $L^Q_{\omega_1}$-substructure of $\mathfrak{B}$, i.e. that $\mathfrak{A} \subseteqq \mathfrak{B}$ and for every $L^Q_{\omega_1}$ formula θ, and every assignment z in $\mathfrak{A}$, $\mathfrak{A} \models \theta[z]$ iff $\mathfrak{B} \models \theta[z]$. If K is a class of structures, $Th_{\omega_1} K$ is the set of $L^Q_{\omega_1}$-sentence true in every $\mathfrak{A} \in K$. If Σ is a set of sentences, $\mathrm{Mod}\,\Sigma$ is the class of structures (of some fixed type) such that $\Sigma \subseteqq Th_{\omega_1}\mathfrak{A}$.

Let $t \subseteqq \tau\mathfrak{A}$ and let $\phi \neq V \subseteqq |\mathfrak{A}|$. Then $\mathfrak{A}\,|\,(V, t)$ is the t-reduct of the substructure of $\mathfrak{A}$ determined by V, i.e. if $\mathfrak{B}$ is the substructure of $\mathfrak{A}$ determined by V, $\mathfrak{A}\,|\,(V, t)$ is the structure $\mathfrak{C}$ with universe $|\mathfrak{B}|$ and type t determined by $\mathsf{R}^{\mathfrak{C}} = \mathsf{R}^{\mathfrak{B}}$ for R in t. We write $\mathfrak{A}\,|\,t$ for $\mathfrak{A}\,|\,(|\mathfrak{A}|, t)$. If V is a unary relation symbol, then we will write $\mathfrak{A}\,|\,(\mathsf{V}, t)$ for (the relativized reduct) $\mathfrak{A}\,|\,(\mathsf{V}^{\mathfrak{A}}, t)$.

If t is a relational type, we can find a relational type $t^* \supseteqq t$, and a set $Sk(t)$ of first order sentences of type t^* with the following properties: (i) if $\tau\mathfrak{A} = t$, then there is an expansion $\mathfrak{A}^*$ of $\mathfrak{A}$ with $\tau\mathfrak{A}^* = t^*$ and $\mathfrak{A}^* \in \mathrm{Mod}\,Sk(t)$ (ii) if $\mathfrak{A}, \mathfrak{B} \in \mathrm{Mod}\,Sk(t)$ and $\mathfrak{A} \subseteqq \mathfrak{B}$, then $\mathfrak{A} \prec \mathfrak{B}$. In fact we may take $Sk(t)$ to be the set of sentences which assert that the Skolem relations satisfy their defining sentences, e.g.

$$\forall z[\forall y(R_\theta(x, y) \longrightarrow \theta(x, y)) \wedge (\exists y\theta(x, y) \longrightarrow \exists y R_\theta(x, y))] .$$

If $\langle \mathfrak{A}_i : i \in I\rangle$ is a family of relational structures all of type t, and having pairwise disjoint universes, then $\sum_{i\in I}\mathfrak{A}_i$ is the structure $\mathfrak{B}$ of type t such that $B = \bigcup_{i\in I} A_i$, and $\mathsf{R}^{\mathfrak{B}} = \bigcup_{i\in I} \mathsf{R}^{\mathfrak{A}_i}$ for each $\mathsf{R} \in t$. If the universes of the $\mathfrak{A}_i$ are not disjoint, then $\sum_{i\in I}\mathfrak{A}_i$ is $\sum_{i\in I}\mathfrak{A}'_i$ where $\mathfrak{A}'_i$ is some isomorphic copy of $\mathfrak{A}_i$, and the universes of the $\mathfrak{A}'_i$ are pairwise disjoint. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have different types, $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ is defined as follows. First expand each to a structure of type $\tau\mathfrak{A}_1 \cup \tau\mathfrak{A}_2$ by adding empty relations, to obtain $\mathfrak{A}'_1, \mathfrak{A}'_2$ respectively. Then

$\mathfrak{A}_1 \oplus \mathfrak{A}_2 = \mathfrak{A}'_1 + \mathfrak{A}'_2$.

Let $\langle \mathfrak{A}_i : i \in I \rangle$ be a family of structures, with $\tau\mathfrak{A}_i = t_i$. Choose $t'_i = \{R^i : R \in t_i\}$ pairwise disjoint copies of the t_i (i.e. $R \mapsto R^i$ is $1-1$ and R, R^i have the same rank). Let A_i $(i \in I)$ be new unary relation symbols. Define $\mathfrak{B} = \mathscr{S}\langle \mathfrak{A}_i : i \in I \rangle$ of type $t = \{A_i : i \in I\} \cup \bigcup\{t'_i : i \in I\}$ as follows: $|\mathfrak{B}| = \bigcup_{i \in I} A_i$, $A_i^{\mathfrak{B}} = A_i$, and $(R_i)^{\mathfrak{B}} = R^{\mathfrak{A}_i}$.

