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SEMICLOSED OPERATORS

S. R. CARADUS

Elementary wellknown examples show that the sum of two closed operators need not even have a closed extension; the same is true for products, as one can see by taking the composition of maps $f \to f'$ followed by $f \to f(0)$ defined on the obvious domains in $C[0, 1]$. The natural question which then arises concerns the complexity of operators which might arise by taking repeated sums and products, starting with the closed operators. Somewhat unexpectedly, the answer is very simple: all can be reduced to products of two closed operators. Because of this, we shall distinguish this latter class by the name "semiclosed".

The most convenient proof of this theorem is obtained by first showing that semiclosed operators can always be decomposed in certain special ways.

LEMMA 1. (Canonical Decomposition). *Let X and Y be Banach spaces and let T: $X \to Y$ denote a semiclosed linear operator. Then there exists a Banach space Z and closed operators U: $Z \to Y$ and V: $X \to Z$ with the following properties*:

(a) $T = UV$

(b) *U is defined and continuous on all of Z and the range of U is exactly the range of T.*

(c) *V has the same domain as T and maps this domain one-to-one onto Z.*

Proof. Since T is semiclosed, there exists some decomposition $T = PQ$ where P: $W \to Y$ and Q: $X \to W$ are closed linear mappings and W is some Banach space. Then define

$$Z = \{(x, Qx, PQx): x \in D(T)\}$$

where $D(T)$ denotes the domain of T and Z is considered as a subspace of $X \times W \times Y$. Next take

$$V: x \longmapsto (x, Qx, PQx)$$

and

$$U: (x, Qx, PQx) \longmapsto PQx$$

with $D(V) = D(T)$ and $D(U) = Z$. Then the properties (a), (b) and (c) are easily verified.

We shall reserve the special notation $T = UV(Z)$ to represent the decomposition described in the above lemma. It is clear that, given T, the space Z is unique up to isomorphism. When T is a bounded operator with $D(T) = X$, we have $T = TI(X)$ and when T is closed, we can write $T = PG(G(T))$ where $G(T)$ denotes the graph of T, G is the mapping $x \mapsto (x, Tx)$ and P the projection $(x, Tx) \mapsto Tx$. Another immediate consequence of Lemma 1 is the fact that a semiclosed operator with closed domain is continuous.

Finally, it is interesting to recall the well-known procedure of "making a closed operator continuous" by renorming it domain with the graph norm $|x|_T = |x| + |Tx|$. Lemma 1 implies that the existence of such a procedure characterises semiclosed operators. We make this precise as follows:

COROLLARY 1. *Let $T\colon X \to Y$ denote an arbitrary linear operator. Then T is semiclosed if and only if there exists a norm $x \to |x|_T$ on $D(T)$ such that* (a) *the normed space $X_T = (D(T), |\cdot|_T)$ is complete* (b) *the induced operator $\tilde{T}\colon X_T \to Y$ is continuous.*

Proof. If T is semiclosed and $T = UV(Z)$, then define $|x|_T = |Vx|$. Conversely, if $|\cdot|_T$ exists with properties (a) and (b), then define $T = UV(X_T)$ with $Vx = x$ and $Ux = Tx$.

THEOREM 1. *If T_1 and T_2 are semiclosed operators, then so are $T_1 + T_2$ and T_1T_2 (whenever the latter are defined).*

Proof. Suppose we have $T_i = U_iV_i(Z_i)$, $i = 1, 2$. Then we simply construct the required decompositions.

(i) Let $W = \{(x, V_1x, V_2x)\colon x \in D(T_1 + T_2)\} \subseteqq X \times Z_1 \times Z_2$; $V\colon X \to W$, $D(V) = D(T_1 + T_2)$, $Vx = (x, V_1x, V_2x)$; $D(U) = W$, $U(x, V_1x, V_2x) = (T_1 + T_2)x$. Then $T_1 + T_2 = UV(W)$.

(ii) Let $\hat{W} = \{(x, V_2x, V_1T_2x)\colon x \in D(T_1T_2)\} \subseteqq X \times Z_2 \times Z_1$; $\hat{V}\colon X \to \hat{W}$, $D(\hat{V}) = D(T_1T_2)$, $\hat{V}x = (x, V_2x, V_1T_2x)$; $D(\hat{U}) = \hat{W}$, $\hat{U}(x, V_2x, V_1T_2x) = T_1T_2x$. Then $T_1T_2 = \hat{U}\hat{V}(\hat{W})$.

