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ON A DECOMPOSITION OF TRANSFORMATIONS IN INFINITE MEASURE SPACES

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A decomposition theorem for a measure preserving transformation T on a σ-finite and infinite measure space $(\Omega, \mathscr{B}, m)$ is proved and ergodic theorems are considered.

1. Introduction. A measure preserving transformation T on $(\Omega, \mathscr{B}, m)$ is called of *zero type*, if

$$\lim_{N\to\infty} m(T^{-N}A \cap A) = 0$$

for any measurable set A with $m(A) < \infty$. The transformation T is called of *positive type*, if

$$\limsup_{N\to\infty} m(T^{-N}A \cap A) > 0$$

for any measurable set A with $m(A) > 0$. Krengel and Sucheston [4] showed that every measure preserving transformation can be decomposed into two measure preserving transformations, acting on disjoint invariant measurable sets, such that one of them is of zero type and the other is of positive type. However it seems that, in order to apply this result to ergodic theory, more detailed considerations are necessary. In this paper, we shall improve the result by introducing new concept of positivity and then, applying the obtained result, extend ergodic theorems of Brunel and Keane [1] to infinite measure spaces.

2. The decomposition theorem. A measure preserving transformation T will be called of *weakly positive type*, if T is of positive type and satisfies

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} m(T^{-k}A \cap A) = 0$$

for any measurable set A with $m(A) < \infty$. The transformation T will be called of *strongly positive type*, if T satisfies

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} m(T^{-k}A \cap A) > 0$$

for any measurable set A with $m(A) > 0$.

THEOREM 1. *If T is a measure preserving transformation on*

$(\Omega, \mathscr{B}, m)$, then Ω uniquely decomposes into three measurable sets Ω_0, Ω_+ and Ω_{++}, invariant under T, such that T restricted to Ω_0 is of zero type, T restricted to Ω_+ is of weakly positive type, and T restricted to Ω_{++} is of strongly positive type. Moreover, Ω_{++} is a union of countably many measurable sets of finite measure which are invariant under T.

Proof. Let $\mathscr{I} = \{A \in \mathscr{B}; T^{-1}A = A$ and $m(A) < \infty\}$. Since m is σ-finite, there exist countably many sets A_n in $\mathscr{I}$ such that $\Omega_{++} = \bigcup_n A_n$ is the union of $\mathscr{I}$. Let $E \subset \Omega_{++}$ be any measurable set of positive measure. Then $m(E \cap A) > \varepsilon$ for some $A \in \mathscr{I}$ and $\varepsilon > 0$. Hence we have

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} m(T^{-k}(E \cap A) \cap A) > 0 ,$$

from which it follows that

$$\lim_{N\to 1} \text{sum} \frac{1}{N} \sum_{k=0}^{N-1} m(T^{-k}E \cap E) > 0 .$$

Thus T restricted to Ω_{++} is of strongly positive type. On the other hand, since $\Omega - \Omega_{++}$ contains no invariant measurable set of finite positive measure, it follows (cf. [2, pp. 40–41] and [3]) that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} m(T^{-k}A \cap A) = 0 \tag{1}$$

for any measurable set $A \subset \Omega - \Omega_{++}$ with $m(A) < \infty$. This together with [4, Theorem 2.1] implies that $\Omega - \Omega_{++}$ decomposes into two measurable sets Ω_0 and Ω_+, invariant under T, such that T restricted to Ω_0 is of zero type and T restricted to Ω_+ is of weakly positive type. The uniqueness of such a triple of measurable sets is easily checked, and hence we omit the details.

We note here that the part Ω_{++} above is nothing less than the seat of the maximal finite equivalent invariant measure. This follows from [5, Theorem 8].

3. An application to ergodic theorems. The following definition is due to Brunel and Keane [1].

DEFINITION. A sequence $k_1, k_2, \cdots$ of strictly increasing nonnegative integers is called *uniform*, if there exist

(i) a strictly L-stable $(X, \mathscr{M}, \mu, \varphi)$;

(ii) a $Y \in \mathscr{M}$ such that $\mu(Y) > 0 = \mu(\partial Y)$, where ∂Y denotes the boundary of Y;

(iii) a point $y \in X$ such that $k_i = k_i(y, Y)$ for each i, where $k_i(y, Y)$ is defined recursively as:

$$k_1(y, Y) = \min\{j \geqq 0 ; \quad \varphi^j y \in Y\} ,$$
$$k_i(y, Y) = \min\{j > k_{i-1}(y, Y) ; \quad \varphi^j y \in Y\} \qquad (i > 1) .$$

Brunel and Keane showed in [1] that if T is a measure preserving transformation on a finite measure space then, for every uniform sequence $k_1, k_2, \cdots$, the average

$$\frac{1}{N} \sum_{i=1}^{N} f(T^{k_i} \cdot)$$

converges in the mean and almost everywhere. In this section we shall extend this result to σ-finite and infinite measure spaces.