Define $\mathrm{P}_{i \in D}(\mathfrak{A}_i, \mathfrak{D}) = (\mathscr{S}(\mathfrak{D}, \sum_{i \in D} \mathfrak{A}_i), K)$, where $K = \{\langle x, i \rangle : i \in D$ and $x \in |\mathfrak{A}_i|\}$.

DEFINITION 1. (a) $\mathfrak{A}$ is maximal iff wherever $\mathfrak{A} \subseteqq \mathfrak{B}$ and $\mathfrak{A} \equiv_{\omega_1} \mathfrak{B}$ then $\mathfrak{A} = \mathfrak{B}$.

(b) $\mathfrak{A}$ is strongly maximal iff $\mathfrak{A} = (\mathfrak{A}', U^{\mathfrak{A}})$, where U is unary, and whenever $\mathfrak{A} \subseteqq \mathfrak{B}$, $\mathfrak{A} \equiv \mathfrak{B}$, and $cU^{\mathfrak{B}} = \aleph_0$, then $\mathfrak{A} = \mathfrak{B}$.

(c) S is the set of cardinals κ which admit a maximal model of countable type; $S' = \{\kappa \in S : \kappa \geqq \beth_1\}$.

(d) $\mathrm{Sp}(T) = \{\kappa : \kappa$ admits a maximal model of $T\}$

REMARK. This notion of strongly maximal is weaker than the notion of strongly maximal introduced in [4], but is all that is needed in this paper.

2. Products and preservation of $L^Q_{\omega_1}$-equivalence.

We will need to know that $L^Q_{\omega_1}$-equivalence is preserved under the operations Σ and p defined above. The results we need follow from Wojciechowska's generalizations of the Feferman-Vaught theorems on generalized products [5]. The following corollary of Wjociechowska's main theorem will suffice for our purpose. In this corollary, $\mathfrak{S}$ is an expansion of $\langle S(I), \cup, \sim \rangle$, $\mathfrak{A} = \langle \mathfrak{A}_i \rangle_{i \in I}$ is a family of structures (of fixed type) indexed on I, and $\mathscr{P}(\mathfrak{A}, \mathfrak{S})$ is the Feferman-Vaught generalized product [1].

COROLLARY 2.1. *Suppore that* $\mathfrak{A}_i \equiv_{\omega_1} \mathfrak{B}_i$, $i \in I$. *Then* $\mathscr{P}(\langle \mathfrak{A}_i \rangle_{i \in I}, \mathfrak{S}) \equiv_{\omega_1} \mathscr{P}(\langle \mathfrak{B}_i \rangle_{i \in I}, \mathfrak{S})$. *Similarly if* $\mathfrak{A}_i \prec_{\omega_1} \mathfrak{B}_i$, $i \in I$ *then* $\mathscr{P}(\langle \mathfrak{A}_i \rangle_{i \in I}, \mathfrak{S}) \prec_{\omega_1} \mathscr{P}(\langle \mathfrak{B}_i \rangle_{i \in I}, \mathfrak{S})$.

From this corollary we prove

COROLLARY 2.2. (a) *If* $\mathfrak{A}_i \equiv_{\omega_1} \mathfrak{B}_i$ *then* $\sum_{i \in I} \mathfrak{A}_i \equiv_{\omega_1} \sum_{i \in I} \mathfrak{B}_i$, *and if* $\mathfrak{A}_i \prec_{\omega_1} \mathfrak{B}_i$ *then* $\sum_{i \in I} \mathfrak{A}_i \prec_{\omega_1} \sum_{i \in I} \mathfrak{B}_i$.

(b) *If* $\mathfrak{A}_i \equiv_{\omega_1} \mathfrak{B}_i$ *then* $\mathrm{P}_{i \in D}(\mathfrak{A}_i, \mathfrak{D}) \equiv_{\omega_1} \mathrm{P}_{i \in D}(\mathfrak{B}_i, \mathfrak{D})$.

Proof of (a). If $c \notin |\mathfrak{A}|$, and U is a unary predicate not in $\tau\mathfrak{A}$, we define $\mathfrak{A}'$ of type $\tau\mathfrak{A} \cup \{U\}$ by

$$\mathfrak{A}' = (|\mathfrak{A}| \cup \{c\}, A, R^{\mathfrak{A}})_{R \in \tau\mathfrak{A}} .$$

In Feferman-Vaught [1] it is shown that the cardinal sum $\sum_{i\in I}\mathfrak{A}_i$ is a (relativized reduct of) a generalized product $\mathscr{P}(\langle\mathfrak{A}_i'\rangle_{i\in I}, \mathfrak{S})$. Thus we can obtain Corollary 2.2a from Corollary 2.1 and the following simple modification of Lemma 4.7 of Feferman-Vaught [1].

