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DISCONTINUOUS CHARACTERS AND SUBGROUPS OF FINITE INDEX

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For brevity's sake, a subgroup of finite index is called "large." A discontinuous torsion character is (clearly) one whose kernel is a large nonopen subgroup. Compact Abelian groups (and certain other LCA groups) have the same number (either none or infinitely many) of large nonopen subgroups and discontinuous torsion characters. Locally compact Abelian groups of which all large subgroups are open include connected, locally connected, and monothetic (possibly totally disconnected) groups. Contrariwise, there are locally compact groups G which have as many as $2^{|G|}$ large nonopen subgroups. These include nondiscrete torsion Abelian groups of bounded order and all totally disconnected, nonmetrizable, compact groups.

1.1. Notation and discussion. In what follows, everything referred to as a locally compact (or compact) group will be assumed to be Hausdorff and nondiscrete, except when it is necessary to refer to a discrete group or a nondiscrete finite group in the course of a proof. If G is a group, G_d denotes G with the discrete topology. If n is a positive integer, $G^{(n)} = \{x^n: x \in G\}$. The symbol e will universally denote the identity of a group. For two topological groups G_1 and G_2, "$G_1 \cong G_2$" means that G_1 and G_2 are topologically isomorphic. The component of e in a topological group G is denoted by C_G. The *weight* of G, denoted by $w(G)$, is the smallest cardinal number for a basis for the topology of G. The symbol T will denote the *circle group* (of complex numbers with modulus one); a *character* on an Abelian group G is a homomorphism (continuous or not) from G into T. If X is a set, $|X|$ denotes the cardinal number of T.

Vol. I of the treatise of Hewitt and Ross [3] will be used as an encyclopedic reference; however, some elementary facts to be found in any text on topological groups will be assumed and used without comment.

It is evident that the closure of a large subgroup (as defined in the synopsis) of any topological group is always a large open subgroup, and that every open subgroup of a compact group is large. As stated in the synopsis, the kernel of a discontinuous character of finite order is a large nonopen subgroup. More generally, a large subgroup of an Abelian topological group is nonopen if and only if it is contained in the kernel of a discontinuous torsion character. (Equivalently, a finite Abelian topological group has discontinuous characters if and only if the group is not discrete.)

The first few results of the sequel have to do with necessary conditions for the existence of large nonopen subgroups. A fundamental relationship between large nonopen subgroups and discontinuous torsion characters is established in Theorem 1.7; the remainder of this section is concerned with consequences of this result.

LEMMA 1.2. *Let G be a locally compact Abelian or compact group. Let L be a large nonopen subgroup of G and let $C = C_G$. Then $C \subset L$ and C is not open.*

Proof. By the results in [7], or by 24.24 of [3] (whichever is appropriate) C is contained in a divisible subgroup H of G. Let $n = [G: L]$; then $x^{n!} \in L$ for all $x \in G$, and thus $C \subset H \subset L$. Since L has void interior, C is not open.

COROLLARY 1.3. *Let G be a group as in 1.2. If G has only finitely many open subgroups, every large subgroup of G is open.*

Proof. C is open in this case.

[*Note*: (1.10.3), below, shows that 1.3 need not be true if G is not Hausdorff.]

THEOREM 1.4. *Every large subgroup of a compact monothetic group is open.*

Proof. Let L be a large subgroup of G, let $n = [G: L]$, let $J = G^{(n)}$, and let H be a dense cyclic subgroup of G. We have $J \subset L$, and J is compact and therefore closed. Now HJ/J is a dense subgroup of G/J, but HJ/J is cyclic and each of its elements has order $\leqq n$; thus $|HJ/J| \leqq n$. Since G/J is a Hausdorff space with a finite dense subset, it is finite. Thus J is open and since $J \subset L$, L is open also.

COROLLARY 1.5. *A character on a compact totally disconnected monothetic group is continuous if and only if it is of finite order.*
[*Note*: Such groups are characterized in [3], 10.1–10.6 and 25.16.]

Proof. The theorem implies that every character of finite order is continuous. The converse is contained in 24.26 of [3].

LEMMA 1.6. *Let A and B be two sets, at least one of which is infinite. Let $\mathscr{A}$ be a collection of finite subsets of A. Then:*

(1) *If* $|B| \leqq |\mathscr{A}|$, $|B| \leqq |A|$.
(2) *If* $\bigcup \mathscr{A} = A$ *and* $|\mathscr{A}| \leqq |B|$, $|A| \leqq |B|$.

Proof. A simple exercise.