THEOREM 2. *Let T be a measure preserving transformation on a σ-finite and infinite measure space $(\Omega, \mathscr{B}, m)$. If $f \in L^1(\Omega, \mathscr{B}, m)$ and if $k_1, k_2, \cdots$ is a uniform sequence, then*

$$f^*(\omega) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} f(T^{k_i} \omega) \tag{2}$$

exists almost everywhere and $f^* \in L^1(\Omega, \mathscr{B}, m)$.

Proof. Let Ω_0, Ω_+ and Ω_{++} be as in Theorem 1. Since Ω_{++} is a union of countably many invariant measurable sets of finite measure, it follows from [1] that (2) exists almost everywhere on Ω_{++}. The almost everywhere convergence of (2) on $\Omega_0 \cup \Omega_+$ can be shown as follows. Here it may and will be assumed without loss of generality that $f \geqq 0$ and $\Omega = \Omega_0 \cup \Omega_+$. The Birkhoff individual ergodic theorem then implies that

$$g(\omega) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j \omega)$$

exists almost everywhere, $g \in L^1(\Omega, \mathscr{B}, m)$, and g is invariant under T. Since $\Omega = \Omega_0 \cup \Omega_+$ by assumption, it follows that $g = 0$ almost everywhere. This together with fact, established by Brunel and and Keane [1], that uniform sequences have positive density implies that Cesàro averages of $f(T^{k_i} \omega)$ converge almost everywhere to zero. The proof is complete.

REMARK. It follows easily from Theorem 2 that if $1 \leqq p < \infty$ and $f \in L^p(\Omega, \mathscr{B}, m)$, then

$$f^*(\omega) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} f(T^{k_i} \omega)$$

exists almost everywhere and $f^* \in L^p(\Omega, \mathscr{B}, m)$.

The proof of the following theorem is similar to the one given by Krengel and Sucheston [4].

THEOREM 3. *Let T be a measure preserving transformation on a σ-finite and infinite measure space $(\Omega, \mathscr{B}, m)$. If $k_1, k_2, \cdots$ is a uniform sequence, $1 < p < \infty$, and $f \in L^p(\Omega, \mathscr{B}, m)$, then*

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{i=1}^{N} f(T^{k_i}\cdot) - f^* \right\|_p = 0 .$$

Proof. Since $f^* = 0$ almost everywhere on $\Omega_0 \cup \Omega_+$, it suffices to show, under the assumption $\Omega = \Omega_0 \cup \Omega_+$, that

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{i=1}^{N} f(T^{k_i}\cdot) \right\|_p = 0 . \tag{3}$$

But clearly it suffices to show (3) for $f = 1_A$ with $m(A) < \infty$. Since

$$\left\| \frac{1}{N} \sum_{i=1}^{N} 1_A(T^{k_i}\cdot) \right\|_p^p \leqq \left\| \frac{1}{N} \sum_{i=1}^{N} 1_A(T^{k}\cdot) \right\|_q^q$$

for $p > q > 1$, it may and will assume without loss of generality that $p < 2$. Set $\delta = p - 1$, and define

$$\begin{cases} a_{N,k} = \dfrac{1}{N} & \text{if} \quad k \in \{k_i; 1 \leqq i \leqq N\} , \\ a_{N,k} = 0 & \text{otherwise} . \end{cases}$$

Then

$$\begin{aligned} \left\| \frac{1}{N} \sum_{i=1}^{N} 1_A(T^{k_i}\cdot) \right\|_p^p &= \int \left(\sum_{k=0}^{\infty} a_{N,k} 1_A(T^k\omega) \right)^{1+\delta} dm(\omega) \\ &= \sum_{k=0}^{\infty} a_{N,k} \int_{T^{-k}A} \left(\sum_{i=0}^{\infty} a_{N,i} 1_A(T^i\omega) \right)^{\delta} dm(\omega) . \end{aligned}$$