LEMMA 2.3. (a) *For every formula* θ *of* $L^Q_{\omega_1}$ *of type* $t\cup\{U\}$ *there is a formula* φ *of type* t *such that* θ *and* φ *have the same free variables and for all* $\mathfrak{A}$ *of type* t,

$$\mathfrak{A}'\models\theta\longleftrightarrow\varphi^U ,$$

(*where* φ^U *is obtained from* φ *by relativizing all quantifiers to* U).

(b) *Hence* $\mathfrak{A}\equiv_{\omega_1}\mathfrak{B}$ *iff* $\mathfrak{A}'\equiv_{\omega_1}\mathfrak{B}'$, *and* $\mathfrak{A}\prec_{\omega_1}\mathfrak{B}$ *iff* $\mathfrak{A}'\prec_{\omega_1}\mathfrak{B}'$.

Proof. The proof of (a) is an easy induction on θ based on the following fact: If φ is any formula of type $\tau\mathfrak{A}'$, and φ^* is obtained from φ by replacing each atomic subformula in which the variable x occurs by $\exists x(Ux\wedge\neg(x=x))$, then $\mathfrak{A}'\models\exists x(\neg Ux\wedge\varphi)\leftrightarrow\varphi^*$. Part (b) follows easily from part (a) using the fact that c is definable in $\mathfrak{A}'$. This proves the lemma.

Proof of Corollary 2.2b. We now consider the product $\mathrm{P}_{i\in D}(\mathfrak{A}_i, \mathfrak{D})$. We may assume that $0\notin D$ and that $i\notin|\mathfrak{A}_i|$, $i\in D$. Then we can form $\mathfrak{A}_i'$ as in Lemma 2.3a with $|\mathfrak{A}_i'|=|\mathfrak{A}_i|\cup\{i\}$, and $\mathfrak{A}_i''$ with $|\mathfrak{A}_i''|=|\mathfrak{A}_i|\cup\{i\}\cup\{0\}$. Let $\mathfrak{S}=\langle SD, \cup, \sim, R^{\mathfrak{S}}\rangle_{R\in\tau\mathfrak{D}}$ where $R^{\mathfrak{S}}=\{\langle\{x_0\},\cdots,\{x_{n-1}\}\rangle:\langle x_0,\cdots,x_{n-1}\rangle\in R^{\mathfrak{D}}\}$. We show that $\mathrm{P}_{i\in D}(\mathfrak{A}_i,\mathfrak{D})$ is isomorphic to a relativized reduct of the generalized product $\mathscr{P}_{i\in D}(\mathfrak{A}_i'', \mathfrak{S})$. Now $\mathfrak{C}=\mathrm{P}_{i\in D}(\mathfrak{A}_i,\mathfrak{D})=(\mathscr{S}(\mathfrak{D}, \sum_{i\in D}\mathfrak{A}_i), K)$ has type $t=(\tau\mathfrak{D})'\cup(\tau\mathfrak{A}_i)'\cup\{D, A, K\}$, where D denotes $|\mathfrak{D}|$ and A denotes $|\sum_{i\in D}\mathfrak{A}_i|$ and $K=\{\langle x, y\rangle: x\in\mathfrak{A}_i \text{ and } y=i\}$. (Thus $C=A\cup D$.) We define $\eta:|\mathfrak{C}|\rightarrow\prod_{i\in D}A_i''$ as follows: For $i\in D$, η_i is the function which is 0 except at i, where $\eta_i(i)=i$. For $a\in|\mathfrak{A}_i|$, η_a is the function which is 0 except at i, where $\eta_a(i)=a$. Clearly η is $1-1$. For $R\in t$ we write R_0 for the relation induced on $\prod_{i\in D}A_i''$ by R via η, i.e., $\mathfrak{S}\cong_\eta\langle D_0\cup A_0, R_0\rangle_{R\in t}$. We show that for each $R\in t$, R_0 is definable in $\mathscr{P}(\langle\mathfrak{A}_i\rangle_{i\in D}, \mathfrak{S})$. For $R\in t$ we define an acceptable sequence ξ_R such that R_0 is easily defined using Q_{ξ_R} (for the definition of acceptable sequence ξ, and of Q_ξ, see Feferman-Vaught [1]). To describe the sequence ξ_R we suppose that $I(x)$, $Z(x)$ are formulas of type $\tau(\mathfrak{A}_i'')$ which define i and 0 respectively, and that $\mathrm{Sing}(x)$ is a formula of type $\tau\mathfrak{S}$ which asserts that $X\subseteq D$ is a singleton.