Verifications of the above assertions are straight forward.

The above theorem shows that the property of being semiclosed is algebraically stable. In addition, it persists in other useful ways.

THEOREM 2. *Let $T\colon X \to Y$ denote a semiclosed operator. Then*

(i) *if X is separable (or more generally, if X admits quasi-complements* [7], [5]) *then T has a densely defined semiclosed extension.*

(ii) *if* X_0 *is a subspace of* X *and* X_0 *is the domain of some closed operator, then* T_0, *the restriction of* T *to* X_0, *is a semiclosed operator.*

Proof. (i) Suppose $D(T)$ is not dense in X and write $D = \overline{D(T)}$. Then if $\hat{D}$ is a quasicomplement of D, we can define the projection map $\pi: D \oplus \hat{D} \to D$. It is easy to verify that π is closed so that $T\pi$ is a semiclosed extension of T. Finally, straight forward arguments show that $T\pi$ is densely defined. (ii) If X_0 is the domain of some closed operator, then X_0 is also the range of a closed operator $S: Z \to X$ for some Z. Hence X_0 is also the range of a one-to-one closed operator $\hat{S}: Z/N(S) \to X$. Now $\hat{S}\hat{S}^{-1}$ is semiclosed and is the restriction I_0 of the identity operator to X_0. Hence $TI_0 = T_0$ is semiclosed.

REMARK. It is known that not every subspace is the domain of a closed operator. Kaashoek [3] draws attention to this point by giving a simple construction in Banach space of a dense subspace of finite codimension. A wellknown theorem [4] indicates that such a subspace cannot be the range of any closed operator and hence cannot be the domain of any closed operator.

We now introduce a topology in the class $\mathscr{S}(X, Y)$ of semiclosed operators $X \to Y$. Let $T \in \mathscr{S}(X, Y)$ and suppose α denotes a canonical decomposition $T = UV(Z)$ for T. Then, for $\varepsilon > 0$, write

$$\begin{aligned}\mathscr{N}(T; \alpha, \varepsilon) = \{S \in \mathscr{S}(X, Y)\colon\ & D(S) = D(T),\\ & S \text{ has canonical decomposition } S = \tilde{U}V(Z),\\ & \| U - \tilde{U} \| < \varepsilon\}.\end{aligned}$$

The topology generated in $\mathscr{S}(X, Y)$ by the semibasic sets $\mathscr{N}(T; \alpha, \varepsilon)$ will be denoted by τ. Since applications visualized and examples in the current literature (e.g. [8]) involve families of operators which are all defined on the same domain, the above definition is not as restrictive as would appear. It is also possible to show that it suffices to choose, for each T, just one canonical decomposition α_T; the topology generated is the same.

The next thing to observe is that, if $B(X, Y)$ denotes the space of bounded operators defined on all of X, then τ restricted to $B(X, Y)$ is the uniform operator topology. For, on the one hand, it is evident that $\{S \in B(X, Y)\colon \| S - T \| < \varepsilon\} \subseteq \mathscr{N}(T; \alpha, \varepsilon)$ where $T \in B(X, Y)$ and α is given by $T = TI(X)$. Conversely if β is any canonical decomposition for $T \in B(X, Y)$ then, by uniqueness, β has the form $T\phi.\ \phi^{-1}(Z)$ where ϕ is an isomorphism of Z onto X. Hence if $S \in \mathscr{N}(T; \beta, \varepsilon) \cap B(X, Y)$, then S can be written $S\phi \cdot \phi^{-1}(Z)$ and thus $\mathscr{N}(T; \beta, \varepsilon) \cap B(X, Y) \subseteqq \{S \in B(X, Y): \| S - T \| < \varepsilon \| \phi^{-1} \|\}$.

The relationship between τ and class of closed operators in $\mathscr{S}(X, Y)$ is not so evident. A well known "generalised convergence" for closed operators is throughly treated in [4]. An application of Theorem IV. 2.29 of [4] shows that if $\{T_n\}$ is a sequence of closed operators converging to closed operator T in the topology τ, then $\{T_n\}$ also converges to T in the generalised sense.