Let $\varepsilon > 0$ be given. It follows from (1) that there exists a subset S of the nonnegative integers having density zero such that

$$\lim_{k\to\infty} m(T^{-k}A \cap A) = 0 ,$$

provided $k \notin S$. Choose a positive integer n_0 such that $|i - j| > n_0$ and $|i - j| \notin S$ imply

$$m(T^{-i}A \cap T^{-j}A) < \varepsilon . \tag{4}$$

Let $D(k, n_0)$ denote the set of nonnegative integers i such that $|i-k| \leqq n_0$. Since $0 < \delta < 1$, we have

$$\begin{aligned}
&\int_{T^{-k}A}\Big(\sum_{i=0}^{\infty} a_{N,i}1_A(T^i\omega)\Big)^{\delta} dm(\omega) \\
&\quad \leqq \int_{T^{-k}A}\Big(\sum_{i\in D(k,n_0)} a_{N,i}1_A(T^i\omega)\Big)^{\delta} dm(\omega) \\
&\qquad + \int_{T^{-k}A}\Big(\sum_{\substack{i\notin D(k,n_0)\\ |i-k|\notin S}} a_{N,i}1_A(T^i\omega)\Big)^{\delta} dm(\omega) \\
&\qquad + \int_{T^{-k}A}\Big(\sum_{\substack{i\notin D(k,n_0)\\ |i-k|\in S}} a_{N,i}1_A(T^i\omega)\Big)^{\delta} dm(\omega) \\
&\quad = I(N, k) + II(N, k) + III(N, k) .
\end{aligned}$$

Since $|a_{N,i}| \leqq 1/N$, we can choose an N_0 such that $N > N_0$ implies

$$I(N, k) \leqq \int_{T^{-k}A}\Big(\frac{2n_0+1}{N}\Big)^{\delta} dm < \varepsilon .$$

On the other hand if we define for any given $\varepsilon_1 > 0$ a measurable set $G(N, k; \varepsilon_1)$ by

$$G(N, k; \varepsilon_1) = \Big\{\sum_{\substack{i\notin D(k,n_0)\\ |i-k|\notin S}} a_{N,i}1_A(T^i\omega) > \varepsilon_1\Big\} ,$$

then

$$\begin{aligned}
II(N, k) &\leqq \int_{T^{-k}A}(\varepsilon_1^{\delta} + 1_{G(N,k;\varepsilon_1)})dm \\
&\leqq \varepsilon_1^{\delta} m(A) + m(T^{-k}A \cap G(N, k; \varepsilon_1) \\
&< \varepsilon_1^{\delta} m(A) + \frac{\varepsilon}{\varepsilon_1} ,
\end{aligned}$$

since $\varepsilon_1 m(T^{-k}A \cap G(N, k; \varepsilon_1) < \varepsilon$ by (4). Hence it is sufficient to show that

$$\lim_{N\to\infty} \sum_{k=0}^{\infty} a_{N,k} III(N, k) = 0 .$$

Clearly,

$$\begin{aligned}
\sum_{k=0}^{\infty} a_{N,k} III(N, k) &\leqq \sum_{k=0}^{\infty} a_{N,k}\int_{T^{-k}A}\Big(\sum_{|i-k|\in S} a_{N,i}\Big)^{\delta} dm \\
&\leqq \sum_{k=0}^{\infty} a_{N,k}\Big(\sum_{\alpha\in S} a_{N,k+\alpha}\Big)^{\delta} m(A) \\
&\qquad + \Big(\sum_{\substack{\alpha\in S\\ \alpha\leqq k_N}} a_{N,k_N-\alpha}\Big)^{\delta} m(A) .
\end{aligned}$$

An argument similar to [3] can be applied to infer that

$$\lim_{N\to\infty} \sum_{k=0}^{\infty} a_{N,k}\Big(\sum_{\alpha\in S} a_{N,k+\alpha}\Big)^{\delta} = 0 .$$

On the other hand, since

$$\lim_{N\to\infty}\frac{k_N}{N} = \mu(Y)^{-1} < \infty$$

by [1], we have

$$\lim_{N\to\infty}\Big(\sum_{\substack{\alpha\in S\\ \alpha\leqq k_N}} a_{N,k_N-\alpha}\Big) = \lim_{N\to\infty}\Big(\frac{k_N}{N}\frac{|\{\alpha\in S;\alpha\leqq k_N\}|}{k_N}\Big) = 0\,.$$

This completes the proof.

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