Note that $f\in D_0$ iff $X_0=\{i: f(i)=0\}$ is a singleton, and $X_0\subseteq X_1=\{i: f(i)=i\}$. Thus $D_0=Q_{\xi_D}$, where ξ_D is the sequence which asserts

$$\mathrm{Sing}\,(X_0) \wedge X_0 \subsetneqq X_1\,.$$
$$X_0 = \left\{i\colon \mathfrak{A}_i'' \models \neg Z(v_0)\begin{bmatrix} v_0 \\ f(i)\end{bmatrix}\right\}$$
$$X_1 = \left\{i\colon \mathfrak{A}_i'' \models I(v_0)\begin{bmatrix} v_0 \\ f(i)\end{bmatrix}\right\}$$

(i.e., $\xi_D = \langle \mathrm{Sing}\,(\mathsf{X}_0) \wedge \mathsf{X}_0 \subsetneqq \mathsf{X}_1, \neg Z(v_0), I(v_0)\rangle$). Similarly A_0 is given by

$$\mathrm{Sing}\,(\mathsf{X}_0) < \mathsf{X}_0 \subsetneqq \mathsf{X}_1$$
$$\mathsf{X}_0\colon \neg Z(v_0)$$
$$\mathsf{X}_1\colon \neg I(v_0)\,.$$

Now $\langle f, g\rangle \in \mathsf{K}_0$ iff $f \in \mathsf{A}_0$, $g \in \mathsf{D}_0$, and $f(i) \neq 0$ exactly when $g(i) = i$. Thus K_0 is definable using the sequences for A, D and the sequence given by

$$\mathsf{X}_0 = \mathsf{X}_1$$
$$\mathsf{X}_0\colon \neg Z(v_0)$$
$$\mathsf{X}_1\colon I(v_1)\,.$$

For $\mathsf{R} \in \tau D$, use

$$\mathsf{R}\mathsf{X}_0\mathsf{X}_1$$
$$\mathsf{X}_0\colon I(v_0)$$
$$\mathsf{X}_1\colon I(v_1)$$

and for $\mathsf{R} \in \tau\mathfrak{A}_i$ use

$$\mathsf{X}_0 \neq 0$$
$$\mathsf{K}_0\colon \mathsf{R}v_0v_1$$

3. Main result.

3.1. *Some maximal structures with many automorphisms.*

Let $\mathscr{T} = \langle {}^{\underline{\omega}}2 \cup {}^{\omega}2, {}^{\omega}2, \subseteqq, {}^{n}2, F\rangle_{n\in\omega}$, where F is a four place relation: $Fabxy$ iff $a, b \in {}^{\omega}2$ and $x \subsetneqq a$, $y \subsetneqq b$ and $x, y \in {}^{n}2$ for some n. The structure $\langle {}^{\underline{\omega}}2, \subsetneqq\rangle$ is the full binary tree, ${}^{\omega}2$ is the set of branches, ${}^{n}2$ the set of nodes at the nth level, and for each pair of branches b, b' the set $\{(x, y)\colon Fbb'xy\}$ is an order preserving function on the nodes contained in b onto the nodes contained in b'. In [4], $\mathscr{T}$ was shown to be maximal.

We now construct two structures $\mathscr{T}_R$ and $\mathscr{T}_S$, both of type $\tau(\mathscr{T}) \cup \{\mathsf{B}\}$; in $\mathscr{T}_R$, B denotes the set R of eventually right turning branches; in $\mathscr{T}_S$, B denotes $R \cup \{c\}$, where c always turns left. More precisely,

$$\mathscr{T}_R = (\mathscr{T}, R) \quad \text{where} \quad R = \left\{b \in {}^{\omega}\{0, 1\}: \lim_{n\to\infty} b_n = 1\right\},$$

and

$$\mathscr{T}_S = (\mathscr{T}, S) \quad \text{where} \quad S = R \cup \{c\} \quad \text{and} \quad c \in {}^{\omega}\{0\}.$$

LEMMA 3.1. *Let $f: {}^{\underline{\omega}}2 \to 2$. Then there is a unique automorphism g of $\mathscr{T}$ such that for all n and $x \in |\mathscr{T}|$,*

$$(gx)_n = \begin{cases} x_n & \text{if } f(x \mid n) = 0 \\ 1 - x_n & \text{if } f(x \mid n) = 1 \quad (\textit{i.e., twist when } f = 1). \end{cases}$$

Proof. Clearly, g is $1-1$ and onto; it is also an automorphism since $x \subseteq y$ iff $g(x) \subseteq g(y)$, and any automorphism of $({}^{\underline{\omega}}2 \cup {}^{\omega}2, \subseteq)$ is an automorphism of $\mathscr{T}$.

LEMMA 3.2. *If $D \subseteq |\mathscr{T}| \sim \{c\}$ and D is finite, then there is an isomorphism g on $\mathscr{T}_R$ onto $\mathscr{T}_S$ such that for all $b \in D$, $g(b) = b$.*

Proof. Clearly we may assume that $D \subseteq {}^{\omega}2$. Let n be chosen so that if $b \in D$ then $b(m) = 1$ for some $m < n$. Let e be the branch such that $e(m) = 0$ for $m < n$ and $e(m) = 1$ when $m \geqq n$. Define $f: {}^{\underline{\omega}}2 \to 2$ by $f(e \mid m) = 1$ if $m \geqq n$, $f(x) = 0$ in all other cases. Let g be the automorphism of $\mathscr{T}$ induced by f as in Lemma 3.1. Clearly, if $b \in R$ and $b \neq e$ then $g(b) \in R$ since $g(b)_p = (b)_p$ except for finitely many p. Similarly, if $b \notin R$ and $b \neq e$, then $f(b) \notin R$. Finally $f(e) = e$, so f takes R to $R \cup \{c\}$.