THEOREM 1.7. *Let G be a locally compact Abelian group, D the set of discontinuous torsion characters on G, and $\mathscr{L}$ the family of large non-open subgroups of G. Then, if either D or $\mathscr{L}$ is infinite, $|D| = |\mathscr{L}|$.*

Proof. Let Γ denote the group of characters on G, and K (respectively, K_c) denote the group of (continuous) torsion characters on G. We may consider Γ to be the dual group of G_d, and K its torsion subgroup. Similarly, K_c is the torsion subgroup of the dual group of G.

We may assume that $\mathscr{L} \neq \varnothing \neq D$. Let ϕ be any member of K, and let $S(\phi)$ denote the set of all members of K which have the same kernel as ϕ. Clearly, if $\phi \in D$, then $S(\phi) \subset D$. Also, $\psi(G) = \phi(G)$ for all $\psi \in S(\phi)$, since both groups are finite subgroups of T and have the same order. For each $\psi \in S(\phi)$, there is an automorphism γ of $\phi(G)$ such that $\psi = \gamma \circ \phi$, namely $\gamma(t) = \psi(\phi^{-1}(t))$ $(t \in \phi(G))$. But $\phi(G)$ has only finitely many automorphisms; thus $S(\phi)$ is finite. Let $\mathscr{S} = \{S(\phi)\colon \phi \in D\}$. Then $\bigcup \mathscr{S} = D$ and $|\mathscr{S}| \leqq |\mathscr{L}|$. By 1.6, if either $\mathscr{L}$ or D is infinite, $|D| \leqq |\mathscr{L}|$, and we may assume henceforth that $\mathscr{L}$ is infinite. Since $D = K \cap (K_c)' \neq \varnothing$, D contains a coset of K_c; thus $|K_c| \leqq |D| \leqq |\mathscr{L}|$. If $|\mathscr{L}| \leqq |K_c|$, we are through. Suppose contrariwise that $|K_c| < |\mathscr{L}|$. For any large subgroup H, let $A(H) = \{\psi \in \Gamma\colon \psi(H) = 1\}$; clearly $A(H)$ is a finite subgroup of K, and $A(H) \subset K_c$ if and only if H is open. Further, $H \rightarrow A(H)$ is a one-to-one correspondence, since $H = \{x \in G\colon \psi(x) = 1$ for all $\psi \in A(H).\}$ (See (24.10) of [3].) Let

$$\mathscr{A} = \{K_c \cap A(H)\colon H \in \mathscr{L}\},$$

a collection of finite subgroups of K_c with $|\mathscr{A}| \leqq |\mathscr{L}|$. If $\mathscr{A}$ is finite, $|\mathscr{A}| < |\mathscr{L}|$ (trivially). If $\mathscr{A}$ is infinite, then K_c is infinite and thus $|\mathscr{A}| \leqq |K_c| < |\mathscr{L}|$. In either case, there exists a subcollection $\mathscr{M}$ of $\mathscr{L}$ such that $|\mathscr{M}| = |\mathscr{L}|$ and $K_c \cap A(H_1) = K_c \cap A(H_2)$ for all $H_1, H_2 \in \mathscr{M}$. But if $H_1 \neq H_2$, $A(H_1) \neq A(H_2)$, and therefore $D \cap (A(H_1)) \neq D \cap (A(H_2))$ (since $D = K \cap (K_c)'$). Consequently, $\mathscr{B} = \{D \cap A(H)\colon H \in \mathscr{M}\}$ has the same cardinality as $\mathscr{M}$; but then $|\mathscr{L}| = |\mathscr{M}| = |\mathscr{B}|$, thus $|\mathscr{L}| \leqq |D|$.

THEOREM 1.8. *Let G be a compact Abelian group, and $\mathscr{U}$ the family of open subgroups of G. If $D \neq \varnothing$, then $|D| \geqq |\mathscr{U}|$. Thus G has either infinitely many large nonopen subgroups, or none.*

Proof. Suppose $D \neq \emptyset$, so that $\mathscr{L} \neq \emptyset$ and (by 1.3) $\mathscr{U}$ is infinite. Now (as in the proof of 1.7) $|D| \geqq |K_c|$ and the mapping $H \to A(H)$ is one-to-one from $\mathscr{U}$ onto the collection of all finite subgroups of K_c; thus $|\mathscr{U}| \leqq |K_c| \leqq |D|$ and (by 1.7) $|D| = |\mathscr{L}|$.