Returning now to the study of τ on $\mathscr{S}(X, Y)$, we can show that τ has all reasonable properties for which one might hope.

THEOREM 3. *τ is a locally convex Hausdorff topology on $\mathscr{S}(X, Y)$.*

Proof. We will show, in fact, that each $\mathscr{N}(T; \alpha, \varepsilon)$ is convex. Let $S_i \in \mathscr{N}(T; \alpha, \varepsilon)$, $i = 1, 2$ with α given by $T = UV(Z)$. Then S_i can be written $U_i V(Z)$ with $\| U - U_i \| < \varepsilon$. If $a_i > 0$ with $a_1 + a_2 = 1$, then $a_1 S_1 + a_2 S_2 = \bar{U} V(Z)$ where $\bar{U}(Vx) = a_1 S_1 + a_2 S_2 x$. Hence

$$\| (U - \tilde{U}) Vx \| = \| a_1(U - U_1) Vx + a_2(U - U_2) Vx \| \leqq \varepsilon \| Vx \| .$$

Secondly, suppose T_1 and T_2 are semiclosed operators not separated by τ. Then in particular, for each $\varepsilon > 0$ and canonical decompositions α_1 and α_2, respectively, for T_1 and T_2, there must exist $S \in \mathscr{N}(T_1; \alpha_1, \varepsilon) \cap \mathscr{N}(T_2; \alpha_2, \varepsilon)$. Evidently $D(T_1) = D(T_2)$ and without loss of generality, we may restrict considerations to α_1 and α_2 acting through the same intermediate space Z. So we have $T_i = U_i V_i(Z)$ and S has two decompositions $S = \tilde{U}_i V_i(Z)$ such that $\| \tilde{U}_i - U_i \| < \varepsilon$, $i = 1, 2$. By uniqueness, there exists an automorphism ϕ of Z such that $V_1 = \phi V_2$ and $\tilde{U}_2 = \tilde{U}_1 \phi$. Hence, for any $x \in D(T_1) = D(T_2)$, we have

$$\begin{aligned} \| (T_1 - T_2)x \| &= \| (U_1 - \tilde{U}_1) V_1 x + (\tilde{U}_2 - U_2) V_2 x \| \\ &\leqq \varepsilon (\| V_1 x \| + \| V_2 x \|) . \end{aligned}$$

By holding the α_i fixed, we see that, for each x, $T_1 x - T_2 x$ can be made arbitrarily small. Hence $T_1 = T_2$.

THEOREM 4. *The mappings $(T_1, T_2) \to T_1 + T_2$ and $(\lambda, T) \to \lambda T$ are jointly continuous in $\mathscr{S}(X, Y)$. Moreover if $X = Y$, then $(T_1, T_2) \to T_1 T_2$ is separately continuous.*

Proof. (Sketch) Using by now familiar methods, we can obtain relations such as

$$\mathscr{N}(T_1; \alpha_1, \varepsilon_1) + \mathscr{N}(T_2; \alpha_2, \varepsilon_2) \subseteqq \mathscr{N}(T_1 + T_2; \alpha_1 + \alpha_2, \varepsilon_1 + \varepsilon_2)$$

where $\alpha_1 + \alpha_2$ is obtained from α_1 and α_2 by the construction of Theorem 1. Similar relations for multiplication are obtained.

We now turn attention toward possible applications. Our ultimate goal would be to subsume portions of the theory of partial differential operators with constant coefficients in a sufficiently general understanding of the topological algebras generated by the closed operators which can be obtained from the differentiation operations. We would therefore be considering commutative topological algebras obtained by fixing n commuting generators $D_1 \cdots D_n$ in $\mathscr{S}(X, X)$, all defined on the same domain. The development of the theory of such algebras has almost exclusively involved additional assumptions which have the effect of making the spectrum of each element in the algebra a compact subset of the complex plane. See for example [1] and [6]. In our case, however, we must allow unbounded spectra if we are to obtain information about spectral properties of differential operators. While it is true that some progress in this direction has been made in [9] and [10], much work is yet required before the applications envisaged above can be carried through.

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QUEEN'S UNIVERSITY
KINGSTON, ONTARIO