3.2. *Main lemma. Next we show that for every $\kappa \in S$, $\kappa \geqq \beth_1$, we can find T with $\{\beth_1, \kappa\} \subseteq \mathrm{Sp}(T)$. In fact what we need is the following*

LEMMA 3.3. *For each $\kappa \in S$, $\kappa \geqq \beth_1$, there are structures $\mathfrak{A}_\kappa$, $\mathfrak{B}_\kappa$ such that*

(i) *$c\mathfrak{A}_\kappa = \beth_1$ and $c\mathfrak{B}_\kappa = \kappa$,*

(ii) *$\tau\mathfrak{A}_\kappa = \tau\mathfrak{B}_\kappa$ is countable and the same for all κ, and $\mathfrak{A}_\kappa \equiv_{\omega_1} \mathfrak{B}_\kappa$. Also, if $\Sigma = \bigcap_{\kappa \in S} Th_{\omega_1} \mathfrak{A}_\kappa$ then*

(iii) *$\mathfrak{C} \in \mathrm{Mod}\, \Sigma$ and $\mathfrak{B}_\kappa \subseteq \mathfrak{C}$ implies $\mathfrak{B}_\kappa = \mathfrak{C}$,*

(iv) *$\mathfrak{C} \in \mathrm{Mod}\, \Sigma$ and $\mathfrak{A}_\kappa \subseteq \mathfrak{C}$ implies $\mathfrak{A}_\kappa = \mathfrak{C}$.*

Proof. We construct $\mathfrak{A}_\kappa$, $\mathfrak{B}_\kappa$ from the structures $\mathscr{T}_R$, $\mathscr{T}_S$ defined above, and $\mathfrak{M}_\kappa$ which we now describe.

In [4] it was shown that for each $\kappa \in S$ there is a strongly maximal structure $\mathfrak{M}_\kappa$ of power κ and countable type. Since any expansion of a strongly maximal model is strongly maximal, we may assume without loss of generality that all $\mathfrak{M}_\kappa$ have the same type $t = \tau\,\mathrm{Sk}\,(t)$, and that $\mathfrak{M}_\kappa \in \mathrm{Mod\,Sk}\,(t)$. Thus for all κ, if $\mathfrak{M}_\kappa \subsetneqq \mathfrak{M}' \in \mathrm{Mod\,Sk}\,(t)$ then $\mathfrak{M}_\kappa \prec \mathfrak{M}'$. Hence there is a $U \in \tau\,\mathrm{Sk}\,(t)$ such that for all κ, $\mathfrak{M}_\kappa \subsetneqq \mathfrak{M}' \in \mathrm{Mod\,Sk}\,(t)$ and $cU^{\mathfrak{M}'} = \omega$ implies that $\mathfrak{M}_\kappa = \mathfrak{M}'$.

We now fix κ and construct $\mathfrak{A}_\kappa, \mathfrak{B}_\kappa$; to simplify notation we drop the subscript κ. By the downward Lowenheim-Skolem theorem for $L^Q_{\omega_1}$ there is $\mathfrak{N} \prec_{\omega_1} \mathfrak{M}$ with $c\mathfrak{N} = \beth_1$. Let $\mathfrak{N}_b$, $b \in R$, be pairwise disjoint copies of $\mathfrak{N}$, each disjoint from $\mathscr{T}$ and $\mathfrak{M}$, and let $\mathfrak{N}_c = \mathfrak{N}$. Let $\mathfrak{A}_1 = \sum_{b \in R} \mathfrak{N}_b$, $\mathfrak{B}_1' = \sum_{b \in R} \mathfrak{N}_b + \mathfrak{N} = \sum_{a \in S} \mathfrak{N}_a$, and $\mathfrak{B}_1 = \sum_{b \in R} \mathfrak{N}_b + \mathfrak{M}$.

Let H be the function on $\mathfrak{B}_1$ into $R \cup \{c\}$ defined by

$$H(x) = \begin{cases} b & \text{if} \quad x \in \mathfrak{N}_b \\ c & \text{if} \quad x \in \mathfrak{M} \end{cases}.$$

Let $\mathscr{T}_0$ be a copy of $\mathscr{T}$ disjoint from the structurres so far mentioned. For each $b \in R$, let G_b be a function on $\mathscr{T}_0$ onto $\mathfrak{N}_b$.