LEMMA 1.9. *Let G be a locally compact Abelian group with a large nonopen subgroup. Then every open subgroup of G has a large nonopen subgroup. If G is compact and totally disconnected, every neighborhood of the identity contains a large nonopen subgroup of G.*

Proof. Let $U \in \mathscr{U}$ and $L \in \mathscr{L}$. Then $[U: U \cap L] = |\{uL: u \in U\}| \leqq [G: L]$. $U \cap L$ is not open since L has void interior. If G is compact and totally disconnected, every neighborhood of e contains a (large) open subgroup of G. If U is one such subgroup, $[G: U \cap L] = [G: U] \cdot [U: U \cap L] \leqq [G: U] \cdot [G: L]$. Thus $U \cap L$ is a large nonopen subgroup of G.

THEOREM 1.10. *Let G be an infinite compact Abelian torsion group.*
Then

(1) *G has $2^{|G|}$ discontinuous torsion characters* (*and, therefore, $2^{|G|}$ large nonopen subgroups*);

(2) *Every neighborhood of the identity contains a large nonopen subgroup of G.*

(3) *Every proper dense subgroup of G is contained in a large proper subgroup of G.*

Proof. (1) By 25.9 of [3], $G^{(n)} = \{e\}$ for some integer n, so every character on G is of finite order. Now the characters on G form a compact group (namely, the dual group of G_d) whose weight is $|G|$ and whose cardinality is therefore $2^{|G|}$. (See 24.15 and 24.47 of [3].) But $|K_c| = w(G)$, where $|G| = 2^{w(G)} > W(G)$; thus G has $2^{|G|}$ discontinuous characters. (The rest of the statement follows from 1.7.)

(2) By 24.21 of [3], G is totally disconnected, so Lemma 1.9 applies.

(3) If H is a proper dense subgroup of G, there is a character ψ on G such that $\psi(H) = 1$ and $\psi(G) \neq 1$. The kernel of ψ is a large proper subgroup containing H.

Note: Statement (3) is a generalization of an example, communicated by K. A. Ross, in which $G = \{0, 1\}^{A}$. (Ross' proof is as follows: Choose $x \in G \cap H'$ and let ψ be the homomorphism of $H \oplus [x]$ into T such that $\psi(H) = 1$ and $\psi(x) = -1$. Extend ψ to a character of order 2 on G.)

COROLLARY 1.11. *Let G be a locally compact Abelian group of bounded order. Then statements* (1) *and* (3) *of* 1.10 *are true.*

Proof. As before (in 1.10.1), G has $2^{|G|}$ characters, all of finite order. Since G is totally disconnected, it has a compact open subgroup. (See 24.21 and 7.7 of [3].) This subgroup has a discontinuous character which may be extended to a character on G. Thus G has $2^{|G|}$ discontinuous characters. (Using the terminology of 1.7, we have $D \neq \emptyset$. Thus $|D| \geqslant |K_c|$, and since $|D \cup K_c| = 2^{|G|}$, $|D| = 2^{|G|}$.) The proof of statement (3) can be applied just as in 1.10.

Note: For an example of a locally compact group which has exactly the same number of open and nonopen large subgroups, let $G = H \times H_d$ where H is a compact Abelian torsion group. Here, G and G_d have the same number of continuous characters, namely $2^{|G|}$ of each.

2. Large nonopen subgroups of compact groups.

Discussion 2.1. Let $\{V_\alpha : \alpha \in A\}$ be an infinite collection of open subgroups of a locally compact group G. Let $\mathscr{F}$ be a free ultrafilter on A. We write

$$L(\mathscr{F}, V_\alpha) = \bigcup_{F \in \mathscr{F}} \bigcap_{\alpha \in F} V_\alpha \,.$$

In this section, we describe sufficient conditions under which $L(\mathscr{F}, V_\alpha)$ is a large nonopen subgroup of G.

The reader may recall that a *free ultrafilter* is a maximal filter with no fixed point (see [2], 2.2–2.6 and 12.2). Thus if $\mathscr{F}$ is a free ultrafilter on A, and $\{F_1, \cdots, F_n\}$ is a finite partition of A, exactly one of the cells F_j is in $\mathscr{F}$. Also, no finite subset of A is a member of $\mathscr{F}$.

LEMMA 2.2. *Let G be a compact group, $\mathscr{U}$ and $\mathscr{V}$ infinite collections of open subgroups of G, with $\mathscr{U} \subset \mathscr{V}$ and $\bigcap \mathscr{U} = \bigcap \mathscr{V}$. Then $|\mathscr{U}| = |\mathscr{V}|$.*

Proof. We may assume that $\mathscr{U}$ is closed under finite intersections. It follows that each member of $\mathscr{V}$ is the union of (finitely many) cosets of a member of $\mathscr{U}$.