Now we define

$$\mathfrak{A} = (\mathscr{S}(\mathscr{T}_R, \mathfrak{A}_1, \mathscr{T}_0), H, G_b)_{b \in R}$$
$$\mathfrak{B}' = (\mathscr{S}(\mathscr{T}_S, \mathfrak{B}_1', \mathscr{T}_0), H, G_b)_{b \in R}$$
$$\mathfrak{B} = (\mathscr{S}(\mathscr{T}_S, \mathfrak{B}, \mathscr{T}_0), H, G_b)_{b \in R}\,.$$

It is evident that $c\mathfrak{A} = \beth_1$ and $c\mathfrak{B} = \kappa$, and that $\tau\mathfrak{A} = \tau\mathfrak{B}$ is countable. Moreover this type is independent of κ because all the $\mathfrak{M}_\kappa$ have the same type. To establish $\mathfrak{A} \equiv_{\omega_1} \mathfrak{B}$, we prove that $\mathfrak{A} \equiv_{\omega_1} \mathfrak{B}'$ and $\mathfrak{B}' \prec_{\omega_1} \mathfrak{B}$.

We now show that $\mathfrak{A} \equiv_{\omega_1} \mathfrak{B}'$. In fact, we show that if t is a finite subset of $\tau\mathfrak{A}$, then $\mathfrak{A} | t \cong \mathfrak{B}' | t$. Given the finite type t, let $D = \{b \in R\colon G_b \in t\}$. By Lemma 3.1, there is an isomorphism f on $\mathscr{T}_R$ onto $\mathscr{T}_S$ such that for all $b \in D$, $f(b) = b$. For each $b, b' \in S$ choose an isomorphism $g_{b,b'}$ on $\mathfrak{N}_b$ onto $\mathfrak{N}_{b'}$, with $g_{b,b'}$ the identity when $b = b'$. Now it is easily seen that we can extend f to an isomorphism on $\mathfrak{A} | t$ onto $\mathfrak{B}' | t$ by defining $f(x) = g_{b,f(b)}(x)$ for all $x \in \mathfrak{N}_b$ and $f(x) = x$ for $x \in \mathscr{T}'$.

We complete the proof that $\mathfrak{A} \equiv_{\omega_1} \mathfrak{B}$ by showing that $\mathfrak{B}' \prec_{\omega_1} \mathfrak{B}$. Let $\mathfrak{C} = (\mathscr{S}(\mathscr{T}_S, \mathfrak{A}_1, \mathscr{T}_0), H, G_b, c)_{b \in R}$ (treat c as the unary relation $\{c\}$). Now let $\mathfrak{D} = \mathfrak{C} \oplus (\mathfrak{M}, W^{\mathfrak{M}})$, $\mathfrak{D}' = \mathfrak{C} \oplus (\mathfrak{N}, W^{\mathfrak{N}})$ where $W^{\mathfrak{M}} = |\mathfrak{M}|$ and $W^{\mathfrak{N}} = |\mathfrak{N}|$. By Corollary 2.2a and the definition of $\oplus$, we have $\mathfrak{D}' \prec_{\omega_1} \mathfrak{D}$. It is enough to show that to every formula φ of type $\tau\mathfrak{B}'$, there is a fomula $\varphi^\#$ of type $\tau\mathfrak{D}'$ such that for all assignments z to $\mathfrak{B}'$, $\mathfrak{B}' \models \varphi[z]$ iff $\mathfrak{D}' \models \varphi^\#[z]$, and $\mathfrak{B} \models \varphi[z]$ iff $\mathfrak{D}^F \models \varphi^\#[z]$. We define $\varphi^\#$ inductively as follows:

$$R^\# u_0 \cdots u_{n-1} = R u_0 \cdots u_{n-1} \quad \text{for all} \quad R \in \tau\mathfrak{B}',\ R \neq H$$
$$H^\# u_0 u_1 = H u_0 u_1 \vee [W u_0 \wedge u_1 \approx c]$$
$$(\neg\varphi)^\# = \neg\varphi^\#$$

$$(\varphi \wedge \psi)^{\#} = \varphi^{\#} \wedge \psi^{\#}$$
$$(\exists u_0 \varphi)^{\#} = \exists u_0 \varphi^{\#}$$
$$(Q u_0 \varphi)^{\#} = Q u_0 \varphi^{\#} .$$

An easy induction on φ shows that the function taking φ into $\varphi^{\#}$ is as required. This completes the proof that $\mathfrak{A} \equiv_{\omega_1} \mathfrak{B}$.

Now we prove (iii). Suppose that $\mathfrak{C} \in \operatorname{Mod} \Sigma$ and $\mathfrak{B} \subseteqq \mathfrak{C}$. We must show that $\mathfrak{B} = \mathfrak{C}$. Since $\mathscr{T}$ is maximal it is easy to see that $\mathfrak{C}$ has the form $(\mathscr{S}(\mathscr{T}_S, \mathfrak{C}_1 \mathscr{T}_0), H^{\mathfrak{C}}, G_b^{\mathfrak{C}})_{b \in R}$, for some $\mathfrak{C}_1 \supseteqq \mathfrak{B}$. Thus for each $b \in R$, domain of $G_b^{\mathfrak{C}} = T_0^{\mathfrak{C}}$, since there is a sentence true in all $\mathfrak{A}$'s which asserts that G_b is a function with domain T_0. Thus $G_b^{\mathfrak{C}} = G_d^{\mathfrak{B}}$. It follows that in $\mathfrak{C}$, range G_b meets $H^{-1}(b)$. But in all $\mathfrak{A}$'s, if range G_b meets $H^{-1}(z)$, then $H^{-1}(z) \subseteqq$ range G_b, and this is expressible by the sentence

$$\forall z[\exists x \exists y(H(x, y) \wedge G_b(x, y)) \longrightarrow \forall y(H(y, z) \longrightarrow \exists x G(x, y))] .$$

Thus for each $b \in R$, $(H^{\mathfrak{C}})^{-1}(b) \subseteqq$ range G_b. Now in all $\mathfrak{A}$, $|\mathfrak{A}_1| \subseteqq \bigcup_{b \in R} H^{-1}(b)$. Since there are unary predicate symbols A_1, B such $(A_1)^{\mathfrak{A}} = |\mathfrak{A}_1|$, $B^{\mathfrak{A}} = R$, this is expressible by a first order sentence. Now $|A_1|^{\mathfrak{C}} = |\mathfrak{C}_1|$, and $B^{\mathfrak{C}} = S = R \cup \{c\}$, so we have

$$|\mathfrak{C}_1| \subseteqq \bigcup_{b \in R} (H^{\mathfrak{C}})^{-1}(b) \cup (H^{\mathfrak{C}})^{-1}(c) .$$

Since we already have $(H^{\mathfrak{C}})^{-1}(b) \subseteqq |\mathfrak{B}|$ for $b \in R$, it remains only to show that $(H^{\mathfrak{C}})^{-1}(c) \subseteqq \mathfrak{M}$. Now each $\mathfrak{M}$, and hence each $\mathfrak{N}_b$, is a model of Sk (t). It follows that if $\sigma \in$ Sk (t), then for each $\mathfrak{A}$ we have

$$\forall z(B(x) \longrightarrow \sigma^z)$$

where σ^z is obtained from σ by relativizing all quantifiers to $H(x, z)$ (treating z as a constant). In particular then,

$$\mathfrak{C}_c = \mathfrak{C}_1 | ((H^{\mathfrak{C}})^{-1}(c), \tau \mathfrak{M}) \in \operatorname{Mod} \operatorname{Sk}(t) .$$

Evidently, we also have $\mathfrak{M} \subseteqq \mathfrak{C}_c$. Also since in each $\mathfrak{A}$, $U^{\mathfrak{A}_z} = U^{\mathfrak{A}} \cap (H^{\mathfrak{A}})^{-1}(z)$ is countable for each $z \in \mathfrak{B}^{\mathfrak{A}}$, there is an $(L^Q_{\omega_1})$ sentence in Σ which asserts this. It follows that $U^{\mathfrak{C}_c} = U^{\mathfrak{C}} \cap (H^{\mathfrak{C}})^{-1}(c)$ is countable. Thus since $\mathfrak{M}$ is strongly maximal, it follows that $(H^{\mathfrak{C}})^{-1}(c) \subseteqq |\mathfrak{M}|$. This completes the proof of (iii); the proof of (iv) is exactly the same; replacing $\mathfrak{B}$ by $\mathfrak{A}$ and deleting reference to $\mathfrak{M}$ and c. This completes the proof of Lemma 3.3.

3.3. *Main theorem.*

THEOREM 3.4. *There is a complete countable $L^Q_{\omega_1}$-theory T such* *that for every* $\kappa \geqq \beth_1$, *T has a maximal model of power* κ *if there is a maximal structure of power* κ, *i.e.*, $\mathrm{Sp}(T) = S \cap \{\kappa : \kappa \geqq \beth_1\}$.

Proof. Let $\mathfrak{A}_\kappa, \mathfrak{B}_\kappa$ be the structures given by Lemma 3.3. Let $\{T_d : d \in \beth_1\} = \{Th_{\omega_1}\mathfrak{A}_\kappa : \kappa \in S'\}$. We now construct $L^Q_{\omega_1}$-equivalent maximal structures $\mathfrak{C}_\kappa$ for each $\kappa \in S'$, with $\mathfrak{C}_\kappa$ of power κ. Taking $T = Th_{\omega_1}\mathfrak{C}_\kappa$ will complete the proof. First let

$$\mathfrak{C}_{\kappa,d} = \begin{cases} \mathfrak{B}_\kappa & \text{if} \quad T_d = Th_{\omega_1}\mathfrak{B}_\kappa \\ \mathfrak{A}_\kappa & \text{otherwise, where} \quad Th_{\omega_1}\mathfrak{A} = T_d \,. \end{cases}$$

Let $\mathfrak{D}$ be any maximal structure with $|\mathfrak{D}| = \beth_1$, and let

$$\mathfrak{C}_\kappa = \underset{d \in D}{P}(\mathfrak{C}_{\kappa,d}, \mathfrak{D}) \,.$$

Evidently $\mathfrak{C}_\kappa$ is of power κ. By Corollary 2.2 for $\kappa, \lambda \in S$, and $\kappa, \lambda \geqq \beth_1$, $\mathfrak{C}_\kappa \equiv_{\omega_1} \mathfrak{C}_\lambda$.