LEMMA 2.3. *Suppose H is large open subgroup of a locally compact group G, J is normal subgroup of H, and L/J is a large nonopen subgroup of H/J (where $J \subset L \subset H$). Then L is a large nonopen subgroup of G.*

Proof. We have

$$[G\colon L] = [G\colon H]\ [H\colon L] = [G\colon H]\ [H/J\colon H/L];$$

i.e., L is large in G. If L is open, L is locally compact. But then L/J is locally compact, which is impossible since L/J is not closed in H/J. ([3], 5.22 and 5.11).

THEOREM 2.4. *Let G be a compact group. Suppose there is an infinite collection $\mathscr{U}$ of open normal subgroups of G and a fixed integer n such that $|G/U| \leqq n$ for all $U \in \mathscr{U}$. Then G has 2^{2^r} large nonopen subgroups, where $r = |\mathscr{U}|$.*

Proof. Case (1). Suppose $G = S^A$ where S is a finite group and A is an infinite set. *For* $\alpha \in A$, define $V_\alpha = \{x \in G\colon x_\alpha = e\}$. (Here, $\mathscr{U} = \{V_\alpha\colon \alpha \in A\}$, $n = |S|$, and $r = |A|$.) Let $\mathscr{F}$ be any free ultrafilter on A, and $L = L(\mathscr{F}, V_\alpha)$. One sees that L is a dense normal subgroup of G, via a minor generalization of the familiar proposition that

$$\{x \in G\colon x_\alpha = e \text{ for all but finitely many } \alpha\}$$

is a dense normal subgroup. Now, given any $x \in G$, there exists exactly one element $s(x)$ of S such that $\{\alpha \in A\colon x_\alpha = s(x)\}$ is a member of $\mathscr{F}$. Thus $[G\colon L] = |S|$; in fact, $(G/L)_d \cong S$. This proves that L is a large, but proper, subgroup of G, and since L is dense it is not open. Now suppose $\mathscr{F}_1$ and $\mathscr{F}_2$ are distinct ultrafilters on A. We will show that $L(\mathscr{F}_1, V_\alpha) \neq L(\mathscr{F}_2, V_\alpha)$. For there is a subset X of A such that $X \in \mathscr{F}_1$ and $A \cap X' \in \mathscr{F}_2$; let $e \neq s \in S$, and define

$$x_\alpha = \begin{cases} e, \alpha \in X, \\ s, \alpha \in A \cap X' . \end{cases}$$

Then $(x_\alpha\colon \alpha \in A)$ is in $L(\mathscr{F}_1, V_\alpha)$ but not $L(\mathscr{F}_2, V_\alpha)$. According to [2], (9F), A has 2^{2^r} ultrafilters, of which only r are not free. Thus G has 2^{2^r} large nonopen subgroups.

Case (2). Suppose all the quotient groups G/U are simple and isomorphic to each other and $\bigcap \mathscr{U} = e$. We can then choose a subfamily $\{U_\alpha\colon \alpha \in A\}$ of $\mathscr{U}$ such that

$$G \cong \prod_{\alpha \in A} G/U_\alpha$$

via the mapping $g \rightarrow (gU_\alpha\colon \alpha \in A)$ (as in Varopoulos [8], § 4, Theorem 2). By 2.2 (above), $|A| = |\mathscr{U}|$. Case (1) now applies. G has 2^{2^r} large nonopen subgroups of the form $L(\mathscr{F}, U_\alpha)$.

Case (3). In general, we may choose a subfamily $\{U_\alpha : \alpha \in A_0\}$ of $\mathscr{U}$, with $|A_0| = |\mathscr{U}|$ and $U_\alpha \neq U_\beta$ for $\alpha \neq \beta$, and such that all of the quotient groups G/U_α are isomorphic; this is possible since there are only finitely many groups (to within isomorphism) of order $\leqq n$. Let

$$U_\alpha = G_\alpha^m \subset G_\alpha^{m-1} \subset \cdots \subset G_\alpha^0 = G$$

be a maximal subnormal series (modulo U_α), chosen so that m is the same for each α, and so that

$$G_\alpha^{j-1}/G_\alpha^j \cong G_\beta^{j-1}/G_\beta^j$$

for all $\alpha, \beta \in A_0$ and each j $(1 \leqq j \leqq m)$. Let k be the smallest integer with the property that there are r distinct subgroups G_α^k. Let $V_\alpha = G_\alpha^k$ for each $\alpha \in A_0$. We now have an open subgroup H of G and a subset A_1 of A_0, such that $|A_1| = r$, $H = G_\alpha^{k-1}$ for all $\alpha \in A_1$, and $V_\alpha \neq V_\beta$ for $\alpha \neq \beta$. Let $J = \bigcap_{\alpha \in A_1} V_\alpha$. Recall that each V_α is an open maximal normal subgroup of H, and all of the groups H/V_α are isomorphic. Case (2) applies to the group H/J, which therefore has 2^{2^r} large nonopen subgroups. Since H is open (and therefore large) in G, each of these subgroups can be lifted to a large nonopen subgroup of G, as in 2.3.

THEOREM 2.5. *Let G be a compact totally disconnected nonmetrizable group. Then*:

(1) *G has $2^{|G|}$ large nonopen subgroups.*

(2) *Every neighborhood of e contains a large nonopen subgroup of G.*

(3) *If G is Abelian, G has $2^{|G|}$ discontinuous torsion characters.*

Proof. Let $\mathscr{B}$ be any collection of open normal subgroups of G, such that $\bigcap \mathscr{B} = e$. By 2.2, and the obvious generalization of Theorem 1.2 of [6], $|\mathscr{B}| = w(G)$. Since $\mathscr{B}$ is uncountable, there must be some integer n and some subcollection $\mathscr{U}$ of $\mathscr{B}$ such that $|\mathscr{U}| = w(G)$ and $|G/U| \leqq n$ for all $U \in \mathscr{U}$. Theorem 2.4 therefore applies, and, since $2^{w(G)} = |G|$ (see [4]), statement (1) follows.

(2) follows from the proof of 1.9; (3) follows from 1.7.

COROLLARY 2.6. *Let G be a locally compact group such that G/C_G is compact and nonmetrizable. Then G has at least 2^c large nonopen subgroups. If G is Abelian, it has at least 2^c discontinuous torsion characters.*

3. Examples and questions.

3.1. Let $G = \prod_{n=1}^{\infty} Z(2^n)$, where $Z(m)$ denotes the additive group of integers modulo m. Let

$$J_1 = \{0\}, \cdots, J_n = \{0, 2, \cdots, 2^n - 2\} \in Z(2^n), \cdots, \text{ and } J = \prod_{n=1}^{\infty} J_n .$$

Then $G/J = \{0, 1\}^{\omega}$, so G has 2^c large nonopen subgroups, by 2.3 and Case (1) of 2.4. (Alternately, G has 2^c discontinuous characters of order 2.)

Note: This G contains an infinite compact subgroup with no large subgroups, for the monothetic group Δ_2 of dyadic integers is topologically isomorphic to the projective limit of the groups $Z(2^n)$. (Let $\Lambda_n = \{x \in \Delta_2 : x_k = 0, 0 \leqq k < n\}$, as in 10.4 of [3]. According to p. 25 of [9], Δ_2 is an inverse limit of the quotient groups Δ_2/Λ_n. It is clear that $\Delta_2/\Lambda_n \cong Z(2^n)$. See also 6.13–6.14 of [3] and the references noted in 1.5, above.)

3.2. Let $G = \prod_{n=5}^{\infty} S_n$, $J = \prod_{n=5}^{\infty} A_n$, where S_n is the symmetric group, and A_n is the alternating group, on n letters. As in the previous example, G has 2^c large nonopen subgroups. Its subgroup J has no large nonopen subgroups, for if L is a subgroup of J with $J^{(k)} \subset L$, we have (assuming $k \geqq 4$)

$$L \supset \{e\}^k \times \prod_{n=k+1}^{\infty} A_n ,$$

which is open in J.

3.3. Suppose G is an infinite, compact, metrizable, non-Abelian torsion group. (By 4.5 of [5], G is totally disconnected.) *Question*: Does G have any large nonopen subgroups?

3.4. Let $G = \prod_{n=1}^{\infty} G_n$, where each G_n is a finite simple group and $\lim_{n\to\infty} |G_n| = \infty$. *Question*: Can G have a large nonopen subgroup?

[The author conjectures that $G_n^{(k)} \neq \{e\}$ for n sufficiently larger than k; this would imply that the answer to the question is in the negative, as in 3.2. If the conjecture should prove to be false, we would then have a special case of 3.3, wherein G would be semisimple and of bounded order.]

3.5. In order for the conclusion of 1.7 not to hold, G must be noncompact, have infinitely many open subgroups, only finitely many large subgroups, and at least one large nonopen subgroup. The groups we have seen in this paper either have no large nonopen subgroups at all, or else 2^{2^r} such subgroups, where r is the cardinal number of some infinite collection of large open subgroups. *Question*: Are there other possibilities?

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