It remains to show that each $\mathfrak{C}_\kappa$ $(\kappa \in S')$ is maximal. To simplify notation we omit the subscript κ from $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ in the remainder of the proof (thus we write $\mathfrak{C}_d$ for $\mathfrak{C}_{\kappa,d}$). Suppose $\mathfrak{C} \equiv_{\omega_1} \mathfrak{C}'$ and $\mathfrak{C} \subseteqq \mathfrak{C}'$. We must show $\mathfrak{C} = \mathfrak{C}'$. Clearly $\mathfrak{D} = \mathfrak{C} \,|\, (\mathsf{D}, t)$ for some type t. It is easy to see that if $\mathfrak{D}' = \mathfrak{C}' \,|\, (\mathsf{D}, t)$ then $\mathfrak{D} \equiv_{\omega_1} \mathfrak{D}'$ and $\mathfrak{D} \subseteqq \mathfrak{D}'$. Since $\mathfrak{D}$ is maximal it follows that $\mathfrak{D}' = \mathfrak{D}$. Notice that for $d \in D$, $\mathfrak{C}_d = \mathfrak{C} \,|\, (K^{-1}(d), t)$, where t is the type of $\mathfrak{A}$. Clearly $\forall x(\mathsf{D}x \vee \exists y(\mathsf{D}y \wedge \mathsf{K}xy))$ is true in $\mathfrak{C}$ and hence in $\mathfrak{C}'$. Thus, putting $\mathfrak{C}'_d = \mathfrak{C}' \,|\, ((\mathsf{K}^{\mathfrak{C}'})^{-1}(d), t)$ we have $|\mathfrak{C}'| = D \cup \bigcup_{d \in D} |\mathfrak{C}'|$. To see $\mathfrak{C} = \mathfrak{C}'$ it suffices to show that $\mathfrak{C}_d = \mathfrak{C}'_d$ for each $d \in D$.

It is evident that $\mathfrak{C}_d \subseteqq \mathfrak{C}'_d$. Although $\mathfrak{C} \equiv_{\omega_1} \mathfrak{C}'$, we cannot immediately conclude that $\mathfrak{C}_d \equiv_{\omega_1} \mathfrak{C}'_d$ (and hence by the maximality of $\mathfrak{C}_d$ that $\mathfrak{C}_d = \mathfrak{C}'_d$) because d may not be definable in $\mathfrak{C}$. However, to conclude that $\mathfrak{C}'_d = \mathfrak{C}_d$, it suffices to show, by parts (iii) and (iv) of Lemma 3.3, that $\mathfrak{C}'_d \in \mathrm{Mod}(\Sigma)$ where $\Sigma = \bigcap_{\kappa \in S} Th_{\omega_1}\mathfrak{A}_\kappa$. Now in $\mathfrak{C}$ we have, for each $\sigma \in \Sigma$,

$$\forall d(\mathsf{D}(d) \longrightarrow \sigma^d)$$

where σ^d is obtained from σ by relativizing all quantifiers to $K(x, d)$ (treating d as a constant). Thus, since $\mathfrak{C} \equiv_{\omega_1} \mathfrak{C}'$, we have for each $d \in D$, $\mathfrak{C}'_d \in \mathrm{Mod}\,\Sigma$. Thus $|\mathfrak{C}'_d| = |\mathfrak{C}_d|$, and hence $\mathfrak{C} = \mathfrak{C}'$, as was to be shown. This completes the proof of Theorem 3.4.

4. Problems.

(1) Is there a set Γ (Γ countable, Γ complete) of $L^Q_{\omega_1}$-sentences such that both $S \cap \mathrm{Sp}(\Gamma)$ and $S \sim \mathrm{Sp}(\Gamma)$ are cofinal with the first

measurable cardinal? I.e. is there a cardinal κ less than the first measurable such that whenever $\bigcup(\kappa \cap \mathrm{Sp}\,\Gamma) = \kappa$ we have $\mathrm{Sp}\,\Gamma \supseteqq S \sim K$?

(2) Is Theorem 3.4 true if we replace $\beth_1$ by ω_1?

(3) What is the least κ such that whenever $\bigcup(\kappa \cap \mathrm{Sp}\,(\Gamma)) = \kappa$ we have $\bigcup \mathrm{Sp}\,(\Gamma) \supseteqq S \sim \kappa$.

(4) More generally, we would like a characterization of those classes of cardinals of the form $\mathrm{Sp}\,(\Gamma)$ (Γ countable, Γ complete).

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